PUBL. I.R.M.A. STRASBOURG, 1991, 476/S-26
ACTES 26^e SÉMINAIRE LOTHARINGIEN, P. 29-42

# Enumeration in Musical Theory

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**Abstract**

Being a mathematician and a musician (I play the flute) I found it very interesting to deal with Pólya's counting theory in my Master's thesis. When reading about Pólya's theory I came across an article, called "Enumeration in Music Theory" by D. L. Reiner [11]. I took up his ideas and tried to enumerate some other "musical objects".

At first I would like to generalize certain aspects of 12-tone music to $n$-tone music, where $n$ is a positive integer. Then I will explain how to interpret intervals, chords, tone-rows, all-interval-rows, rhythms, motifs and tropes in $n$-tone music. Transposing, inversion and retrogradation are defined to be permutations on the sets of "musical objects". These permutations generate permutation groups, and these groups induce equivalence relations on the sets of "musical objects". The aim of this article is to determine the number of equivalence classes (I will call them patterns) of "musical objects". Pólya's enumeration theory is the right tool to solve this problem.

In the first chapter I will present a short survey of parts of Pólya's counting theory. In the second chapter I will investigate several "musical objects".

## 1 Preliminaries

There is a lot of literature about Pólya's counting theory. For instance see [1], [2], [3], [9] or [10]. Let $M$ be a set with $|M| = m$. You should know the definition of the type $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ of a permutation $\pi \in S_M$ and the definition of the cycle index $\mathrm{CI}(\Gamma; x_1, \ldots, x_m)$ of a permutation group $\Gamma \leq S_M$. In particular we will use the cycle index of the cyclic group and of the dihedral group.

## 2 Applications of Pólya's Theory in Musical Theory

Some parts of this chapter were already discussed by D.L.Reiner in [11]. Now we are going to calculate the number of patterns of chords, intervals, tone-rows, all-interval-rows, rhythms, motifs and tropes. Proving any detail would carry me too far. For further information see [6].

*The author thanks Jens Schwaiger for helpful comments.

## 2.1 Patterns of Intervals and Chords

### 2.1.1 Number of Patterns of Chords

**Definition 1 ($n$-Scale)** 1. If we divide one octave into $n$ parts, we will speak of an $n$-scale. The objects of an $n$-scale are designated as $0, 1, \ldots, n-1$.

2. In twelve tone music we usually identify two tones which are 12 semi-tones apart. For that reason we define an $n$-scale as the cyclic group $(Z_n, +)$ of order $n$.

**Definition 2 (Transposing, Inversion)** 1. Let us define $T$ the operation of transposing as a permutation $T: Z_n \to Z_n$, $a \mapsto T(a) := 1 + a$. The group $\langle T \rangle$ is the cyclic group $\zeta_n^{(E)}$.

2. Let us define $I$ the operation of inversion as $I: Z_n \to Z_n$, $a \mapsto I(a) := -a$. The group $\langle T, I \rangle$ is the dihedral group $\vartheta_n^{(E)}$.

**Definition 3 ($k$-Chord)** 1. Let $k \leq n$. A $k$-chord in an $n$-scale is a subset of $k$ elements of $Z_n$. An interval is a 2-chord.

2. Let $G = \zeta_n^{(E)}$ or $G = \vartheta_n^{(E)}$. Two $k$-chords $A_1, A_2$ are called equivalent iff there is some $\gamma \in G$ such that $A_2 = \gamma(A_1)$.

**Remark 4** 1. We want to work with Pólya's Theorem, therefore I identify each $k$-chord $A$ with its characteristic function $\chi_A$. Two $k$-chords $A_1, A_2$ are equivalent iff the two functions $\chi_{A_1}$ and $\chi_{A_2}$ are equivalent in the sense of Pólya's Theorem.

2. Let us define two finite sets: $P := Z_n$ and $F := \{0, 1\}$. Each function $f \in F^P$ will be identified with $A_f := \{k \in Z_n \mid f(k) = 1\}$.

3. Let $w: F \to \mathcal{R} := \mathbf{Q}[x]$ be a mapping with $w(1) := x$ and $w(0) := 1$, where $x$ is an indeterminate. Define the weight $W(f)$ of a function $f \in F^P$ as

$$W(f) := \prod_{k \in Z_n} w(f(k)).$$

We see that the weight of a $k$-chord is $x^k$. The weight of a pattern $W([f]) := W(f)$ is well defined.

Applying Pólya's Theorem of [2], we derive:

**Theorem 5 (Patterns of $k$-Chords)** *1. Let $G$ be a permutation group on $Z_n$. The number of patterns of $k$-chords in the $n$-scale $Z_n$ is the coefficient of $x^k$ in*

$$\mathrm{CI}(G; 1 + x, 1 + x^2, \ldots, 1 + x^n).$$

*2. If $G = \zeta_n^{(E)}$, the number of patterns of $k$-chords is* $\frac{1}{n} \sum_{j \mid \gcd(n,k)} \varphi(j) \binom{\frac{n}{j}}{\frac{k}{j}}$, *where $\varphi$ is Euler's $\varphi$-function.*

*3. If $G = \vartheta_n^{(E)}$, the number of patterns of $k$-chords is*

$$\begin{cases} \frac{1}{2n}\Big(\sum\limits_{j|\gcd(n,k)} \varphi(j)\binom{\frac{n}{j}}{\frac{k}{j}} + n\binom{\frac{(n-1)}{2}}{[\frac{k}{2}]}\Big) & \text{if } n \equiv 1 \bmod 2 \\ \frac{1}{2n}\Big(\sum\limits_{j|\gcd(n,k)} \varphi(j)\binom{\frac{n}{j}}{\frac{k}{j}} + n\binom{\frac{n}{2}}{\frac{k}{2}}\Big) & \text{if } n \equiv 0 \bmod 2 \text{ and } k \equiv 0 \bmod 2 \\ \frac{1}{2n}\Big(\sum\limits_{j|\gcd(n,k)} \varphi(j)\binom{\frac{n}{j}}{\frac{k}{j}} + n\binom{\frac{n}{2}-1}{[\frac{k}{2}]}\Big) & \text{if } n \equiv 0 \bmod 2 \text{ and } k \equiv 1 \bmod 2. \end{cases}$$

*4. In the case $n = 12$ and $G = \zeta_n^{(E)}$, we get the numbers in table 1 on page 31.*

*5. In the case $n = 12$ and $G = \vartheta_n^{(E)}$, we get the numbers in table 2 on page 32.*

### 2.1.2 The Complement of a $k$-Chord

**Definition 6 (Complement of a $k$-Chord)** Let $A \subseteq Z_n$ with $|A| = k$ be a $k$-chord. The complement of $A$ is the $(n-k)$-chord $Z_n \setminus A$.

