Characterisations of Perfect Graphs

Martin Grötschel

If $G = [V, E]$ is a graph then we denote by $\alpha(G)$ the stability number of G, by $\omega(G)$ the clique number, by $\chi(G)$ the chromatic number, and by $\rho(G)$ the clique covering number of G. If $W \subseteq V$ is a set of nodes, then $G[W]$ denotes the subgraph of G induced by W.

It is obvious that $\alpha(G) \le \rho(G)$ and $\omega(G) \le \chi(G)$ holds for all graphs G. C. Berge (1961) defined the following concepts. If G is a graph then G is called *α-perfect* if

$$\alpha(G[W]) = \rho(G[W]) \text{ for all } W \subseteq V,$$

G is called *χ-perfect* if

$$\omega(G[W]) = \chi(G[W]) \text{ for all } W \subseteq V,$$

and G is called *perfect* if G is α-perfect and χ-perfect. Berge conjectured that a graph is α-perfect if and only if it is χ-perfect. This is equivalent to saying that a graph G is α-perfect if and only if its complementary graph $\overline{G}$ is α-perfect.

This conjecture was settled in the affirmative by L. Lovász (1972a). His result is often called "the perfect graph theorem". Lovász (1972b) gave a further characterization by showing that a graph is perfect if and only if

$$\alpha(G[W]) \cdot \omega(G[W]) \ge |W| \text{ for all } W \subseteq V.$$

Perfect graphs also have a polyhedral characterization. Let $P(G)$ be the convex hull of the incidence vectors of all stable sets in a graph G.

Then it is easy to see that the nonnegativity constraints $x_v \ge 0$, $v \in V$, and the clique constraints $x(C) := \sum_{v \in C} x_v \le 1$ for all cliques $C \subseteq V$ are valid inequalities for $P(G)$. This means that the so-called *fractional stable set polyhedron*

$$P^*(G) := \{x \in \mathbb{R}^V \mid x_v \ge 0 \text{ for all } v \in V,$$
$$\sum_{v \in C} x_v \le 1 \text{ for all cliques } C \subseteq V\}$$

contains the *stable set polyhedron* $P(G)$. Denote by $\alpha_c(G)$ resp. $\alpha_c^*(G)$ the optimum values of the linear programs $\max c^T x, x \in P(G)$ resp. $\max c^T x, x \in P^*(G)$, where $c : V \to \mathbb{Z}$ is an objective function, then Fulkerson (cf. Trotter (1973)) and Chvátal (1975) showed that a graph G is perfect if and only if $P(G) = P^*(G)$ which is equivalent to $\alpha_c(G) = \alpha_c^*(G)$ for all $c : V \to \mathbb{Z}$.

If $e \in \mathbb{R}^V$ is the vector such that all its components are one, then $\alpha_e^*(G)$ is usually denoted by $\alpha^*(G)$ and called the *fractional stability number*. A sharpening of the Fulkerson-Chvátal result above states that G is perfect if and only if

$$\alpha(G[W]) = \alpha^*(G[W]) \text{ for all } W \subseteq V.$$

Books treating perfect graphs and containing some of the concepts introduced so far are Berge (1973), Golumbic (1980), and Berge & Chvátal (1984).

In connection with studies on the Shannon capacity of a graph L. Lovász (1979) introduced a number, denoted by $\vartheta(G)$, which can be defined as follows:

Let $G = [V, E]$ be a graph, $V = \{1, 2, ..., n\}$, then denote by $(\mathcal{A}(G)$ the set of symmetric (n,n)-matrices $A = (a_{ij})$ which satisfy $a_{ij} = 1$ if $i = j$ or if $ij \notin E$. If $\Lambda(A)$ denotes the maximum eigenvalue of A, then

$$\vartheta(G) := \min\{\Lambda(A) \mid A \in \mathcal{A}(G)\}.$$

Suppose $\alpha(G) = k$ for a graph G, and assume that $\{1, ..., k\}$ is a stable set in G. Then by definition every matrix $A \in \mathcal{A}(G)$ contains the (k,k)-matrix A_k all whose entries are one as the principal submatrix consisting of the first k rows and columns of A. It is well-known that $\Lambda(A_k) = k$ and that $\Lambda(A_k) \leq \Lambda(A)$ which implies $\alpha(G) \leq \vartheta(G)$. Similarly one can see that $\vartheta(G) \leq \alpha^*(G)$. In Grötschel, Lovász, Schrijver (1984) perfect graphs could be characterized by means of $\vartheta(G)$ in the following way. A graph $G = [V, E]$ is perfect if and only if

