Probabilistic Interpretations of q-Analogues

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1. Abstract: In some instances, the parameter q of a q-analogue may be directly identified as the probability of tails occurring in a Bernoulli trials scheme. The simple coin-tossing game presented in the next section gives rise to a q-analogue of a standard limit formula for the exponential function and to a q-analogue of Euler's product formula for the Riemann zeta function. The context in which these q-identities arise bears some resemblance to the one Gilbert Labelle used in obtaining a q-analogue of Euler's gamma function. Slight variations of the same game also lead to probabilistic interpretations of the inversion number and of the major index.

2. Game I: Players $1, 2, \ldots, n$ all compete for ranks 1 through r. All players begin at rank 1. In turn, a coin is passed from player to player. When tossed, the coin lands tails up with probability q and heads up with probability $p = (1 - q)$. Upon receiving the coin, a player tosses until heads occurs. Each time a player tosses a tails, he/she advances a rank with one exception: If tails occurs at the highest rank r, then the player tossing the coin returns to rank 1. Upon tossing heads, a player goes out of the game with his/her achieved rank and passes the coin to the next player. Play ends when all players have had a turn. The player(s) with the highest rank is (are) deemed the winner(s).

Let $F(n\ ;\ r)$ denote the set of functions mapping $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, r\}$. The outcome of Game I may be viewed as a function mapping the set $\{1, 2, \ldots, n\}$ of players to the set of ranks $\{1, 2, \ldots, r\}$, or, in other words, as an element of $F(n\ ;\ r)$. Because of q's prominent role, an outcome of Game I is said to be a *q-random mapping*.

As an illustration, suppose that $n = 4$ and $r = 3$. If players 1, 2, 3 and 4 respectively toss TTTTTTTH, H, TTH, and TTTH, then the outcome may be conveniently displayed as follows:

TT		H	T	3	ranks
TTH		T	T	2	
TTT	H	T	T H	1	
1	2	3	4		
players					

The corresponding function is $f = 2\ 1\ 3\ 1 \in F(4\ ;\ 3)$.

3. q-Random Mappings: The probability that a given $f \in F(n\ ;\ r)$ is the outcome of Game I is

$$P_q\{f\} \;=\; \frac{q^{f(1) + \ldots + f(n) - n}}{[r]^n}$$

where $[m] = 1 + q + \ldots + q^{m-1} = (1 - q^m)/(1 - q)$. Furthermore, let NFP($n$) denote the subset of functions in $F(n\ ;\ n)$ having no fixed points, that is,

$$\mathrm{NFP}(n) = \{f \in F(n\ ;\ n) : f(j) \neq j \text{ for } 1 \leq j \leq n\}.$$

It is not too difficult to prove that

$$\lim_{n \to \infty} P_q\,\{f \in \mathrm{NFP}(n)\} \;=\; \frac{1}{[e]}$$

where $[e]$ denotes a q-analog of the real number e defined by

$$[e] \;=\; \sum_{n \geq 0} \frac{1}{[n]!}\ .$$

The q-analog of e satisfies the limit formula

$$[e] \;=\; \lim_{n \to \infty} \prod_{j=1}^{n} \left(1 \;-\; \frac{q^{j-1}}{[n]}\right)^{-1}.$$

This follows as a corollary from the more general fact that

$$\sum_{n \geq 0}^{\infty} \frac{t^n}{[n]!} \;=\; \lim_{n \to \infty} \prod_{j=1}^{n} \left(1 \;-\; \frac{q^{j-1}\,t}{[n]}\right)^{-1}$$

which holds for $-1 < q \leq 1$ and $|\,t\,| < (1 - q)^{-1}$.

4. A q-Riemann Zeta Function: Suppose there are $s > 1$ players and p ranks (where p is prime). Game I is said to end in an s-way tie at rank p if all players go out with rank p.

TH	H	TTH	H	
TT	T	TTT	T	$p = 3$ ranks
TT	T	TTT	T	

$s = 4$
players

The probability that such an event occurs is $(q^{p-1}/[p])^s$.

