THE ABC's OF CLASSICAL ENUMERATION

by

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Accepté pour publication dans:
"Les Annales des Sciences Mathématiques du Québec", en 1986.

Résumé. En utilisant la décomposition des permutations en "arbres binaires croissants", on établit une formule de récurrence qui donne des généralisations et plusieurs q-analogues des polynômes eulériens, des nombres d'Euler, des nombres de Catalan, des polynômes de Stirling de première et de deuxième espèces, des polynômes d'Hermite, des polynômes de Bell, etc. Dans plusieurs cas, les séries génératrices des q-analogues obtenus peuvent être exprimées sous forme de produits infinis.

Abstract. Using the "arbre binaire croissant" permutation decomposition, a relatively simple recurrence relationship is derived that provides refinements, generalizations, and several q-analogs of the Eulerian polynomials, the Euler numbers, the Catalan numbers, the Stirling polynomials of both the first and second kinds, the Hermite polynomials, the Bell polynomials, and others. In several cases, the generating functions for the q-analogs may be expressed in the form of infinite products.

Avec l'appui financier du programme FCAR (Québec, EQ 1608) et du CRSNG (Canada, A5660).

1. **Introduction**. The similarity in form of the following recurrence relationships, which respectively define $(n+1)!$, the Eulerian polynomials, the Catalan numbers, and the Stirling polynomials of the first and second kinds,

(1.1) (a) $$L_{n+1} = \sum_{k=0}^{n} \binom{n}{k} L_k L_{n-k}, \qquad L_0 := 1$$

(b) $$A_{n+1}(t) = A_n(t) + t \sum_{k=1}^{n} \binom{n}{k} A_k(t) A_{n-k}(t), \qquad A_0(t) := 1$$

(c) $$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \qquad C_0 := 1$$

(d) $$s_{n+1}(y) = y \sum_{k=0}^{n} \binom{n}{k} (n-k)!\, s_k(y), \qquad s_0(y) := 1$$

(e) $$S_{n+1}(y) = y \sum_{k=0}^{n} \binom{n}{k} S_k(y), \qquad S_0(y) := 1$$

lies in the fact that essentially the same counting argument may be used in deriving each one of them. The argument is elementary and is based on the "arbre binaire croissant" permutation decomposition, henceforth referred to as the ABC decomposition, which may be described as follows:

A permutation σ of a set D of n integers will be written as a list $\sigma = \sigma_1 \sigma_2 \ldots \sigma_n$ and the symbol L[D] will denote the set of such lists. For simplicity, L[n] will signify the set of lists of $\{1,2,\ldots,n\}$.

For $\sigma \in L[C]$ where C is a non-empty set of (n+1) integers, let (k+1) be the unique index such that σ_{k+1} is equal to the minimum element in C. Further, let

(1.2) (a) $A := \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$

(b) $B := \{\sigma_{k+2}, \sigma_{k+3}, \ldots, \sigma_{n+1}\}$.

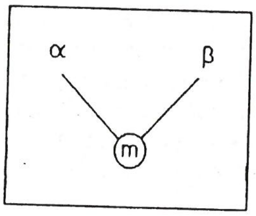
Then, the ABC decomposition of $\sigma \in L[C]$ is defined to be the unique factorization of σ into the sublists indicated by

(1.3) $$\sigma = \alpha\, m\, \beta$$

where $\alpha := \sigma_1 \sigma_2 \ldots \sigma_k \in L[A]$, $m :=$ minimum element of C, and $\beta := \sigma_{k+2} \sigma_{k+3} \ldots \sigma_{n+1} \in L[B]$.

The reason for referring to (1.3) as the ABC decomposition becomes clear if one views $\sigma = \alpha\, m\, \beta$ geometrically as follows:

(1.4)

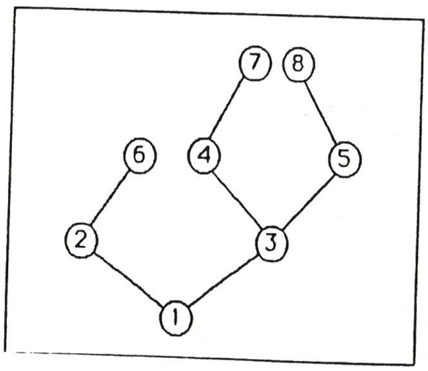


Iteration of (1.4) will produce a unique rooted binary planar tree in which each vertex has a different label from the set C, such that the labels appear in increasing order as one moves up and away from the root. This unique tree is the so-called "arbre binaire croissant" associated to σ as described by Foata and Schützenberger [13]. For example, the ABC corresponding to

(1.5) $$\sigma = 26147385 \in L[8]$$

is the following one:

(1.6)



Note that, given any ABC, the corresponding permutation may be recovered by projecting the labels of the vertices onto a horizontal axis in such a way that the left (respectively right) subtree above a vertex h falls to the left (resp. right) of h.

The essential counting argument underlying the relationships listed in (1.1) is exemplified by the derivation of recurrence (1.1a) which goes as follows: First, observe that (1.3) may be viewed as a bijection from L[C] to the set of 4-tuples

$$(1.7) \qquad \bigcup_{k=0}^{n} \{(A,B;\alpha,\beta) : |A| = k,\ A + B = C\setminus\{m\},\ \alpha \in L[A],\ \beta \in L[B]\}$$

where $|A|$ denotes the cardinality of the set A and A + B signifies the disjoint union of A and B. Also, note that $|L[D]|$ depends only on the cardinality of D. In fact, $|L[D]| = n!$ if $|D| = n$. Then, if we let

$$(1.8) \qquad L_n := |L[n]|,$$

the following calculation based on (1.3) and (1.7)

$$(1.9) \qquad L_{n+1} = \sum_{\sigma \in L[C]} 1 = \sum_{k=0}^{n} \sum_{|A|=k} \sum_{\alpha \in L[A]} \sum_{\beta \in L[B]} 1$$

$$= \sum_{k=0}^{n} \binom{n}{k} L_k L_{n-k}$$

establishes the fact that the recurrence for (n+1)! is indeed (1.1a). As will become apparent later, to obtain the remaining recurrences of (1.1), one merely places various restrictions and weights on σ and then uses the same decomposition.

While it is interesting that the simple argument in (1.9) may be modified and exploited in several settings, the remarkable fact is that the ABC decomposition may be used to derive a single recurrence which contains all of the relationships listed in (1.1). In fact, in addition to containing all of (1.1), the "master" recurrence (3.3) given in Theorem (3.1) of §3 provides refinements, generalizations, and several q-analogs of a multitude of other recurrences pertaining to partitions of sets and to the descent set, the inversion set, the pattern sets, and the cycle type of permutations (see §2 for all definitions). Surprisingly, the proof of Theorem (3.1) is not significantly more difficult than the one given for (1.1a): One begins by observing that certain permutation statistics are compatible with the ABC decomposition and then just follows then through (1.9).