**Remark 7** 1. Let $G = \zeta_n^{(E)}$ or $G = \vartheta_n^{(E)}$ be a permutation group on $Z_n$ and let $1 \le k < n$. There exists a bijection between the sets of patterns of $k$-chords and $(n-k)$-chords.

2. If $n \equiv 0 \bmod 2$, the complement of an $\frac{n}{2}$-chord is an $\frac{n}{2}$-chord. Now I want to figure out the number of patterns of $\frac{n}{2}$-chords $[A]$ with the property $A \sim Z_n \setminus A$. Applying the Theorem of [1] we get:

**Theorem 8** *1. Let $n \equiv 0 \bmod 2$. The number of patterns of $\frac{n}{2}$-chords which are equivalent to their complement, is $\mathrm{CI}(G; 0, 2, 0, 2, \ldots)$.*

*2. If $n = 12$ and $G = \zeta_n^{(E)}$, there are 20 patterns of 6-chords which are equivalent to their complement.*

*3. If $n = 12$ and $G = \vartheta_n^{(E)}$, there are 8 patterns of 6-chords which are equivalent to their complement.*

### 2.1.3 The Interval Structure of a $k$-Chord

In this section we use $\vartheta_n^{(E)}$ as the permutation group acting on $Z_n$. The set of all possible intervals between two differnet tones in $n$-tone music will be called $\mathrm{Int}(n)$, thus

$$\mathrm{Int}(n) := \{x - y \mid x, y \in Z_n,\ x \neq y\} = \{1, 2, \ldots, n-1\}.$$

**Definition 9 (Interval Structure)** On $Z_n$ we define a linear order $0 < 1 < 2 < \ldots < n-1$. Let $A := \{i_1, i_2, \ldots, i_k\}$ be a $k$-chord. Without loss of generality let $i_1 < i_2 < \ldots < i_k$. The interval structure of $A$ is defined as the pattern $[f_A]$, wherein the function $f_A$ is defined as

$$f_A : \{1, 2, \ldots, k\} \to \mathrm{Int}(n)$$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| # of patterns | 1 | 6 | 19 | 43 | 66 | 80 | 66 | 43 | 19 | 6 | 1 | 1 |

Table 1: Number of patterns of $k$-Chords in 12-tone music with regard to $\zeta_n^{(E)}$.

$$f_A(1) := i_2 - i_1,\ f_A(2) := i_3 - i_2, \ldots,\ f_A(k-1) := i_k - i_{k-1},\ f_A(k) := i_1 - i_k,$$

and two functions $f_1, f_2 \colon \{1, 2, \ldots, k\} \to \mathrm{Int}(n)$ are called equivalent, iff there exists some $\varphi \in \vartheta_k^{(E)}$ such that $f_2 = f_1 \circ \varphi$. The group $\vartheta_k^{(E)}$ is generated by $\tilde{T}$ and $\tilde{I}$ with $\tilde{T}(i) := i+1 \bmod k$ and $\tilde{I}(i) := k+1-i$ for $i = 1, \ldots, k$. The differences $i_{j+1} - i_j$ must be interpreted as differences in $Z_n$. They are the intervals between the tones $i_j$ and $i_{j+1}$.

**Theorem 10** *Let $A_1 := \{i_1, i_2, \ldots, i_k\}$ and $A_2 := \{j_1, j_2, \ldots, j_k\}$ be two $k$-chords with $i_1 < i_2 < \ldots < i_k$ and $j_1 < j_2 < \ldots < j_k$. Furthermore let $f := f_{A_1}$ and $g := f_{A_2} \colon \{1, 2, \ldots, k\} \to \mathrm{Int}(n)$ be constructed as in Definition 9. Then*

$$[f] = [g] \Longleftrightarrow [\{i_1, i_2, \ldots, i_k\}] = [\{j_1, j_2, \ldots, j_k\}].$$

I omit the proof of this theorem.

**Remark 11** If the permutation group acting on $Z_n$ is the cyclic group $\zeta_n^{(E)}$, then the interval structure of $A := \{i_1, i_2, \ldots, i_k\}$ must be defined as the pattern $[f_A]$ in regard to $\zeta_k^{(E)} := \langle \tilde{T} \rangle$ with $\tilde{T}(i) := i + 1 \bmod k$. The function $f_A$ is defined as in Definition 9.

**Remark 12** Let $f$ be a function $f \colon \{1, 2, \ldots, k\} \to \mathrm{Int}(n)$. The pattern $[f]$ is the interval structure of a $k$-chord, iff $\sum_{i=1}^k f(i) = n$. One must interpret this sum as a sum of intervals, thus as a sum of positive integers.

