COMMENTS ON THE PROOF OF THE

q-DYSON THEOREM

by

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Dyson conjectured [3] and Gunson [5] and Wilson [7] proved that

(1) $$[1] \prod_{1\le i<j\le n} \left(1-\frac{X_i}{X_j}\right)^{a_i}\left(1-\frac{X_j}{X_i}\right)^{a_j} = \binom{a_1+a_2+\dots+a_n}{a_1,\dots,a_n}$$

$$= \frac{(a_1+\dots+a_n)!}{a_1!\dots a_n!},$$

where [1] denotes the constant term. Good [4] has given a very simple proof of this, observing that both the constant term on the left and the multinomial coefficient on the right satisfy the same boundary conditions and the same recurrence. In fact, if we set

(2) $$f(\underline{X};a_1,\dots,a_n) = \prod_{i<j} \left(1-\frac{X_i}{X_j}\right)^{a_i} \left(1-\frac{X_j}{X_i}\right)^{a_j},$$

then

(3) $$f(\underline{X};a_1,\dots,a_n) = \sum_{i=1}^{n} f(\underline{X};a_1,\dots,a_i-1,\dots,a_n).$$

Equation (3) is easily proved. By the Lagrange interpolation formula for the polynomial of degree n-1 taking on the value 1 at each of $X_1,\dots,X_n$, we have

(4) $$1 = \sum_{i=1}^{n} \prod_{\substack{j=1\\ j\ne i}}^{n} \frac{X-X_j}{X_i-X_j}.$$

Set $X = 0$ and we obtain

(5) $$1 = \sum_{i=1}^{n} \prod_{\substack{j=1\\ j\ne i}}^{n} \frac{1}{1-X_i/X_j}.$$

Equations (3) results when each side of (5) is multiplied by $f(\underline{X};a_1,\dots,a_n)$.

The q-Dyson theorem, conjectured by Andrews [1] and proved by Zeilberger and Bressoud [8], states that :

(6) $$[1] \prod_{1\le i<j\le n} \left(\frac{X_i}{X_j}\right)_{a_i} \left(q\frac{X_j}{X_i}\right)_{a_j} = \begin{bmatrix} a_1+\dots+a_n \\ a_1,\dots,a_n \end{bmatrix}$$

$$= \frac{(1-q)(1-q^2)\dots(1-q^{a_1+\dots+a_n})}{(1-q)\dots(1-q^{a_1})\dots(1-q)\dots(1-q^{a_n})},$$

where $(X)_a = (1-X)(1-Xq)\dots(1-Xq^{a-1})$. The proof given below comes from § 5 of joint work with I.P. Goulden [2]. The q-multinomial coefficient satisfies the recurrence

(7) $$\begin{bmatrix} a_1+\dots+a_n \\ a_1,\dots,a_n \end{bmatrix} = \sum_{k=1}^{n} q^{a_{k+1}+\dots+a_n} \begin{bmatrix} a_1+\dots+a_n-1 \\ a_1,\dots,a_k-1,\dots,a_n \end{bmatrix}.$$

To mimic Good's proof of the Dyson conjecture, we need to verify that the constant term on the left side of (6) has the same boundary conditions (easily verified) and satisfies the same recurrence relation. One hopes that, as in Good's proof, the product of the left side of (6) as a function of $X_1,\dots,X_n$ satisfies the recurrence given in (7). Unfortunately, this is not the case. If we set

(8) $$g(\underline{X}; a_1,\dots,a_n) = \prod_{i<j} \left(\frac{X_i}{X_j}\right)_{a_i} \left(q\frac{X_j}{X_i}\right)_{a_j},$$

then

(9) $$g(\underline{X}; a_1,\dots,a_n) = \sum_{k=1}^{n} q^{a_{k+1}+\dots+a_n} g(\underline{X}; a_1,\dots,a_k-1,\dots,a_n) + E(\underline{X}; a_1,\dots,a_n)$$

and it remains to show that

(10) $$[1]\, E(\underline{X}; a_1,\dots,a_n) = 0 .$$

We derive equation (9) and an explicit form for $E(\underline{X}; \underline{a})$ as follows.

(11) $$g(\underline{X}\,;\,a_1,\ldots,a_n) = \prod_{i<j} \left(\frac{X_i}{X_j}\right)_{a_i} \left(q\,\frac{X_j}{X_i}\right)_{a_j-1} \prod_{i<j} \left(1 - q^{a_j}\,\frac{X_j}{X_i}\right)$$

$$= \prod_{i<j} \left(\frac{X_i}{X_j}\right)_{a_i} \left(q\,\frac{X_j}{X_i}\right)_{a_j-1} \sum_{T} \prod_{i,j} \left(-q^{a_j}\,\frac{X_j}{X_i}\right)^{\chi(i<j,(j,i)\in T)}$$

where the summation is over all tournaments T (complete directed graphs) on n vertices. For $1 \le i < j \le n$, $(i,j) \in T$ means that i beats j or the edge between i and j is directed towards j. $\chi(A)$ is 1 if A is true, 0 if false.