The agenda for this paper is as follows: After providing the necessary background in §2 and proving the master recurrence in §3, a variety of corollaries to Theorem (3.1) will be discussed in sections 4 through 9.

Section 4 is devoted to the presentation of some of the generalized Catalan numbers considered by Carlitz and Riordan [5] and by Fürlinger and Hofbauer [18].

In section 5, examples concerning the descent set of a permutation are given. In particular, the (p,r)-Eulerian polynomials of Gessel [20] and the p-Euler numbers found in [15,20] are considered.

In sections 6 through 8, we present a number of recurrences which pertain to what at this point are best described as "left-to-right" and "right-to-left" cycle type results on permutations. Among the many recurrences of these sections, one will find Gould's [22] p-Stirling numbers of the first type, the p-analog of the number of derangements given in [32], both of the q-Hermite polynomials of Cigler [8], a q-analog of the double Stirling numbers considered by Carlitz and Scoville [7], and others. Interestingly, the "left-to-right" and "right-to-left" cycle types lead to two classes of infinite products that are, in a sense that will become clear, complementary.

In the final section, results concerning "left-to-right" and "right-to-left" partitions of a set C are considered. Recurrence relationships are given for new analogs of the Stirling numbers of the second kind and of the Bell polynomials.

A few remarks are in order at this point:

Remark 1. In selecting and specializing the parameters in the master recurrence to obtain the corollaries of sections 4 through 9, it should be noted that one is at the same time choosing and placing weights on a class of increasing rooted binary trees. Only in some cases will the particular class of ABC's be described.

Remark 2. The approach used in this paper provides a common combinatorial setting for many classic q-analogs that have previously been studied in a variety of contexts.

Remark 3. The ABC decomposition is certainly one of the most basic in combinatorics and has been used by a number of mathematicians to generate various refinements of the Eulerian numbers. Although it would be difficult to give a complete set of references, the influence on the present paper by the works of Foata and Schützenberger [12,13], Françon [16,17], Viennot [17,35], and, in particular, by the work [4] of Bergeron and Reutenauer, should be acknowledged. In fact, the generation of several refinements of the Eulerian numbers in [4] provided the direct stimulus for this paper.

Remark 4. Rather than working with recurrences, the modern school of combinatorics [23, 24, 26, 28, 29, 30] derives a functional or differential equation for the generating series directly from the decomposition being considered. For instance, using the language of UQAM [24, 29, 30], if L denotes the species of lists on linearly ordered sets, then decomposition (1.3) may be written in the compact form

$$L' = L^2. \tag{1.10}$$

Furthermore, if

$$L(x) := \sum_{n\geq 0} L_n x^n/n! \tag{1.11}$$

denotes the exponential generating series of the species L, then (1.10) implies immediately that

$$L'(x) = L^2(x) \tag{1.12}$$

which is equivalent to (1.1.a).

Certainly, this derivation of (1.12) is much clearer and more direct than the one given in (1.9) of (1.1a). However, it is not clear to what extent the generality of Theorem (3.1) can be lifted to the level of generating series, and, for this reason, the focus of this paper will be on recurrences. A starting point for such a lifting perhaps lies in the theory of non-commutative generating functions as developed by Longtin [31] in the study of descent set problems.

2. **Preliminary definitions and facts**. Throughout this section, D will be an n element set of integers. The notions of descent set, "cycle" type, and three letter patterns are defined as follows: The set of descents and the number of descents of a permutation $\sigma \in L[D]$ are respectively defined to be

$$\mathrm{Des}(\sigma) := \{i : \sigma_i > \sigma_{i+1},\ 0 < i < n\}$$

and

$$\mathrm{des}(\sigma) = |\mathrm{Des}(\sigma)|.$$

As an example, for σ in (1.5), note that $\mathrm{Des}(\sigma) = \{2,5,7\}$ and $\mathrm{des}(\sigma) = 3$. For later use, observe that for $\sigma \in L[C]$ as factorized in (1.3) we have

(2.1) (a) $\mathrm{Des}(\sigma) = \{i+1 : i \in \mathrm{Des}(\beta)\}$ if $k = 0$

(b) $\mathrm{Des}(\sigma) = \{k\} \cup \mathrm{Des}(\alpha) \cup \{i + k + 1 : i \in \mathrm{Des}(\beta)\}$ if $0 < k \le n$.

Because of a certain incompatibility with the ABC decomposition, the notion of a cycle in a permutation will be replaced by the enumeratively equivalent notion of a left-to-right (or right-to-left) minimum component [32], which is compatible with (1.3): An integer σ_j in the list $\sigma \in L[D]$ is said to be a left-to-right (respectively right-to-left) minimum if $\sigma_i > \sigma_j$ for $1 \le i < j$ (resp. $\sigma_i < \sigma_j$ for $j < i \le n$). In the terminology of [7], a left-to-right minimum would be referred to as a left lower record. In terms of the associated ABC, the left-to-right (resp. right-to-left) minimums of σ correspond to the labels of the vertices on the extreme left (resp. right) branch. For instance, in (1.6) the left-to-right minimums of σ are $\{1,2\}$ and the right-to-left minimums are $\{1,3,5\}$.

The unique factorization of $\sigma \in L[D]$ as

(2.2) $$\sigma = w_1 w_2 \ldots w_j$$

where each sublist w_i, $1 \le i \le j$, begins with and contains only one left-to-right minimum is referred to as the left-to-right minimum component factorization of σ. For σ in (1.5), we have $\sigma = w_1 w_2$ where $w_1 = 26$ and $w_2 = 147385$.

On the other hand, the factorization

(2.3) $$\sigma = v_1 v_2 \ldots v_r$$

where each sublist v_i ends with and contains only one right-to-left minimum will be referred to as the right-to-left minimum component factorization of σ. For σ in (1.5), observe that $\sigma = v_1 v_2 v_3$ where $v_1 = 261$, $v_2 = 473$, and $v_3 = 85$.

Note that, in the corresponding ABC, each left-to-right (resp. right-to-left) minimum component corresponds to a vertex h on the extreme left (resp. right) branch together with the upper right (resp. left) subtree attached to h.