**Remark 13** Let $x, y_1, y_2, \ldots, y_n$ be indeterminates over $\mathbf{Q}$ and let $\mathcal{R}$ be the ring $\mathcal{R} := \mathbf{Q}[x, y_1, y_2, \ldots, y_n]$. Now I want to define a weight function $w \colon \mathrm{Int}(n) \to \mathcal{R},\ i \mapsto w(i) := x^i y_i$. The weight of a function $f \colon \{1, 2, \ldots, k\} \to \mathrm{Int}(n)$ is the product weight

$$W(f) := \prod_{i=1}^{k} w(f(i)) = \prod_{i=1}^{k} x^{f(i)} y_{f(i)} = x^{\sum_{i=1}^k f(i)} \prod_{i=1}^{k} y_{f(i)}.$$

Now we can define $W([f]) := W(f)$. According to Remark 12 the pattern $[f]$ is the interval structure of a $k$-chord, iff $\sum_{i=1}^k f(i) = n$.This is true, iff $W(f) = x^n \prod_{i=1}^k y_{f(i)}$. The indices of the $y$'s in $W(f)$ show, which intervals occur in the $k$-chord.

An Application of Pólya's Theorem of [2] is

**Theorem 14** *The inventory of interval structures of $k$-chords in $n$-tone music is the coefficient of $x^n$ in* $\mathrm{CI}\Big(\vartheta_k^{(E)}; \sum_{i=1}^{n-1} x^i y_i, \sum_{i=1}^{n-1} x^{2i} {y_i}^2, \sum_{i=1}^{n-1} x^{3i} {y_i}^3, \ldots, \Big)$.

**Example 15** The inventory of the interval structures of 3-chords in 12-tone music is the coefficient of $x^{12}$ in

$$\mathrm{CI}\Big(\vartheta_3^{(E)}; \sum_{i=1}^{11} x^i y_i, \sum_{i=1}^{11} x^{2i} {y_i}^2, \sum_{i=1}^{11} x^{3i} {y_i}^3\Big).$$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| # of patterns | 1 | 6 | 12 | 29 | 38 | 50 | 38 | 29 | 12 | 6 | 1 | 1 |

Table 2: Number of patterns of $k$-Chords in 12-tone music with regard to $\vartheta_n^{(E)}$.

This is

$y_1{}^2 y_{10} + y_1(y_2y_9 + y_3y_8 + y_4y_7 + y_5y_6) + y_2{}^2y_8 + y_2(y_3y_7 + y_4y_6 + y_5{}^2) + y_3{}^2y_6 + y_3y_4y_5 + y_4{}^3$

If you are interested in the number of patterns of 3-chords with intervals $\geq k$, then put $y_1 := y_2 := \ldots := y_{k-1} := 0$ and $y_k := y_{k+1} := \ldots := y_n := 1$. In the case $k = 2$ there are 7 patterns of 3-chords with intervals greater or equal 2.

## 2.2 Patterns of Tone-Rows

**Definition 16 (Tone-Row, $k$-Row)** 1. Arnold Schönberg introduced the so called tone-rows. Here I am going to give a mathematical form of his definition. Let $n \geq 3$. A tone-row in an $n$-scale is a bijectiv mapping $f: \{0, 1, \ldots, n-1\} \to Z_n$, $i \mapsto f(i)$. $f(i)$ is the tone which occurs in $i^{\text{th}}$ position in the tone-row.

2. Let $n \geq 3$ and $2 \leq k \leq n$. A $k$-row in $n$-tone music is an injective mapping $f: \{0, 1, \ldots, k-1\} \to Z_n$.

**Remark 17** 1. A $k$-row with $k = n$ is a tone-row.

2. Two $k$-rows $f_1, f_2$ are equivalent if $f_1$ can be written as transposing, inversion, retrogradation or an arbitrary sequence of these operations of $f_2$.
Transposing of a $k$-row $f$ is $T \circ f$, Inversion of $f$ is $I \circ f$. According to Definition 2, we know that $T$ and $I$ are permutations on $Z_n$, and that $\langle T, I\rangle = \vartheta_n^{(E)}$. Actually inversion of a $k$-row $f$ should be defined as $T^{f(0)} \circ I \circ T^{-f(0)} \circ f$. Retrogradation $R$, is a permutation $R \in S_{\{0,1,\ldots,k-1\}}$ defined as:

$$R := \begin{cases} (0, k-1) \circ (1, k-2) \circ \ldots \circ (\frac{k}{2} - 1, \frac{k}{2}) & \text{if } k \equiv 0 \bmod 2 \\ (0, k-1) \circ (1, k-2) \circ \ldots \circ (\frac{k-3}{2}, \frac{k+1}{2}) \circ (\frac{k-1}{2}) & \text{if } k \equiv 1 \bmod 2. \end{cases}$$

Let $\Pi := \langle R\rangle \leq S_{\{0,1,\ldots,k-1\}}$, then $|\Pi| = 2$. Retrogradation of a $k$-row $f$ is defined as $f \circ R$.

3. Since $\Pi := \langle R\rangle$, the cycle index of $\Pi$ is

$$\mathrm{CI}(\Pi; y_1, y_2, \ldots, y_k) = \begin{cases} \frac{1}{2}(y_1{}^k + y_2{}^{\frac{k}{2}}) & \text{if } k \equiv 0 \bmod 2 \\ \frac{1}{2}(y_1{}^k + y_1 y_2{}^{\frac{k-1}{2}}) & \text{if } k \equiv 1 \bmod 2. \end{cases}$$

Thus two $k$-rows $f_1, f_2$ are equivalent iff $\exists \varphi \in \vartheta_n^{(E)} \exists \sigma \in \Pi$ such that $f_1 = \varphi \circ f_2 \circ \sigma$.