$$\alpha(G[W]) = \vartheta(G[W]) \text{ for all } W \subseteq V$$

which is equivalent to

$$\alpha^*(G[W]) = \vartheta(G[W]) \text{ for all } W \subseteq V.$$

Furthermore, it was shown in Grötschel, Lovász, Schrijver (1984) that for critically imperfect graphs G, $\alpha(G) < \vartheta(G) < \alpha^*(G)$ holds and that $\vartheta(G) - \alpha(G)$ and $\alpha^*(G) - \vartheta(G)$ can be polynomially bounded away from zero. This makes it possible to recognize the imperfectness of a critically imperfect graph in polynomial time. Since every imperfect graph contains a critically imperfect graph, imperfectness of a graph can thus be checked in nondeterministic polynomial time.

Another characterization of perfect graphs has recently been discovered by László Lovász, Alexander Schrijver and the author (unpublished).

Let $G = [V, E]$ be a graph and let $w \in \mathbb{R}^*$ be a nonnegative integral vector. Denote by G_w the graph obtained from G by replacing each node $v \in V$ by w_v nonadjacent copies, i. e. each node $v \in V$ is replaced by a stable set S_v of size w_v, and two nodes of G_w, say $s \in S_v$ and $t \in S_u$, are adjacent if and only if their "originals" u and v are different adjacent nodes of G. For every $w \in \mathbb{Z}_+^V$ we define a w-weighted version of $\vartheta(G)$ as follows:

$$\vartheta_w(G) := \vartheta(G_w).$$

Set

$$\text{THETA}(G) := \{x \in \mathbb{R}^V \mid 0 \leq x_v \leq 1 \text{ for all } v \in W$$
$$\sum_{v \in V} w_v x_v \leq \vartheta_w(G) \text{ for all } w \in \mathbb{Z}_+^v\}.$$

THETA(G) is the intersection of (infinitely many) halfspaces and thus a convex set. It is clear from the definition that

$$P(G) \subseteq \text{THETA}(G) \subseteq P^*(G)$$

and hence, if G is perfect equality holds in the two inclusions above. In particular, if G is a perfect graph then THETA(G) is a polytope, since $P(G)$ and $P^*(G)$ are polytopes. The (quite surprising) new characterization of perfect graphs states:

G is perfect if and only if THETA(G) is a polytope.

Summing up the discussion above we obtain the following

Theorem. *Let $G = [V, E]$ be a graph. Then the following statements are equivalent.*

(a) *G is perfect.*

(b) *$\overline{G}$ is perfect.*

(c) $\alpha(G[W]) = \rho(G[W])$ *for all* $W \subseteq V$.

(d) $\omega(G[W]) = \chi(G[W])$ *for all* $W \subseteq V$.

(e) $\alpha(G[W]) \cdot \omega(G[W]) \geq |W|$ *for all* $W \subseteq V$.

(f) $P(G) = P^*(G)$.

(g) $\alpha_c(G) = \alpha_c^*(G)$ *for all* $c : V \to \mathbb{Z}$.

(h) $\alpha_c(G) = \alpha_c^*(G)$ *for all* $c : V \to \{0, 1\}$.

(i) $\alpha(G[W]) = \alpha^*(G[W])$ *for all* $W \subseteq V$.

(j) $\alpha(G[W]) = \vartheta(G[W])$ *for all* $W \subseteq V$.

(k) $\alpha^*(G[W]) = \vartheta(G[W])$ *for all* $W \subseteq V$.

(l) THETA(G) *is a polytope.*

Thus there is a large number of quite different characterizations of perfect graphs available.

A long standing open problem – the so-called *strong perfect graph conjecture* due to C. Berge – is the following. A graph G is perfect if and only if neither G nor $\overline{G}$ contain an odd hole (a cycle of odd length at least five without chord).

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Martin Grötschel
Institut für Mathematik
Universität Augsburg
Memminger Str. 6
D-8900 Augsburg