In order to keep track of the minimum component type of $\sigma \in L[D]$, the statistics

$$\ell ri(\sigma) := |\{w : w \text{ is a sublist in (2.2) of length } i\}|$$

and

$$r\ell i(\sigma) := |\{v : v \text{ is a sublist in (2.3) of length } i\}|$$

will be used. In (1.5), $\ell r2(\sigma) = 1$ and $r\ell 3(\sigma) = 2$. Again for later use, note that for $\sigma = \alpha m\beta$ as in (1.3) we have

(2.4) (a) $m\beta$ is a left-to-right min. component of length $n-k+1$

(b) αm is a right-to-left min. component of length $k+1$.

As in [33], an ordered triple (i, j, h) where $1 \le i < j < h \le n$ is said to be a $2\underline{1}3$ pattern in $\sigma \in L[D]$ if $\sigma_j < \sigma_i < \sigma_h$ and $\sigma_j \le \sigma_r$ for $i \le r \le h$. The triple (4, 6, 7) is just such a pattern for σ in (1.5). Similarly, (i, j, h) is said to be a $3\underline{1}2$ pattern of σ if $\sigma_j < \sigma_h < \sigma_i$ and $\sigma_j \le \sigma_r$ for $i \le r \le h$. The statistics defined by

$$213(\sigma) := |\{\tau : \tau \text{ is a } 2\underline{1}3 \text{ pattern in } \sigma\}|$$

and

$$312(\sigma) := |\{\tau : \tau \text{ is a } 3\underline{1}2 \text{ pattern in } \sigma\}|$$

will be utilized to record the number of $2\underline{1}3$ and $3\underline{1}2$ patterns in σ.

For A and B two disjoint sets of integers, the number of inversions from A to B is defined to be

$$\mathrm{Inv}(A,B) := |\{(i,j) : i \in A,\ j \in B,\ i > j\}|.$$

It is not difficult to verify that the numbers of patterns in $\sigma \in L[C]$ are related to those of $\alpha \in L[A]$ and $\beta \in L[B]$ in (1.3) according to the relationships

(2.5) (a) $213(\sigma) = 213(\alpha) + 213(\beta) + k(n-k) - \mathrm{Inv}(A,B)$

(b) $312(\sigma) = 312(\alpha) + 312(\beta) + \mathrm{Inv}(A,B)$.

The various q-analogs which appear in later sections arise not only in connection with the previously described patterns but also in connection with the well known statistics defined on $\sigma \in L[D]$ by

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{Des}(\sigma)} i$$

and

$$\mathrm{inv}(\sigma) := |\{(i,j) : 1 \le i < j \le n,\ \sigma_i > \sigma_j\}|,$$

which are respectively known as the major index and inversion number. In the setting of (1.3) we have

(2.6) (a) $\mathrm{inv}(\sigma) = k + \mathrm{inv}(\alpha) + \mathrm{inv}(\beta) + \mathrm{Inv}(A,B)$

(b) $\mathrm{maj}(\sigma) = k + \mathrm{maj}(\alpha) + \mathrm{maj}(\beta) + (k+1)\,\mathrm{des}(\beta)$.

Some rudiments of the q-calculus [2,27] will be needed. The q-analog, q-factorial, and q-binomial coefficient of a non-negative integer n are respectively defined to be

$$(n)_q := 1 + q + q^2 + \dots + q^{n-1}$$

$$(n)_q! := (1)_q\,(2)_q \dots (n)_q$$

$$\binom{n}{k}_q := (n)_q! \,/\, (k)_q!\,(n-k)_q!$$

where, by convention, $(0)_q := 0$ and $(0)_q! := 1$. The usual two exponential functions in this setting are

$$(2.7) \qquad \text{(a)} \quad \mathrm{Exq}(x) := \sum_{n\geq 0} q^{n(n-1)/2}\, x^n / (n)_q!$$

$$\text{(b)} \quad \mathrm{exq}(x) := \sum_{n\geq 0} x^n / (n)_q!$$

which have the infinite product expansions

$$(2.8) \qquad \text{(a)} \quad \mathrm{Exq}(x) = \prod_{k\geq 0} [1 + (1-q)\, xq^k]$$

$$\text{(b)} \quad \mathrm{exq}(x) = \prod_{k\geq 0} [1 - (1-q)xq^k]^{-1}$$

for $0 < q < 1$ (see [3, p.19]). The q-exponential generating function $F(x)$ of a q-sequence $f_n(q)$ is defined as

$$(2.9) \qquad F(x) := \sum_{n\geq 0} f_n(q)\, x^n / (n)_q!$$

and the q-derivative of $F(x)$ is given by

$$(2.10) \qquad [d/dx]\, F(x) := \frac{F(x) - F(xq)}{(1-q)x}.$$

It is not difficult to show that

$$(2.11) \qquad [d/dx]\, F(x) := \sum_{n\geq 0} f_{n+1}(q)\, x^n / (n)_q!.$$

In §6, the left-to-right and right-to-left minimum component factorizations give rise respectively to the q-separable differential equations

$$(2.12) \qquad \text{(a)} \quad [d/dx]\, F(x) = W(x)\, F(xq)$$

$$\text{(b)} \quad [d/dx]\, G(x) = W(xq)\, G(x).$$

The solutions of these equations are given by

$$(2.13) \qquad \text{(a)} \quad F(x) = F(0) \prod_{k\geq 0} [1 + (1-q)\, xq^k\, W(xq^k)]$$

$$\text{(b)} \quad G(x) = G(0) \prod_{k\geq 0} [1 - (1-q)\, xq^k\, W(xq^{k+1})]^{-1}$$

provided that $0 < q < 1$. The proof of (2.13a) is straightforward and goes as follows: By (2.10), equation (2.12a) may be rewritten in the equivalent form

$$F(x) = F(xq)\, [1 + (1-q)\, x\, W(x)],$$

which, when iterated, yields (2.13a). The proof of (2.13b) is similar.

For a fixed set D of n integers, the fact (see [23, p. 98]) that

$$(2.14) \qquad \sum q^{\mathrm{Inv}\,(A,B)} = \binom{n}{k}_q,$$

where the sum is over all ordered pairs (A,B) such that $|A| = k$ and $A + B = D$, will be crucial in the proof of Theorem (3.1).

3. **The master recurrence**. The parameters $T_j := (t_{1+j}, t_{2+j}, t_{3+j}, \ldots)$, $Y := (y_1, y_2, y_3, \ldots)$ and $Z := (z_1, z_2, z_3, \ldots)$ are respectively associated with the descent set, the left-to-right minimum components, and the right-to-left minimum components of a permutation $\sigma \in L[D]$ according to (3.1) below:

(3.1) (a) $$T_j^{Des(\sigma)} := \prod_{i \in Des(\sigma)} t_{i+j}$$

(b) $$Y^{\ell r(\sigma)} := \prod_{i \geq 1} y_i^{\ell r i(\sigma)}$$

(c) $$Z^{r\ell(\sigma)} := \prod_{i \geq 1} z_i^{r\ell i(\sigma)}$$

By convention, an empty product will be equal to 1. For convenience, an expression of the form $Z := 1$ will mean that $z_i := 1$ for $i \geq 1$.