Applying Theorem 5.2 of [2], we get

**Theorem 18 (Number of Patterns of $k$-Rows)** *The number of patterns of $k$-rows in $Z_n$ is* $\mathrm{CI}\left(\Pi; \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_k}\right) \mathrm{CI}(\vartheta_n^{(E)}; 1 + x_1, 1 + 2x_2, \ldots, 1 + nx_n)\big|_{x_1 = x_2 = \ldots = x_n = 0}$. *This is*

*1.*
$$\frac{1}{2}\left(\frac{1}{4}\left((2)_k + 2^{\frac{k}{2}}\left(\frac{k}{2}\right)!\left(\binom{\frac{n}{2}}{\frac{k}{2}} + \binom{\frac{n-2}{2}}{\frac{k}{2}}\right)\right) + \frac{1}{2n}\left(\binom{n}{k}k! + 2^{\frac{k}{2}}\left(\frac{k}{2}\right)!\binom{\frac{n}{2}}{\frac{k}{2}}\right)\right),$$

*if* $n \equiv 0 \bmod 2$ *and* $k \equiv 0 \bmod 2$. *For integers* $k, v$, $v \geq 0$ *the expression* $(k)_v$ *is defined as:*

$$(k)_v := k \cdot (k-1) \cdot \ldots \cdot (k - (v-1)).$$

*2.*

$$\frac{1}{2}\Big(\frac{1}{4}\cdot 2\cdot 2^{\frac{k-1}{2}}\binom{\frac{n-2}{2}}{\frac{k-1}{2}}(\frac{k-1}{2})! + \frac{1}{2n}\binom{n}{k}k!\Big),$$

*if* $n \equiv 0 \bmod 2$ *and* $k \equiv 1 \bmod 2$.

*3.*

$$\frac{1}{2}\Big(\frac{1}{2n}\binom{n}{k}k! + \frac{1}{2}2^{\frac{k}{2}}\binom{\frac{n-1}{2}}{\frac{k}{2}}(\frac{k}{2})!\Big),$$

*if* $n \equiv 1 \bmod 2$ *and* $k \equiv 0 \bmod 2$.

*4.*

$$\frac{1}{2}\Big(\frac{1}{2n}\binom{n}{k}k! + \frac{1}{2}2^{\frac{k-1}{2}}\binom{\frac{n-1}{2}}{\frac{k-1}{2}}(\frac{k-1}{2})!\Big),$$

*if* $n \equiv 1 \bmod 2$ *and* $k \equiv 1 \bmod 2$.

*In the case* $n = 12$ *the number of patterns of* $k$*-rows is in table 3 on page 34.*

The special case of Theorem 18 for $k = n$ is

**Theorem 19 (Number of patterns of Tone-Rows)** *Let* $n \geq 3$. *The number of patterns of tone-rows in* $n$*-tone music is*

$$\begin{cases} \frac{1}{4}\big((n-1)! + (n-1)!!\big) & \text{if } n \equiv 1 \bmod 2 \\ \frac{1}{4}\big((n-1)! + (n-2)!!(\frac{n}{2}+1)\big) & \text{if } n \equiv 0 \bmod 2. \end{cases}$$

*If* $n$ *is in* $\mathbb{N}$ *then*

$$n!! = \begin{cases} n\cdot(n-2)\cdot\ldots\cdot 2 & \text{if } n \equiv 0 \bmod 2 \\ n\cdot(n-2)\cdot\ldots\cdot 1 & \text{if } n \equiv 1 \bmod 2. \end{cases}$$

*Especially there are 9 985 920 patterns of tone-rows in 12-tone music.*

## 2.3 Patterns of All-Interval-Rows

Let $A$ and $B$ be two finite sets. The set of all injective functions $f: A \to B$ will be denoted by $\mathrm{Inj}(A, B)$. For that reason the set of all tone-rows is $\mathrm{Inj}(\{0, 1, \ldots, n-1\}, Z_n)$. In this chapter let $n \geq 3$.

**Definition 20 (All-Interval-Rows)** Let us define a mapping

$$\alpha: \mathrm{Inj}(\{0,1,\ldots,n-1\}, Z_n) \to \{g \mid g: \{1,2,\ldots,n-1\} \to \mathrm{Int}(n)\}$$

$$f \mapsto \alpha(f)$$

| $k$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| # of patterns | 6 | 30 | 275 | 2 000 | 14 060 | 83 280 |

| $k$ | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|
| # of patterns | 416 880 | 1 663 680 | 4 993 440 | 9 980 160 | 9 985 920 |

Table 3: Number of patterns of $k$-rows in 12-tone music.

and $\alpha(f)(i) := f(i) - f(i-1)$ for $i = 1, 2, \ldots, n-1$. This is subtraction in $Z_n$. The function $\alpha(f)$ is called all-interval-row, iff $\alpha(f)$ is injective, that means $\alpha(f) \in \mathrm{Inj}(\{1, 2, \ldots, n-1\}, \mathrm{Int}(n))$. In other words a tone-row induces an all-interval-row, iff all possible intervals occur as differences between two successive tones of the tone-row. The set of all all-interval-rows will be denoted as $\mathrm{Allint}(n)$.

Let's define some mappings:

1.
$$\beta: \mathrm{Inj}(\{1, 2, \ldots, n-1\}, \mathrm{Int}(n)) \to \{g \mid g: \{0, 1, \ldots, n-1\} \to Z_n\}$$
$$f \mapsto \beta(f)$$
$\beta(f)(0) := 0$ and $\beta(f)(i) := \beta(f)(i-1) + f(i) \bmod n$ for $i = 1, 2, \ldots, n-1$. You can easily derive that for $i = 0, 1, \ldots, n-1$
$$\beta(f)(i) \equiv \sum_{j=1}^{i} f(j) \bmod n.$$

2. Let $l \in Z_n$.
$$\tilde{\beta}: \mathrm{Inj}(\{1, 2, \ldots, n-1\}, \mathrm{Int}(n)) \to \{g \mid g: \{0, 1, \ldots, n-1\} \to Z_n\}$$
$$f \mapsto \tilde{\beta}(f), \qquad \tilde{\beta}(f)(i) \equiv \sum_{j=1}^{i} f(j) + l \bmod n.$$

**Theorem 21** *Let $f$ be a mapping $f: \{1, 2, \ldots, n-1\} \to \mathrm{Int}(n)$. The following statements are equivalent:*

1. *$f$ is an all-interval-row.*
2. *$f \in \mathrm{Inj}(\{1, 2, \ldots, n-1\}, \mathrm{Int}(n))$ and $\beta(f) \in \mathrm{Inj}(\{0, 1, \ldots, n-1\}, Z_n)$.*
3. *$f \in \mathrm{Inj}(\{1, 2, \ldots, n-1\}, \mathrm{Int}(n))$ and $\tilde{\beta}(f) \in \mathrm{Inj}(\{0, 1, \ldots, n-1\}, Z_n)$.*

The proof is omitted.
You can easily prove the following results:

1. If $n \equiv 1 \bmod 2$, there are no all-interval-rows.
2. If $n \equiv 0 \bmod 2$ the function $f$ defined as
$$f(i) := \begin{cases} i & \text{if } i \equiv 1 \bmod 2 \\ -i & \text{if } i \equiv 0 \bmod 2 \end{cases}$$
is an all-interval-row.

