Symmetric functions and P-recursiveness

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Let $c_k(n)$ be the number of k-regular simple labelled graphs on n vertices. If $\{c_k(n) \mid n \ge 0\}$ satisfies a linear recurrence equation with a fixed number of terms then we say that $\{c_k(n) \mid n \ge 0\}$, for fixed k, is P-recursive [R.P. Stanley, Europ. J. Combinatorics, 1 (1980) 175-188]. Loosely speaking, P-recursive problems are problems for which there is a simple computational scheme. The P-recursiveness of $c_k(n)$ for $k = 2$ has been shown by Anand Dumir and Gupta [Duke Math. J. 33 (1966) 757-770], for $k = 3$ by Read [J. London Math. Soc. 35 (1960) 344-351] for $k = 4$ by Read and Wormald [J. Graph Theory 4 (1980) 203-212], and for $k = 5$ by Goulden and Jackson [unpublished work - differential equation available upon request].

The purpose of this talk is to give a result about symmetric functions, which has application to the calculation of $c_k(n)$, and to raise the question of establishing the P-recursiveness of $c_k(n)$ for larger values of k [see also Goulden, Jackson and Reilly, SIAM J. Alg. Disc. Math. 4 (1983) 179-193].

Let $s_j = t_1^j + t_2^j + \dots$, $\underset{\sim}{s} = (s_1, s_2, \dots)$ where $t_1, t_2, \dots$ are commutative indeterminates. Let $\Gamma: \mathbb{Q}[[\underset{\sim}{s}(\underset{\sim}{t})]] \to \mathbb{Q}[[\underset{\sim}{y}]]:$ $\sum_{\underset{\sim}{i} \geq 0} c_{\underset{\sim}{i}} A_{\underset{\sim}{i}} \to \sum_{\underset{\sim}{i} \geq 0} c_{\underset{\sim}{i}} \frac{\underset{\sim}{y}^{\underset{\sim}{i}}}{\underset{\sim}{i}!}$ where $A_{\underset{\sim}{i}}$ is a monomial symmetric function. We call $\Gamma(f)$ the Hammond series of f. This $\Gamma(f)$ encodes the coordinate sequence of f with respect to the monomial symmetric function basis of $Q[[\underset{\sim}{s}(\underset{\sim}{t})]]$.

Let $G(\underset{\sim}{s}(\underset{\sim}{t})) = \prod_{1 \le i < j} (1 + t_i t_j)$. Then it is easy to show that

$$c_k(n) = [t_1^k \dots t_n^k] G(\underset{\sim}{s}(\underset{\sim}{t})) = \left[\frac{y_k^n}{n!} \right] \Gamma(G)\ (0, \dots, 0, y_k).$$

We show that $\Gamma(G)$ can be obtained indirectly as the solution of a system of formal partial differential series by the following theorem.

Theorem: Let $\phi(\underset{\sim}{y}) = (\Gamma(f))(\underset{\sim}{y})$ be the Hammond series for f. Then

$$(\Gamma(\frac{\partial f}{\partial s_n}))(\underset{\sim}{y}) = \sum_{\underset{\sim}{i} \geq \underset{\sim}{0}} (-1)^{m-1} \frac{1}{\underset{\sim}{i}!} (m-1)! \frac{\partial^{\underset{\sim}{i}}}{\partial \underset{\sim}{y}^{\underset{\sim}{i}}} \phi(\underset{\sim}{y})$$

where $m = i_1 + i_2 + \ldots$ and the summation is over $\underset{\sim}{i} = (i_1, i_2, \ldots)$ such that $i_1 + 2i_2 + \ldots = n$.

$$(\Gamma(s_n f))(\underset{\sim}{y}) = \{y_n + \sum_{i \geq 1} y_{n+i} \frac{\partial}{\partial y_i}\} \phi(\underset{\sim}{y}). \square$$

We can use this theorem to show, for example, that $c_4(n) = [\frac{y_4^n}{n!}]V(0,0,0,y_4)$ where V satisfies

$$\left.\begin{aligned}
\frac{\partial V}{\partial y_1} &= y_1 V + y_2 \frac{\partial V}{\partial y_1} + y_3 \frac{\partial V}{\partial y_2} + y_4 \frac{\partial V}{\partial y_3} \\
2\frac{\partial V}{\partial y_2} - \frac{\partial^2 V}{\partial y_1^2} &= -(1+y_2)V - y_3 \frac{\partial V}{\partial y_1} - y_4 \frac{\partial V}{\partial y_2} \\
3\frac{\partial V}{\partial y_3} - 3\frac{\partial^2 V}{\partial y_1 \partial y_2} + \frac{\partial^3 V}{\partial y_1^3} &= y_3 V + y_4 \frac{\partial V}{\partial y_1} \\
4\frac{\partial V}{\partial y_4} - 4\frac{\partial^2 V}{\partial y_1 \partial y_3} - 2\frac{\partial^2 V}{\partial y_1^3} + 4\frac{\partial^3 V}{\partial y_1^2 \partial y_2} - \frac{\partial^4 V}{\partial y_1^4} &= (1-y_4)V.
\end{aligned}\right\}$$

The general system for $c_k(n)$ can be written down in the same way.

__Open Problem:__ Show that there is a scheme for eliminating indeterminate $y_1, \ldots, y_{k-1}$, to obtain an ordinary differential equation for $V(0,\ldots,0,y_k)$. This will establish the P-recursiveness of $\{c_k(n) \mid n \geq 0\}$ for arbitrary k.