For D an n element set of integers, the expression $L_n(u,v,p;\ T_j,Y,Z)$ will be used to denote the generating polynomial

(3.2) $$\sum_{\sigma \in L[D]} u^{213(\sigma)}\, v^{312(\sigma)}\, p^{inv(\sigma)}\, T_j^{Des(\sigma)}\, Y^{\ell r(\sigma)}\, Z^{r\ell(\sigma)}$$

for permutations by patterns, inversions, descent set, and minimum components. Note that $L_n(u,v,p;\ T_j,Y,Z)$ is indeed well defined since (3.2) depends only on the cardinality of D.

For expedience, the parameters u,v, and p will sometimes be suppressed. For instance the symbol $L_n(T_j,Y,Z)$ will occasionally be used to denote the polynomial defined in (3.2). The main theorem of this paper may now be stated and easily proven.

<u>Theorem 3.1.</u> The polynomial defined in (3.2) satisfies the following "master" recurrence:

(3.3) $$L_{n+1}(T_0,Y,Z) = \sum_{k=0}^{n} \binom{n}{k}_q F(k)\, L_k(T_0,Y,1)\, L_{n-k}(T_{k+1},1,Z)$$

where $q := pvu^{-1}$, $F(k) := u^{k(n-k)}\, p^k\, t_k\, y_{n-k+1}\, z_{k+1}$, $t_0 := 1$, and $L_0 := 1$.

<u>Proof.</u> With respect to the ABC decomposition of $\sigma \in L[C]$ given in (1.3), it is clear from (2.1), (2.4), (2.5), and (2.6) that

(3.4) (a) $$T_0^{Des(\sigma)} = T_0^{Des(\alpha)}\, t_k\, T_{k+1}^{Des(\beta)}$$

(b) $$Y^{\ell r(\sigma)} = y_{n-k+1}\, Y^{\ell r(\alpha)}$$

(c) $$Z^{r\ell(\sigma)} = z_{k+1}\, Z^{r\ell(\beta)}$$

(d) $$u^{213(\sigma)} = u^{213(\alpha)}\, u^{213(\beta)}\, u^{k(n-k)}\, u^{-Inv(A,B)}$$

(e) $v^{312(\sigma)} = v^{312(\alpha)}\, v^{312(\beta)}\, v^{\mathrm{Inv}(A,B)}$

(f) $p^{\mathrm{inv}(\sigma)} = p^k\, p^{\mathrm{inv}(\alpha)}\, p^{\mathrm{inv}(\beta)}\, p^{\mathrm{Inv}(A,B)}$.

Placing the expressions $u^{213(\sigma)}$, $v^{312(\sigma)}$, $p^{\mathrm{inv}(\sigma)}$, $T_0^{\mathrm{Des}(\sigma)}$, $y^{\ell r(\sigma)}$, and $z^{r\ell(\sigma)}$ into derivation (1.9), making use of the identities in (3.4), regrouping terms appropriately, and, finally, utilizing (2.14) with $q := p v u^{-1}$ yields (3.3).

4. **Generalized Catalan numbers**. It is known [25,33,34] that, for a fixed $\tau \in \{2\underline{1}3,\ 3\underline{1}2\}$, the number of permutations $\sigma \in L[n]$ that have no τ patterns is equal to the Catalan number C_n. Thus, to obtain recurrences for generalized Catalan numbers one only needs to set u := 0 or v := 0 in (3.3). Three such recurrences, which respectively define the first and second p-Catalan numbers of Riordan and Carlitz [5] and a (t,r)-Catalan number of Fürlinger and Hofbauer [18], are presented in Corollaries (4.1), (4.2), and (4.3). It should be remarked that the combinatorial setting used here differs from the models in [5,18].

Noting that (1) if q := 0 then the q-binomial coefficient of n equals 1, and that (2) there is no problem in setting u := 0 in (3.3) since the q-binomial coefficient of n is a monic polynomial of degree k(n-k); the following consequences of (3.3) become immediate:

Corollary 4.1. Let $C_n(p) := L_n(1,0,p;\ 1,1,1)$. Then

$$C_{n+1}(p) = \sum_{k=0}^{n} p^k\, C_k(p)\, C_{n-k}(p)$$

where $C_0(p) := 1$.

Corollary 4.2. Set $c_n(p) := L_n(0,1,p;\ 1,1,1)$. Then

$$c_{n+1}(p) = \sum_{k=0}^{n} p^{(k+1)(n-k)}\, c_k(p)\, c_{n-k}(p)$$

where $c_0(p) := 1$

Corollary 4.3. Define $K_n(t) := L_n(1,0,1;\ T_0,1,1)$ in the case when $t_i := tr^i$ for $i \geq 1$. Thus,

$$K_n(t) = \sum t^{\mathrm{des}(\sigma)}\, r^{\mathrm{maj}(\sigma)}$$

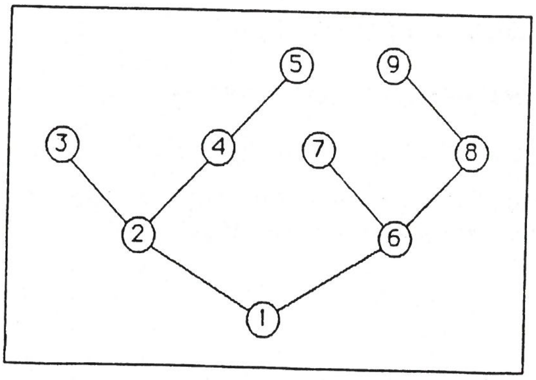
where the summation is over $\sigma \in L[n]$ that have no $3\underline{1}2$ patterns. From Theorem (3.1) it follows that

$$K_{n+1}(t) = K_n(tr) + t \sum_{j=1}^{n} r^j\, K_j(t)\, K_{n-j}(tr^{j+1})$$

with the initial condition $K_0 := 1$.

The set of arbres binaires croissants associated to Corollaries (4.1), (4.2), and (4.3) are easily described. For instance, setting v := 0 selects the subset of ABC's that satisfy the following condition : If h is any vertex in such a tree and A_h (resp. B_h) is the set of vertices in the upper left (resp. right) subtree attached to h, then $\mathrm{Inv}(A_h,B_h) = 0$. Such a tree is sketched in (4.1).