   For the rest of this chapter let $n \geq 4$ and $n \equiv 0 \bmod 2$.
3. $f \in \mathrm{Allint}(n)$ implies $\beta(f)(n-1) = \frac{n}{2}$.
4. $f \in \mathrm{Allint}(n)$ implies $f(1) \neq \frac{n}{2}$ and $f(n-1) \neq \frac{n}{2}$.

**Remark 22** 1. On $\mathrm{Int}(n)$ we have the following permutations:

$$I: \mathrm{Int}(n) \to \mathrm{Int}(n), \qquad j \mapsto I(j) := n - j.$$

$I$ stands for inversion. $I$ is of the type $(1, \frac{n}{2} - 1, 0, \ldots)$.
In the case $n = 12$ there is a further permutation called

$$Q: \mathrm{Int}(n) \to \mathrm{Int}(n), \qquad j \mapsto Q(j) :\equiv 5 \cdot j \bmod 12.$$

$Q$ stands for quartcircle symmetry. Since $\gcd(5, 12) = 1$, $Q$ is a permutation on $Z_n$, and since $5 \cdot 0 = 0$, $Q$ is a permutation on $\mathrm{Int}(n)$. $Q$ is of the type $(3, 4, 0, \ldots, 0)$. You can easily prove that $(I \circ Q)(j) = (Q \circ I)(j) = 7 \cdot j \bmod 12$ and that it is of the type $(5, 3, 0, \ldots, 0)$. $I \circ Q$ is called quintcircle symmetry.

2. On the set $\{1, 2, \ldots, n-1\}$ retrogradation $R$ is a permutation, defined as

$$R := (1, n-1) \circ (2, n-2) \circ \ldots \circ (\frac{n}{2} - 1, \frac{n}{2} + 1) \circ (\frac{n}{2}).$$

3. If $f \in \mathrm{Allint}(n)$, then $I \circ f$, $f \circ R$ are in $\mathrm{Allint}(n)$. Furthermore if $n = 12$ then $Q \circ f \in \mathrm{Allint}(12)$.

4. For that reason we can define the following permutations on $\mathrm{Allint}(n)$.

$$\varphi_I, \varphi_R, \varphi_Q : \mathrm{Allint}(n) \to \mathrm{Allint}(n)$$

$$f \mapsto \varphi_I(f) := I \circ f, \; f \mapsto \varphi_R(f) := f \circ R, \; f \mapsto \varphi_Q(f) := Q \circ f.$$

For $\varphi_Q$ we need the assumption that $n = 12$.

5. It is easy to prove that these permutations commute in pairs and that ${\varphi_I}^2 = {\varphi_R}^2 = {\varphi_Q}^2 = \mathrm{id}$.

6. In [4] there is a further permutation $E$ called exchange at $\frac{n}{2}$. It is defined as

$$E: \mathrm{Allint}(n) \to \mathrm{Allint}(n), \qquad f \mapsto E(f)$$

and

$$E(f)(i) := \begin{cases} f(f^{-1}(\frac{n}{2}) + i) & \text{if } i < n - f^{-1}(\frac{n}{2}) \\ \frac{n}{2} & \text{if } i = n - f^{-1}(\frac{n}{2}) \\ f(i - n + f^{-1}(\frac{n}{2})) & \text{if } i > n - f^{-1}(\frac{n}{2}). \end{cases}$$

I have already mentioned, that $f(1) \neq \frac{n}{2}$ and $f(n-1) \neq \frac{n}{2}$. Since $f \in \mathrm{Allint}(n)$ is bijective, there exists exactly one $j$, such that $1 < j < n - 1$ and $f(j) = \frac{n}{2}$. The values of the function $E(f)(i)$ for $i = 1, 2, \ldots, n-1$ are $f(j+1), f(j+2), \ldots, f(n-1), f(j) = \frac{n}{2}, f(1), f(2), \ldots, f(j-1)$. The permutation $E$ is defined for $n \geq 4$, but in the case $n = 4$ we have $E = \varphi_R$.

7. The following formulas hold: $E \circ \varphi_I = \varphi_I \circ E$, $E \circ \varphi_Q = \varphi_Q \circ E$, $E \circ \varphi_R = \varphi_R \circ E$ and $E^2 = \mathrm{id}$.

8. Let us define three permutation groups on $\mathrm{Allint}(n)$.
$G_1 := \langle \varphi_I, \varphi_R \rangle, G_2 := \langle \varphi_I, \varphi_R, E \rangle$ und $G_3 := \langle \varphi_I, \varphi_R, E, \varphi_Q \rangle$. For $G_2$ we must assume $n \geq 6$, and for $G_3$ we must assume $n = 12$. We calculate that $|G_1| = 4$, $|G_2| = 8$, $|G_3| = 16$.