Suppose that Game I is repeated for each prime p. If $s > 1$, then the probability that an s-way tie at rank p does not occur for any p is

$$\prod_p \left(1 - \frac{q^{s(p-1)}}{[p]}\right) = \frac{1}{\zeta_q(s)}$$

where $\zeta_q(s)$ denotes a q-analog of the Riemann zeta function defined by

$$\zeta_q(s) = \sum_{n\geq 1} \frac{q^{s\,v(n)}}{\{n\}^s}.$$

The expressions $v(n)$ and $\{n\}$ are defined as follows: By convention, $v(1) = 0$ and $\{1\} = 1$. For $n > 1$ canonically factored as $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$,

$$\{n\} = [p_1]^{\alpha_1} [p_2]^{\alpha_2} \dots [p_k]^{\alpha_k}$$

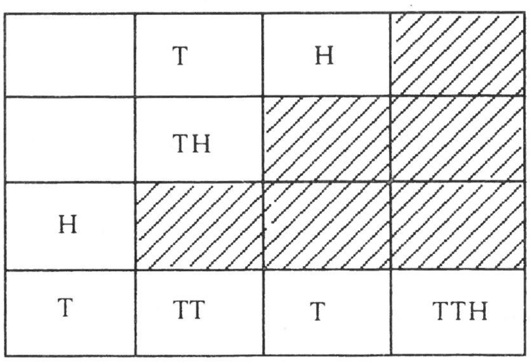
and $v(n) = \alpha_1(p_1 - 1) + \alpha_2(p_2 - 1) + \dots + \alpha_k(p_k - 1)$.

The q-Riemann zeta function satisfies the following two theorems. Theorem 2 is q-analog of Euler's product formula for $\zeta_1(s)$.

<u>Theorem 1</u> For $0 \leq q \leq 1$ and $s > 1$, $\zeta_q(s)$ is convergent.

<u>Theorem 2</u> For $0 \leq q \leq 1$ and $s > 1$, $\zeta_q(s) = \prod_p \left(1 - \frac{q^{s(p-1)}}{[p]^s}\right)^{-1}$ where the product is over all primes p.

5. **Game II:** Suppose Game I is played under the following conditions: (a) $r = n$ and (b) No ties are allowed. Condition (b) is accomplished by simply removing an "occupied" rank from the board.



$r = 4$

$n = 4$

The outcome may be viewed as an element of the symmetric group S_n. This variation on Game I will henceforth be referred to as Game II.

The probability that a given permutation $\sigma \in S_n$ is the outcome of Game II is

$$P_q\{\sigma\} = \frac{q^{\mathrm{inv}\,\sigma}}{[n]!}$$

where $\mathrm{inv}\,\sigma = |\{(i,j) : 1 \le i < j \le n \text{ and } \sigma(i) > \sigma(j)\}|$. Forthermore, it may be shown that

$$\mathrm{inv}\,\sigma = \left\{ \begin{array}{l} \text{number of tails in the shortest} \\ \text{Bernoulli sequence that results in } \sigma \end{array} \right. .$$

It is conjectured that

$$\lim_{n\to\infty} P_q\{\sigma \text{ is a derangement}\} = \begin{cases} \frac{1}{e} & \text{if } q = 1 \\ 0 & \text{otherwise} \end{cases} .$$

6. Game III: Consider adding the stipulation to Game II that "each player begins at the next available rank after the rank achieved by the preceding player." Referring to this variation as Game III, all statements in Section 5 remain true if the word "Game II" is replaced by "Game III" and if the inversion number is replaced by the comajor index:

$$\mathrm{comaj}\,\sigma = \sum_{k=1}^{n-1} (n-k)\,\chi\{\sigma(k) > \sigma(k+1)\}.$$

References

1. G. E. Andrews, The Theory of Partitions, Addison-Wesley Publishing Co., 1976.

2. G. Labelle, "A propos d'un q-analogue pour la Fonction Gamma D'Euler," Ann. Sc. Math. Quebéc (2) 6 (1982), 163-196.

3. R. H. Moritz and R. C. Williams, "A coin-tossing game and some related combinatorics," Mathematics Magazine (1) 61 (1988), 24-29.

4. D. P. Rawlings, "Bernoulli trials and permutation statistics," Inter. J. of Math. and Math. Sciences (2) 15 (1992), 291-312.

5. D. P. Rawlings, "q-Random Mappings," submitted to Math. Magazine.

6. D. P. Rawlings, "Bernoulli trials and number theory," Amer. Math. Monthly (to appear).