(4.1)



Note that each ABC satisfying the condition $\mathrm{Inv}(A_h,B_h) = 0$ for all h is uniquely labelled by what is commonly known as prefix order. Thus, the act of removing the labels is a bijection from the set of ABC's with no $3\underline{1}2$ patterns to the set of unlabelled binary rooted planar trees. The latter set is a well known combinatorial model of the Catalan numbers. Of course, the ABC's corresponding to setting $u := 0$ are just the planar reflections of the type sketched in (4.1).

For further details concerning generalized Catalan numbers, the interested reader is referred to [5,18].

5. **Descent set results**. By specializing the parameter T_j, it is possible to obtain from Theorem (3.1) a recurrence relationship for any enumerative descent set problem on permutations. As examples, the Eulerian polynomials and the Euler numbers are considered in this section.

The Eulerian polynomial [12] is combinatorially defined to be

$$(5.1) \qquad \sum_{\sigma \in L[n]} t^{\mathrm{des}(\sigma)} .$$

Corollary (5.1) below defines Gessel's [20] (p,r)-analog of the Eulerian polynomials in terms of the major index and inversion number. Of course, when $p := 1$ and $r := 1$ the recurrence relationship given in Corollary (5.1) reduces to (1.1b).

Corollary 5.1. Let $A_n(t) := L_n(1,1,p;\, T_0,1,1)$ in the case when $t_i := tr^i$ for $i \geq 1$. Thus,

$$A_n(t) = \sum_{\sigma \in L[n]} t^{\mathrm{des}(\sigma)}\, p^{\mathrm{inv}(\sigma)}\, r^{\mathrm{maj}(\sigma)} .$$

In this setting, Theorem (3.1) reduces to

$$A_{n+1}(t) = A_n(tr) + t \sum_{k=1}^{n} p^k r^k \binom{n}{k}_p A_k(t)\, A_{n-k}(tr^{k+1})$$

where $A_0 := 1$.

The Euler number E_n of André [1] is combinatorially defined to be the cardinality of the set of "down-up" permutations in L[n], that is, permutations $\sigma \in L[n]$ with

$$(5.2) \qquad \mathrm{Des}(\sigma) = \{1,3,5, \ldots, m\}$$

where m is the greatest odd integer less than n. To obtain a recurrence for the Euler numbers, one needs to extract the appropriate terms from (3.3).

This may be done by setting $t_{2i} := 0$ for $i \geq 1$, $t_{2i+1} := t$ for $i \geq 0$, and then inductively extracting the coefficient of maximum degree in t from recurrence (3.3). Note that when this done, only odd indices remain in the summation in (3.3) since $t_{2i} := 0$ for $i \geq 1$ and $t_0 := 1$. A recurrence for the p-Euler numbers of [15,20] is given in the following corollary:

Corollary 5.2. Set $t_{2i} := 0$ for $i \geq 1$ and $t_{2i+1} := t$ for $i \geq 0$. In this case, let $E_n(p)$ denote the coefficient in $L_n(1,1,p; T_0,1,1)$ of maximum degree in t. It then follows from Theorem (3.1) that

$$E_{n+1}(p) = \sum_{k \text{ odd}} p^k \binom{n}{k}_p E_k(p)\, E_{n-k}(p)$$

where $E_0 := 1$ and $E_1 := 1$.

The trees associated with Corollary (5.2) are well known and may be described as follows: Since k is odd in the summation, and therefore in the ABC decomposition, it inductively follows that the left upper subtree attached to a non-leaf is non-empty. Furthermore, as $t_{2i} := 0$ for $i \geq 1$, the right upper subtree attached to a non-leaf (except perhaps the non-leaf on the extreme right branch) is also non-empty. Thus, in this case, the ABC's are complete (or nearly complete).

From the derivation of Corollary (5.2), it is clear that one can obtain a recurrence relationship for the number $A(n; d_1, d_2, \ldots, d_m)$ of permutations $\sigma \in L[n]$ having a fixed descent set $\{d_1 < d_2 < \ldots < d_m\}$ by simply extracting the appropriate terms from identity (3.3). This is left as an exercise. For an explicit formula for $A(n; d_1, d_2, \ldots, d_m)$ see [21].

6. Minimum component results. Analogs of results concerning permutations by cycle type may be obtained from Theorem (3.1) by specializing either the parameter Y or the parameter Z respectively associated with left-to-right and right-to-left minimum components. After the derivation of two p-exponential functions for permutations by minimum component "type", the classic enumeration problems concerning the Stirling numbers of the first kind, derangements, and involutions are considered in Corollaries (6.1), (6.2) and (6.3). It should be noted that part (d) of Corollary (6.1) defines Gould's [22] p-Stirling numbers of the first kind and that Corollaries (6.2b) and (6.3b) respectively give the recurrences for the p-derangements and the p-involutions that appear in [32].

Let $LRC_n(Y) := L_n(1,1,p; 1,Y,1)$ and $RLC_n(Z) := L_n(1,1,p; 1,1,Z)$. From the fact (see [3, p. 41] that

$$\sum_{\sigma \in L[n]} p^{inv(\sigma)} = (n)_p!$$

and from (3.3) it follows that

$$(6.1) \qquad (a) \quad LRC_{n+1}(Y) = \sum_{k=0}^{n} p^k\, y_{n-k+1} \binom{n}{k}_p (n-k)_p!\, LRC_k(Y)$$

$$(b) \quad RLC_{n+1}(Z) = \sum_{k=0}^{n} p^k\, z_{k+1} \binom{n}{k}_p (k)_p!\, RLC_{n-k}(Z)$$

where $LRC_0 := RLC_0 := 1$. If $LRC(x; Y)$ and $RLC(x; Z)$ denote the respective p-exponential functions (see 2.9) of the p-sequences defined in (6.1), then it is not difficult to show that

$$\text{(6.2)} \quad \text{(a)} \quad [d/dx]\, LRC(x; Y) = J(x; Y)\, LRC(xp; Y)$$

$$\text{(b)} \quad [d/dx]\, RLC(x; Z) = J(xp; Z)\, RLC(x; Z)$$

where $q := p$ in (2.10) and

$$J(x; Y) := \sum_{n\geq 0} y_{n+1}\, x^n .$$

By (2.13), the solutions of the equations in (6.2) are

$$\text{(6.3)} \quad \text{(a)} \quad LRC(x; Y) = \prod_{k\geq 0} [1 + (1-p)\, xp^k\, J(xp^k; Y)]$$

$$\text{(b)} \quad RLC(x; Z) = \prod_{k\geq 0} [1 - (1-p)\, xp^k\, J(xp^{k+1}; Z)]^{-1}$$

which provide p-exponential generating functions for permutations by minimum component types. As one might expect, the functions in (6.3) may in a sense be viewed as respective compositions of the two basic p-exponential functions in (2.8) with the function J. As corollaries of (6.1) and (6.3), we have:

<u>Corollary 6.1 (Stirling numbers of the first kind).</u> Let $\ell rs_n(y) := LRC_n(Y)$ when $y_1 := y$ and let $r\ell s_n(z) := RLC_n(Z)$ when $z_1 := z$. Then

$$\text{(a)} \quad \ell rs_{n+1}(y) = y \sum_{k=0}^{n} p^k \binom{n}{k}_p (n-k)_p!\, \ell rs_k(y)$$

$$\text{(b)} \quad \ell rs_{n+1}(y) = \prod_{k=0}^{n} [yp^k + (k)_p]$$

$$\text{(c)} \quad r\ell s_{n+1}(z) = z \sum_{k=0}^{n} p^k \binom{n}{k}_p (k)_p!\, r\ell s_{n-k}(z)$$

$$\text{(d)} \quad r\ell s_{n+1}(z) = \prod_{k=0}^{n} [z + p(k)_p]$$

where $\ell rs_0(y) := r\ell s_0(z) := 1$. Furthermore, if $\ell rs(x)$ and $r\ell s(x)$ are the respective p-exponential generating functions in x for the p-sequences of (a) and (c), then

$$\text{(e)} \quad \ell rs(x) = \prod_{k\geq 0} [1 + (1-p)\, xp^k\, y(1-xp^k)^{-1}]$$

$$\text{(f)} \quad r\ell s(x) = \prod_{k\geq 0} [1 - (1-p) xp^k\, z(1-xp^{k+1})^{-1}]^{-1} .$$

<u>Proof.</u> Parts (a), (c), (e), and (f) are obvious in view of (6.1) and (6.3). The following calculation shows that (a) implies (b):

$$\ell rs_{n+1}(y) = y \sum_{k=0}^{n} p^k \binom{n}{k}_p (n-k)_p! \, \ell rs_k(y)$$

$$= yp^n \, \ell rs_n(y) + y\,(n)_p \sum_{k=0}^{n-1} \binom{n-1}{k}_p \, p^k \,(n-1-k)_p! \, \ell rs_k(y)$$

$$= yp^n \, \ell rs_n(y) + (n)_p \, \ell rs_n(y)$$

$$= \ell rs_n(y) \, [yp^n + (n)_p] = \prod_{k=0}^{n} [yp^k + (k)_p] \; .$$

In a similar manner, (d) follows from (c).

Corollary 6.2 (Derangements). Let $LRD_n := LRC_n(Y)$ when $y_1 := 0$ and $y_i := 1$ for $i > 1$. Further, let $RLD_n := RLC_n(Z)$ when $z_1 := 0$ and $z_i := 1$ for $i > 1$. Then, for $n \geq 1$,

(a) $$LRD_{n+1} = \sum_{k=0}^{n-1} p^k \binom{n}{k}_p (n-k)_p! \, LRD_k = (n)_p \, [LRD_n + p^{n-1} \, LRD_{n-1}]$$

(b) $$RLD_{n+1} = \sum_{k=1}^{n} p^k \binom{n}{k}_p (k)_p! \, RLD_{n-k} = p(n)_p \, [RLD_n + RLD_{n-1}]$$

(c) $$LRD(x) = \prod_{k\geq 0} [1 + (1-p)x^2 \, p^{2k} \, (1-xp^k)^{-1}]$$

(d) $$RLD(x) = \prod_{k\geq 0} [1 - (1-p)x^2 \, p^{2k+1} \, (1-xp^{k+1})^{-1}]^{-1}$$

where $LRD_0 := RLD_0 := 1$, $LRD_1 := RLD_1 := 0$, and $LRD(x)$ and $RLD(x)$ are the respective p-exponential generating functions.

Corollary 6.3 (Involutions). Let $LRI_n := LRC_n(Y)$ when $y_1 := y_2 := 1$ and $y_i := 0$ for $i \geq 3$. Also, let $RLI_n := RLC_n(Z)$ when $z_1 := z_2 := 1$ and $z_i := 0$ for $i \geq 3$. Then, for $n \geq 1$,

(a) $$LRI_{n+1} = p^n \, LRI_n + p^{n-1} \, (n)_p \, LRI_{n-1}$$

(b) $$RLI_{n+1} = RLI_n + p(n)_p \, RLI_{n-1}$$

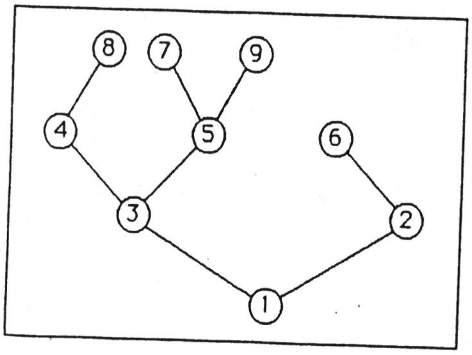
(c) $$LRI(x) = \prod_{k\geq 0} [1 + (1-p)\, xp^k \, (1+xp^k)]$$

(d) $$RLI(x) = \prod_{k\geq 0} [1 - (1-p)\, xp^k \, (1 + xp^{k+1})]^{-1}$$

where $LRI_0 := LRI_1 := RLI_0 := RLI_1 := 1$, and, $LRI(x)$ and $RLI(x)$ are the corresponding p-exponential generating functions.

The ABC's will only be described here for Corollary (6.2a). Setting $y_1 := 0$ selects the ABC's satisfying the following property: If h is a vertex on the extreme left branch of such a tree, then the upper right subtree attached to h must be non-empty. Such a tree is sketched in (6.4) below:

(6.4)



An example of a tree that does not satisfy this condition is sketched in (4.1): The right upper subtree attached to the vertex labeled 3 in (4.1) is empty.

7. **Generalized Hermite polynomials.** It is a well known fact [8,11,14] that the Hermite polynomials have a combinatorial interpretation in terms of weighted involutions. To be specific, a 1-cycle (resp. 2-cycle) is given the weight y (resp. -1). Thus, by assigning the appropriate weights to minimum components, recurrences for the Hermite polynomials may be easily obtained from (3.3).