**Remark 23 (Counting of All-Interval-Rows)** Let

$$x_1, x_2, \dots, x_{n-1}, y_1, y_2, \dots, y_{n-1}, z_1, z_2, \dots, z_{n-1}$$

be indeterminates over $\mathbf{Q}$. Furthermore let $f$ be a mapping $f: \{1, 2, \dots, n-1\} \to \mathrm{Int}(n)$. We define $\mathcal{R} := \mathbf{Q}[x_1, x_2, \dots, x_{n-1}, z_1, z_2, \dots, z_{n-1}]$ and

$$W(f) := \prod_{i:=1}^{n-1} w_i(f(i)).$$

The functions $w_i$ are defined as $w_i: \mathrm{Int}(n) \to \mathcal{R}$, $j \mapsto w_i(j) := z_j \prod_{\nu:=i}^{n-1} x_\nu{}^j$. After calculating $W(f)$ you have to replace terms of the form $x_\nu{}^j$ by $y_{j \bmod n}$. Then you get $\tilde{W}(f) \in \mathbf{Q}[y_1, y_2, \dots, y_{n-1}, z_1, z_2, \dots, z_{n-1}]$. According to Theorem 21 $f$ is an all-interval-row, if and only if, $\tilde{W}(f) = \prod_{i=1}^{n-1} y_i z_i$. Consequently the number of all-interval-rows in $n$-tone music is the coefficient of $\prod_{i=1}^{n-1} y_i z_i$ in

$$\prod_{i=1}^{n-1} \Big( \sum_{j=1}^{n-1} z_j \prod_{k=i}^{n-1} x_k{}^j \Big) \Big|_{x_\nu{}^j = y_{j \bmod n}}.$$

**Remark 24** For $\varphi \in G_1$ or $G_2$ or $G_3$ we want to calculate

$$\chi(\varphi) := |\{ f \in \mathrm{Allint}(n) \mid \varphi(f) = f \}|.$$

After some calculations we can derive that there are only 4 permutions $\varphi$ such that $\chi(\varphi) \neq 0$. In Remark 23 we calculated $\chi(\mathrm{id})$. The value of $\chi(\varphi_I \circ \varphi_R)$ is the coefficient of $\prod_{i=1}^{n-1} y_i z_i$ in

$$\prod_{i=1}^{\frac{n}{2}-1} \Big( \sum_{\substack{j=1 \\ j \neq \frac{n}{2}}}^{n-1} z_j z_{n-j} \prod_{k=i}^{n-1} x_k{}^j \prod_{k=n-i}^{n-1} x_k{}^{n-j} \Big) z_{\frac{n}{2}} \prod_{k=\frac{n}{2}}^{n-1} x_k{}^{\frac{n}{2}} \Big|_{x_\nu{}^j = y_{j \bmod n}}.$$

Now let $n \geq 6$. The value of $\chi(\varphi_I \circ V)$ is the coefficient of $\prod_{i=1}^{n-1} y_i z_i$ in

$$\prod_{i=1}^{\frac{n}{2}-1} \Big( \sum_{\substack{j=1 \\ j \neq \frac{n}{2}}}^{n-1} z_j z_{n-j} \prod_{k=i}^{n-1} x_k{}^j \prod_{k=(\frac{n}{2}+i)}^{n-1} x_k{}^{n-j} \Big) z_{\frac{n}{2}} \prod_{k=\frac{n}{2}}^{n-1} x_k{}^{\frac{n}{2}} \Big|_{x_\nu{}^j = y_{j \bmod n}}.$$

Now let $n = 12$. In order to calculate $\chi(\varphi_Q \circ V \circ \varphi_R)$ you must compute

$$\sum_{i=1}^{5} \Bigg( z_6 \prod_{j=2i}^{11} x_j{}^6 z_3 z_9 \Big( \prod_{j=i}^{11} x_j{}^3 \prod_{j=i+6}^{11} x_j{}^9 + \prod_{j=i}^{11} x_j{}^9 \prod_{j=i+6}^{11} x_j{}^3 \Big) \cdot$$

$$\cdot \prod_{j=1}^{i-1} \Big( \sum_{\substack{k=1 \\ k \notin \{3,6,9\}}}^{n-1} z_k z_{5k \bmod 12} \prod_{l=j}^{11} x_l{}^k \prod_{l=2i-j}^{11} x_l{}^{5k \bmod 12} \Big) \cdot$$

$$\cdot \prod_{j=2i+1}^{i+5} \Big( \sum_{\substack{k=1 \\ k \notin \{3,6,9\}}}^{n-1} z_k z_{5k \bmod 12} \prod_{l=j}^{11} x_l{}^k \prod_{l=12+2i-j}^{11} x_l{}^{5k \bmod 12} \Big) \Bigg).$$

Then substitute $y_{j \bmod 12}$ for $x_\nu{}^j$ and find the coefficient of $\prod_{i=1}^{11} y_i z_i$.

**Theorem 25 (Number of Patterns of All-Interval-Rows)** *For $i = 1, 2, 3$ the number of patterns of all-interval-rows in regard to $G_i$ is*

1. $\frac{1}{4}(\chi(\mathrm{id}) + \chi(\varphi_I \circ \varphi_R))$ *for* $i = 1$.
2. $\frac{1}{8}(\chi(\mathrm{id}) + \chi(\varphi_I \circ \varphi_R) + \chi(\varphi_I \circ V))$ *for* $i = 2$.
3. *For* $i = 3$ *we calculate*

$$\frac{1}{16}(\chi(\mathrm{id}) + \chi(\varphi_I \circ \varphi_R) + \chi(\varphi_I \circ V) + \chi(\varphi_Q \circ \varphi_R \circ V)) =$$

$$= \frac{1}{16}(3\,856 + 176 + 120 + 120) = 267.$$

This is an application of the Lemma of Bunside of [2].