We observe that

(12) $$\left(\frac{X_i}{X_j}\right)_{a_i} \left(q\,\frac{X_j}{X_i}\right)_{a_j-1} \left(-\,\frac{X_j}{X_i}\right) =$$

$$= \left(-\,\frac{X_j}{X_i}\right)\left(1 - \frac{X_i}{X_j}\right)\left(q\,\frac{X_i}{X_j}\right)_{a_i-1} \left(q\,\frac{X_j}{X_i}\right)_{a_j-1}$$

$$= \left(1 - \frac{X_j}{X_i}\right) \left(q\,\frac{X_i}{X_j}\right)_{a_i-1} \left(q\,\frac{X_j}{X_i}\right)_{a_j-1}$$

$$= \left(\frac{X_j}{X_i}\right)_{a_j} \left(q\,\frac{X_i}{X_j}\right)_{a_i-1} .$$

Thus, multiplying by $\left(-\,\frac{X_j}{X_i}\right)$ has the effect of reversing the order of i and j. Since we multiply by this factor if and only if $i < j$ and $(j,i) \in T$, the subscripts are put into the order of winner first, loser second for each pair i,j. That is to say,

(13) $$g(\underline{X}\,;\,a_1,\ldots,a_n) = \sum_{T} \prod_{(i,j)\in T} \left(\frac{X_i}{X_j}\right)_{a_i} \left(q\,\frac{X_j}{X_i}\right)_{a_j-1} q^{a_i \chi(i>j)} .$$

If we now group our tournaments, collecting all those where k loses to everyone else, $k = 1,2,\ldots,n$ and letting Σ^* be the summation over all tournaments where everyone wins at least one match, then

$$(14)\qquad g(\underline{X}\,;\,a_1,\ldots,a_n)$$

$$= \sum_{k=1}^{n} q^{a_{k+1}+\cdots+a_n} \prod_{\substack{i<j\\ i,j\neq k}} \left(\frac{X_i}{X_j}\right)_{a_i} \left(q\,\frac{X_j}{X_i}\right)_{a_j-1} \left(1-q^{a_j}\frac{X_j}{X_i}\right)$$

$$\times \prod_{i\neq j} \left(\frac{X_i}{X_k}\right)_{a_i} \left(q\,\frac{X_k}{X_i}\right)_{a_k-1}$$

$$+ \Sigma^{*} \prod_{(i,j)\in T} \left(\frac{X_i}{X_j}\right)_{a_i} \left(q\,\frac{X_j}{X_i}\right)_{a_j-1} q^{a_i\chi(i>j)}$$

$$= \sum_{k=1}^{n} q^{a_{k+1}+\cdots+a_n}\, g(\underline{X}\,;\,a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_n,a_k-1)$$

$$+ \Sigma^{*} q^{h(\underline{a},T)} \prod_{(i,j)\in T} \left(\frac{X_i}{X_j}\right)_{a_i} \left(q\,\frac{X_j}{X_i}\right)_{a_j-1} .$$

If the constant term of Σ^* is zero, then the symmetry of the constant term in $g(\underline{X}\,;\,a)$ will give us the desired recurrence for the constant term.

We observe that any tournament T counted in Σ^* is non-transitive (contains a cycle). The following theorem, proved in joint work with Goulden [2] and based on ideas from joint work with Zeilberger [8], implies that $E(\underline{X}\,;\,a_1,\ldots,a_n)$ has constant term zero and thus proves the q-Dyson theorem.

THEOREM. Let T be non-transitive tournament on n vertices, then

$$(15)\qquad [1] \prod_{(i,j)\in T} \left(\frac{X_i}{X_j}\right)_{a_i} \left(q\,\frac{X_j}{X_i}\right)_{a_j-1} = 0 .$$

The theorem given above is much stronger than the q-Dyson theorem, and has a number of other corollaries given in the paper with Goulden [2]. Among the other corollaries are

(16) $$[1] \prod_{1\le i<j\le n} \left(\frac{X_i}{X_j}\right)_{a_i} \left(q\,\frac{X_j}{X_i}\right)_{a_j-1} = \left[\begin{matrix} a_1+\dots+a_n \\ a_1,\dots,a_n \end{matrix}\right] \prod_{i=1}^{n} \frac{(1-q^{a_i})}{(1-q^{a_1+\dots+a_i})} ,$$

(17) $$[1] \prod_{1\le i<j\le n} \left(\frac{X_i}{X_j}\right)_{a_i} \left(q\,\frac{X_j}{X_i}\right)_{a_j-\chi(j\ge m)} = \left[\begin{matrix} a_1+\dots+a_n \\ a_1,\dots,a_n \end{matrix}\right] \prod_{i=m}^{n} \frac{(1-q^{a_i})}{(1-q^{a_1+\dots+a_n})} ,$$

both of which had been conjuctured by Kadell [6] .

REFERENCES

[1] G.E. ANDREWS : Problems and prospects for basic hypergeometric functions, pp. 191-224 in "Theory and Applications of Special Functions", ed. R. A. Askey, Academic Press, N.Y.

[2] D.M. BRESSOUD and I.P. GOULDEN : Constant term identities extending the q-Dyson theorem, Trans. Amer. Math. Soc.

[3] F.J. DYSON : Statistical theory of the energy levels of complex systems, I, J. Math. Physics 3 (1962), 140-156.

[4] I.J. GOOD : Short proof of a conjecture of Dyson, J. Math. Physics, 11 (1970), 1884.

[5] J. GUNSON : Proof of a conjecture by Dyson in the statistical theory of energy levels, J. Math. Physics 3 (1962), 752-753.

[6] K.W. KADELL : Andrew's q-Dyson conjecture : n = 4 , Trans. Amer. Math. Soc.

[7] K. WILSON : A proof of a conjecture by Dyson, J. Math. Physics 3 (1962), 1040-1043.

[8] D. ZEILBERGER and D.M. BRESSOUD : A proof of Andrew's q-Dyson conjecture, Discrete Mathematics 54 (1985), 201-224.