Moreover, in leaving the parameters u, v, and p in the recurrence, several q-analogs of the Hermite polynomials arise, including both types considered by Cigler [8]. In fact, the combinatorial setting used here provides several new interpretations of Cigler's q-Hermite polynomials. The following corollary is an immediate consequence of Theorem (3.1):

Corollary 7.1 (Hermite polynomials). Let $LRH_n(u,v,p;\, y) := L_n(u,v,p;\, 1,Y,1)$ when $y_1 := y$, $y_2 := -1$, and $y_i := 0$ for $i \geq 3$. Also, set $RLH_n(u,v,p;\, z) := L_n(u,v,p;\, 1,1,Z)$ when $z_1 := z$, $z_2 := -1$, and $z_i := 0$ for $i \geq 3$. Then, for $n \geq 1$,

(a) $$LRH_{n+1}(y) = p^n y\, LRH_n(y) - p^{n-1} u^{n-1} (n)_q\, LRH_{n-1}(y)$$

(b) $$RLH_{n+1}(z) = z\, RLH_n(z) - pu^{n-1} (n)_q\, RLH_{n-1}(z)$$

where the parameters u, v, and p have been suppressed in the expressions $LRH_j(u,v,p;\, y)$ and $RLH_j(u,v,p;\, z)$, and, where $LRH_0 := RLH_0 := 1$, $LRH_1 := y$, $RLH_1 := z$, and $q := pvu^{-1}$.

The recurrence relationships for $LRH_n(1,v,1;\, y)$ and $RLH_n(u,u,u;\, z)$, which may be obtained from Corollary (7.1), respectively define Cigler's first and second q-Hermite polynomials. It is interesting to note that Cigler's q-Hermite polynomials of the first kind arise in 3 other ways in this setting: The sequences $LRH_n(v,1,1;\, y)$, $RLH_n(1,v,1;\, y)$, and $RLH_n(v,1,1;\, y)$ have the same recurrence as $LRH_n(1,v,1;\, y)$.

The exponential generating functions for the q-sequences of Corollary (7.1) may be derived in some cases. For instance, using calculations similar to the proof of (2.13), one obtains

(7.1) (a) $$\mathrm{LRH}(x) = \prod_{k\geq 0} [1 + (1-p)xp^k (y-xp^k)]$$

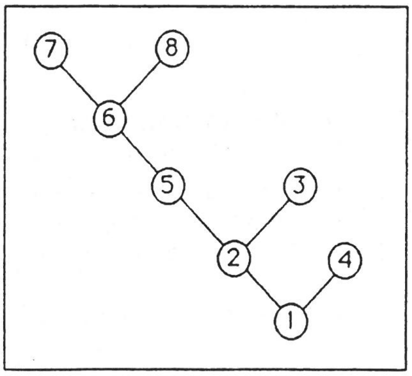
(b) $$\mathrm{RLH}(x) = \prod_{k\geq 0} [1 - (1-pv)xp^k v^k (z-xv^k p^{k+1})]^{-1}$$

(c) $$\mathrm{RH}(x) = \prod_{k\geq 0} [1 - (1-u)\, x^2 u^{2k+1}]\, [1 - (1-u)\, xz\, u^k]^{-1}$$

where LRH(x), RLH(x), and RH(x) are the respective q-exponential generating functions in x for $\mathrm{LRH}_n(1,1,p;\, y)$ with $q := p$, $\mathrm{RLH}_n(1,v,p;\, z)$ with $q := pv$, and $\mathrm{RLH}_n(u,u,u;\, z)$ with $q := u$.

The trees corresponding to Corollary (7.1a) are the so-called "combs" [35] as sketched below:

(7.2)



That is, each upper right subtree attached to a vertex on the extreme left branch has at most 1 vertex. Of course, the ABC's associated with Corollary (7.1b) are just the planar reflections of those of the form sketched in (7.2).

8. **Double minimum component results**. In [7], Carlitz and Scoville studied a sequence of numbers that may be thought of as "double" Stirling numbers of the first kind. By simultaneously working with Y and Z in (3.3), the result of Carlitz and Scoville may be extended in a number of directions. Corollary (8.1) provides a q-analog for the previously mentioned sequence of [7]. As Corollaries (8.1), (8.2), and (8.3) are immediate in light of Theorem (3.1), they are stated without further comment.

Corollary 8.1 (Double Stirling numbers of the first kind). Let $DS_n(y,z) := L_n(1,1,p;\, 1,Y,Z)$ when $y_i := y$ and $z_i := z$ for $i \geq 1$. Then

$$DS_{n+1}(y,z) = yz \sum_{k=0}^{n} p^k \binom{n}{k}_p \ell rs_k(y)\, r\ell s_{n-k}(z)$$

where $DS_0 := 1$, and, $\ell rs_j(y)$ and $r\ell s_j(z)$ are the polynomials defined in Corollary (6.1).

Corollary 8.2 (Double derangements). Let $DD_n := L_n(1,1,p;\ 1,Y,Z)$ when $y_1 := z_1 := 0$ and $y_i := z_i := 1$ for $i \geq 2$. Then, for $n \geq 2$,

$$DD_{n+1} = \sum_{k=1}^{n-1} p^k \binom{n}{k}_p LRD_k\, RLD_{n-k}$$

where $DD_0 := 1$, $DD_1 := DD_2 := 0$, and, LRD_j and RLD_j are defined in Corollary (6.2).

Corollary 8.3 (Stirling derangements). Let $SD_n(y) := L_n(1,1,p;\ 1,Y,Z)$ when $y_i := y$ for $i \geq 1$, $z_1 := 0$, and $z_i := 1$ for $i \geq 2$. Then, for $n \geq 1$,

$$SD_{n+1}(y) = y \sum_{k=1}^{n} p^k \binom{n}{k}_p \ \ell rs_k(y)\, RLD_{n-k}$$

where $SD_0 := 1$, $SD_1 := 0$, and, $\ell rs_j(y)$ and RLD_j are defined in Corollaries (6.1) and (6.2).

9. **Set partition results**. By simultaneously specializing the parameters T_j and Y (or T_j and Z), Theorem (3.1) will yield the solution to the problem of counting partitions of a set C by type. The focus of this section will be on the derivation of the q-exponential generating functions for some analogs of the classic Bell polynomials as defined in [9].

There are two possibilities for obtaining a recurrence for the Bell polynomials from Theorem (3.1):

(9.1) (a) Extract the terms from (3.3) in which the degree with respect to T_0 is equal to the degree with respect to Y minus 1.

(b) Extract the terms in (3.3) in which the degree with respect to T_0 plus the degree with respect to Z is equal to $(n + 1)$.