## 2.4 Patterns of Rhythms

**Definition 26 ($n$-Bar, Entry-time, $k$-Rhythm)** A bar is an important contribution in a composition. Usually a lot of bars of the same form follow one another. If you know the smallest rhythmical subdivision of a bar, you can figure out how many entry-times (think of rhythmical accents played on a drum) a bar holds. If there are $n$ entry-times in a bar, I call it an $n$-bar. In mathematical terms an $n$-bar is expressed as the cyclic group $Z_n$. We can define cyclic temporal shifting $S$ as a permutation $S: Z_n \to Z_n$, $t \mapsto S(t) := t + 1$. Retrogradation $R$ (temporal inversion) is defined as $R: Z_n \to Z_n$, $t \mapsto R(t) := -t$. The group $\langle S \rangle$ is $\zeta_n^{(E)}$ and $\langle S, R \rangle = \vartheta_n^{(E)}$. A $k$-rhythm in an $n$-bar is a subset of $k$ elements of $Z_n$. The permutation groups $\zeta_n^{(E)}$ or $\vartheta_n^{(E)}$ induce an equivalence relation on the set of all $k$-rhythms. Now we want to calculate the number of patterns of $k$-rhythms. We get the same numbers as in Theorem 5.

## 2.5 Patterns of Motifs

**Definition 27 ($k$-Motif)**

1. Now I want to combine both rhythmical and tonal aspects of music.
2. Assume we have an $n$-scale and an $m$-bar, then the set $M$

$$M := \{(x, y) \mid x \in Z_m, y \in Z_n\} = Z_m \times Z_n$$

is the set of all possible combinations of entry-times in the $m$-bar $Z_m$ and pitches in the $n$-scale $Z_n$. Furthermore let $G$ be a permutation group on $M$. In Remark 29 we are going to study two special groups $G$. The group $G$ defines an equivalence relation on $M$:

$$(x_1, y_1) \sim (x_2, y_2) :\Longleftrightarrow \exists g \in G \text{ with } (x_2, y_2) = g(x_1, y_1).$$

In addition to this we have $|M| = m \cdot n$.

3. Let $1 \leq k \leq m \cdot n$. A **$k$-motif** is a subset of $k$ elements of $M$.

**Theorem 28 (Number of Patterns of $k$-Motifs)** *The number of patterns of $k$-motifs in an $n$-scale and in an $m$-bar is the coefficient of $x^k$ in*

$$\mathrm{CI}(G; 1 + x, 1 + x^2, \ldots, 1 + x^{m \cdot n}).$$

This completely follows from Pólya's Theorem of [2].

**Remark 29 (Special Permutation Groups)** Now I want to demonstrate two examples for group $G$.

1. In Definition 2 we had a permutation group $G_2 = \zeta_n^{(E)}$ or $G_2 = \vartheta_n^{(E)}$ acting on the $n$-scale $Z_n$. Moreover in Definition 26 there was a permutation group $G_1 = \zeta_m^{(E)}$ or $G_1 = \vartheta_m^{(E)}$ defined on the $m$-bar $Z_m$. For that reason, we define the group $G$ as $G := G_1 \otimes G_2$. Two elements $(x_1, y_1), (x_2, y_2) \in M$ are called equivalent with respect to $G$, iff there exist $\varphi \in G_1$ and $\psi \in G_2$, with
$$(x_2, y_2) = (\varphi, \psi)(x_1, y_1) = (\varphi(x_1), \psi(y_1)).$$
Because of the fact that we know how to calculate the cycle index of $G_1 \otimes G_2$, we can compute the number of patterns of $k$-motifs.

2. In the case $m = n$, we can define another permutation group $G$, as it is done in [8]. The group $G$ is defined as $G := \langle T, S, \varphi_A \mid A \in \mathrm{Gl}(2, Z_n) \rangle$, with
$$T: M \to M, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto T\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} x \\ y+1 \end{pmatrix}$$
$$S: M \to M, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto S\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} x+1 \\ y \end{pmatrix}$$
$$\varphi_A: M \to M, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \varphi_A\begin{pmatrix} x \\ y \end{pmatrix} := A\begin{pmatrix} x \\ y \end{pmatrix}.$$
The multiplication $A \cdot \begin{pmatrix} x \\ y \end{pmatrix}$ stands for matrix multiplication. The set $\mathrm{Gl}(2, Z_n)$ is the group of all regular $2 \times 2$-matrices over $Z_n$.
You can easily derive the following results:
   (a) $T^n = S^n = \mathrm{id}_M$ and $T^j \neq \mathrm{id}_M$ and $S^j \neq \mathrm{id}_M$ for $1 \leq j < n$.
   (b) $T \circ S = S \circ T$. In addition to this $T \notin \langle S \rangle$ and $S \notin \langle T \rangle$.
   (c) Let $0 \leq i, j < n$, then: $T^i \circ S^j \notin \langle \varphi_A \mid A \in \mathrm{Gl}(2, Z_n) \rangle$, iff $i \neq 0$ or $j \neq 0$.
   (d) Let $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then: $\varphi_A \circ T^k \circ S^l = T^{(cl+dk)} \circ S^{(al+bk)} \circ \varphi_A$.
   (e) $G$ is the group of all affine mappings ${Z_n}^2 \to {Z_n}^2$.

   Although we know quite a lot about the group $G$, I could not find a formula for the cycle index of $G$ for arbitrary $n$.

**Example 30** Let us consider the case, that $n = m = 12$.

1. If $G$ is defined as $G := \vartheta_n^{(E)} \otimes \vartheta_n^{(E)}$, then we derive
$$\mathrm{CI}(G; x_1, x_2, \ldots, x_{144}) =$$
$$= \tfrac{1}{576}(x_1^{144} + 12x_1^{24}x_2^{60} + 36x_1^4x_2^{70} + 147x_2^{72} + 8x_3^{48} + 24x_3^8x_6^{20} + 60x_4^{36} + 96x_6^{24} + 192x_{12}^{12}).$$

By applying Theorem 28, the number of patterns of $k$-motifs is the coefficient of $x^k$ in $1+x+48x^2+937x^3+31\,261x^4+840\,006x^5 19\,392\,669x^6+381\,561\,281x^7+6\,532\,510\,709x^8+98\,700\,483\,548x^9+1\,332\,424\,197\,746x^{10}+\ldots$.

2. If $G:=\langle T,S,\varphi_A \mid A\in \mathrm{Gl}(2,Z_n)\rangle$, I computed the cycle index of $G$ with a Turbo Pascal program as

$$\mathrm{CI}(G;x_1,x_2,\ldots,x_{144})=\frac{1}{663\,552}(x_1^{144}+18x_1^{72}x_2^{36}+36x_1^{48}x_2^{48}+\ldots).$$

By applying Theorem 28, the number of patterns of $k$-motifs is the coefficient of $x^k$ in $1+x+5x^2+26x^3+216x^4+2\,024x^5+27\,806x^6+417\,209x^7+6\,345\,735x^8+90\,590\,713x^9+1\,190\,322\,956x^{10}+\ldots$.
For $k=1,2,3,4$ these numbers are the same as in [8]. In the case $k=5$ however, it is stated that there exist 2 032 different patterns of 5-motifs, while here we get 2 024 of these patterns.

## 2.6 Patterns of Tropes

**Definition 31 (Trope)** 1. If you divide the set of 12 tones in 12-tone music into 2 disjointed sets, each containing 6 elements, and if you label these sets as a first and a second set, we will speak of a trope. This definition goes back to Josef Matthias Hauer. Two tropes are called equivalent, iff transposing, inversion, changing the labels of the two sets or arbitrary sequences of these operations transform one trope into the other.

2. For a mathematical definition let $n\geq 4$ and $n\equiv 0 \bmod 2$. A trope in $n$-tone music is a function $f\colon Z_n\to F:=\{1,2\}$ such that $|f^{-1}(\{1\})|=|f^{-1}(\{2\})|=\frac{n}{2}$. $f(i)=k$ is translated into: The tone $i$ lies in the set with label $k$. Furthermore $T$ and $I$ are permutations on $Z_n$ as in Definition 2. The group $\langle T,I\rangle$ is $\vartheta_n^{(E)}$. Two tropes $f_1,f_2$ are called equivalent, if and only if, $\exists \pi\in\vartheta_n^{(E)}\ \exists\varphi\in S_2$ such that $f_2=\varphi^{-1}\circ f_1\circ\pi$.

3. Let $x$ and $y$ be indeterminates over $\mathbf{Q}$. Define a function $w\colon F\to\mathbf{Q}[x,y]$ by $w(1):=x$ and $w(2):=y$. For $f\in F^{Z_n}$ the weight of $f$ is defined as product weight

$$W(f):=\prod_{x\in Z_n} w(f(x)).$$

A function $f\colon Z_n\to F:=\{1,2\}$ is a trope, iff $W(f)=x^{\frac{n}{2}}y^{\frac{n}{2}}$.

**Theorem 32 (Patterns of Tropes)** *Let $\varphi$ be Euler's $\varphi$-function. The number of patterns of tropes in regard to $\vartheta_n^{(E)}$ is*

$$\begin{cases} \frac{1}{4}\left(\frac{1}{n}\Big(\sum\limits_{t\mid\frac{n}{2}}\varphi(t)\binom{\frac{n}{t}}{\frac{n}{2t}}+\sum\limits_{\substack{t\mid n\\ t\equiv 0 \bmod 2}}\varphi(t)2^{\frac{n}{t}}\Big)+\binom{\frac{n}{2}}{\frac{n}{4}}+2^{\frac{n}{2}-1}\right) & \text{if } n\equiv 0 \bmod 4\\ \frac{1}{4}\left(\frac{1}{n}\Big(\sum\limits_{t\mid\frac{n}{2}}\varphi(t)\binom{\frac{n}{t}}{\frac{n}{2t}}+\sum\limits_{\substack{t\mid n\\ t\equiv 0 \bmod 2}}\varphi(t)2^{\frac{n}{t}}\Big)+\binom{\frac{n-2}{2}}{\frac{n-2}{4}}+2^{\frac{n}{2}-1}\right) & \text{if } n\equiv 2 \bmod 4. \end{cases}$$

*In 12-tone music there are 35 patterns of tropes. (See [5].) Hauer himself calculated that there are 44 patterns of tropes, because in his work the permutation group acting on $Z_n$ was the cyclic group $\langle T\rangle$.*

This is an application of of the *Power Group Enumeration Theorem in polynomial Form* of [7].

## 2.7 Special Remarks on 12-tone music

In addition to the operations of transposing $T$ and of inversion $I$ we can study quartcircle- and quintcircle symmetry in 12-tone music.

**Remark 33 (Quartcircle Symmetry)** The quartcircle symmetry $Q$ is defined as

$$Q: Z_{12} \to Z_{12}, \qquad x \mapsto Q(x) := 5x.$$

$Q$ is a permutation on $Z_{12}$, since $\gcd(5,12) = 1$. Furthermore $Q \notin \langle I,T \rangle$, $Q \circ T = T^5 \circ Q$, $Q^2 = \mathrm{id}_{Z_{12}}$ and $Q \circ I = I \circ Q = 7x$, which is called the quintcircle symmetry.
Let $G$ be $G := \langle I,T,Q \rangle$. Each element $\varphi \in G$ can be written as $\varphi = T^k \circ I^j \circ Q^l$ such that $k \in \{0,1,\ldots,n-1\}$, $j \in \{0,1\}$, and $l \in \{0,1\}$. The cycle index of $G := \langle I,T,Q \rangle$ is

$$\mathrm{CI}(G; x_1, x_2, \ldots, x_{12}) =$$

$$= \frac{1}{48}\Big(\sum_{t|12} \varphi(t) x_t^{\frac{12}{t}} + 2x_1^6 x_2^3 + 3x_1^4 x_2^4 + 6x_1^2 x_2^5 + 11x_2^6 + 4x_3^2 x_6 + 6x_4^3 + 4x_6^2\Big).$$

This group $G$ is an other permutation group acting on $Z_{12}$ with a musical background. The question arises, how to generalize the quartcircle symmetry of 12-tone music to $n$-tone music. Should we take any unit in $Z_n$ or only those units $e$ such that $e^2 = 1$ ?

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