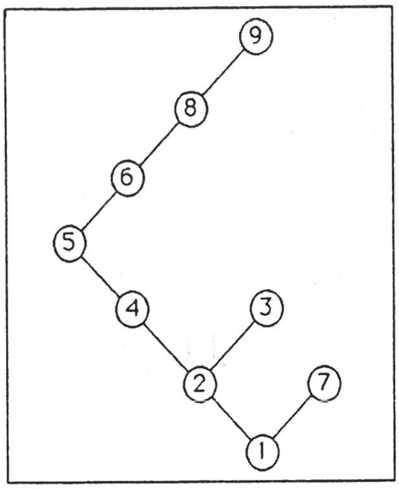
However, the following combinatorial extractions are perhaps more illuminating. To do this, it is important to keep in mind that α and β of (1.3) respectively give rise to the terms L_k and L_{n-k} in (3.3).

First, for D a fixed set of n integers, let LRT[D] denote the set of permutations σ of D that have ABC's that satisfy the following condition:

(9.2) Every upper right subtree attached to the extreme left branch looks like a line segment.

For example, the permutation $\sigma = 568942317 \in L[9]$ is an element of LRT[9] since the corresponding ABC sketched in (9.3) satisfies condition (9.2).

(9.3)



Note that each $\sigma \in LRT[D]$ may be associated to a partition of D in the following natural way: Each right "tilted" line segment of the corresponding ABC may be viewed as a subset in the partition of D. For instance, the tree in (9.3) corresponds to the partition $\{1,7\}$, $\{2,3\}$, $\{4\}$, and $\{5,6,8,9\}$ of $\{1,2,\dots,9\}$. Furthermore, note that the monomial $T_0^{Des(\sigma)} Y^{\ell r(\sigma)} = t_4t_5t_7y_1y_2^2y_4$ for σ of (9.3) is of the type described in (9.1a).

Now consider the generating polynomial for weighted "left-to-right" partitions of the set D defined to be

(9.4) $$LRP_n(T_j,Y) := \sum v^{312(\sigma)} p^{inv(\sigma)} T_j^{Des(\sigma)} Y^{\ell r(\sigma)}$$

where the summation is over $\sigma \in LRT[D]$. To extract a recurrence from Theorem (3.1) for this polynomial, begin by observing that $LRP_n(T_j,Y)$ is equal to the sum of the terms in $L_n(1,v,p;\ T_j,Y,1)$ that correspond to permutations having ABC's of the form sketched in (9.3). In this setting, $\beta \in L[B]$ in decomposition (1.3) is restricted to the unique increasing list of B. Thus, the only term to be extracted from $L_{n-k}(1,v,p;\ T_{k+1},1,1)$ in (3.3) is 1. Theorem (3.1) then yields

(9.5) $$LRP_{n+1}(T_0,Y) = \sum_{k=0}^{n} p^k\, t_k\, y_{n-k+1} \binom{n}{k}_q LRP_k(T_0,Y)$$

where $LRP_0 := 1$ and $q := pv$. Note that (9.5) is precisely the recurrence that is obtained when one extracts from (3.3) the terms that satisfy (9.1a).

On the other hand, consider the generating polynomial for weighted "right-to-left" partitions of the set D defined to be

(9.6) $$RLP_n(T_j,Z) := \sum v^{312} p^{inv(\sigma)} T_j^{Des(\sigma)} Z^{r\ell(\sigma)}$$

where the summation is over $\sigma \in L[D]$ having ABC's which are planar reflections of the type sketched in (9.3). In this case, $\alpha \in L[A]$ in (1.3) is restricted to the unique decreasing list of A. Thus, extracting the appropriate terms from (3.3) yields

(9.7) $$RLP_{n+1}(T_0,Z) = \sum_{k=0}^{n} p^k\, z_{k+1} \binom{n}{k}_q t_1t_2\dots t_k\, RLP_{n-k}(T_{k+1},Z)$$

where $RLP_0 := 1$ and $q := pv$. Note that the permutation corresponding to the planar reflection of (9.3) gives rise to a monomial of the form described in (9.1b).

Recurrences for generalized Bell polynomials may now be obtained from (9.5) and (9.7) by setting $T_0 := 1$. This is done in the following corollary:

Corollary 9.1. Let $LRB_n(v,p; Y) := LRP_n(1,Y)$ and $RLB_n(v,p; Z) := RLP_n(1,Z)$. Then recurrences (9.5) and (9.7) respectively reduce to

$$\text{(a)}\qquad LRB_{n+1}(v,p;Y) = \sum_{k=0}^{n} p^k y_{n-k+1} \binom{n}{k}_q LRB_k(v,p;Y)$$

$$\text{(b)}\qquad RLB_{n+1}(v,p;Z) = \sum_{k=0}^{n} p^k z_{k+1} \binom{n}{k}_q RLB_{n-k}(v,p;Z)$$

where $LRB_0 := RLB_0 := 1$ and $q := pv$.

Note that the recurrences for $LRB_n(1,1;\ Y)$ and $RLB_n(1,1;\ Z)$ each define the classic Bell polynomials. Further, observe that $LRB_n(v,p;\ Y)$ when $y_i := y$ for $i \geq 1$ and that $RLB_n(v,p;\ Z)$ when $z_i := z$ for $i \geq 1$ both provide q-analogs of the Stirling numbers of the second kind.

As before, using calculations similar to those used in proving (2.13), some q-exponential generating functions may be obtained for the sequences in Corollary (9.1). Let $LRB(x)$, $LB(x)$, and $RLB(x)$ denote the q-exponential generating functions in x respectively of $LRB_n(1,p;\ Y)$ with $q := p$, $LRB_n(v,1;\ Y)$ with $q := v$, and $RLB_n(v,p;\ Z)$ with $q := pv$. Then

$$\text{(9.8)}\qquad \text{(a)}\qquad LRB(x) = \prod_{k\geq 0} [1 + (1-p)\, xp^k\, W(xp^k;\ Y)]$$

$$\text{(b)}\qquad LB(x) = \prod_{k\geq 0} [1 - (1-v)\, xv^k\, W(xv^k;\ Y)]^{-1}$$

$$\text{(c)}\qquad RLB(x) = \prod_{k\geq 0} [1 - (1-pv)\, xp^k v^k\, W(xv^k p^{k+1};\ Z)]^{-1}$$

where

$$W(x;\ Y) := \sum_{n\geq 0} y_{n+1}\, x^n/(n)_q!\,.$$

Acknowledgements. I would like to thank Professor Pierre Leroux for making several valuable comments on the first few drafts of this paper. I would also like to express my appreciation for the generous support given to me by UQAM during my sabbatical leave.

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