New Euler–Mahonian permutation statistics

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Abstract

We define or redefine new Mahonian permutation statistics, called MAD, MAK and ENV. Of these, ENV is shown to equal the classical INV, that is the number of inversions, while MAK has been defined in a slightly different way by Foata and Zeilberger. It is shown that the triple statistics (des, MAK, MAD) and (exc, DEN, ENV) are equidistributed over $\mathcal{S}_n$. Here DEN is Denert's statistic. In particular, this implies the equidistribution of (exc, INV) and (des, MAD). These bistatistics are not equidistributed with the classical Euler-Mahonian statistic (des,MAJ). The proof of the main result is by means of a bijection which is essentially equivalent to several bijections in the literature (or inverses of these). These include bijections defined by Foata and Zeilberger, by Françon and Viennot and by Biane, between the symmetric group and sets of weighted Motzkin paths. These bijections are used to give a continued fraction expression for the generating function of (exc, INV) or (des, MAD) on the symmetric group.

1 Introduction

The subject of permutation statistics, it is frequently claimed, dates back at least to Euler [7]. However, it was not until MacMahon's extensive study [19] at the turn of the century that this became an established discipline of mathematics, and it was to take a long time before it developed into the vast field that it is today.

In the last three decades or so, much progress has been made in discovering and analyzing new statistics. See for example [9, 10, 12, 13, 14, 22, 24, 26]. Inroads have also been made in connecting permutation statistics to various geometric structures and to the classical theory of hypergeometric functions, as in [8, 15, 16, 20].

MacMahon considered four different statistics for a permutation π: The number of descents ($\operatorname{des}\pi$), the number of excedances ($\operatorname{exc}\pi$), the number of inversions (INV π), and the major index (MAJ π). These are defined as follows: A descent in a permutation $\pi = a_1a_2\cdots a_n$ is an i such that $a_i > a_{i+1}$, an excedance is an i such that $a_i > i$, an inversion is a pair (i, j) such that $i < j$ and $a_i > a_j$, and the major index of π is the sum of the descents in π. (In fact, MacMahon studied these statistics in greater generality, namely over the rearrangement class of an arbitrary word w with possibly repeated letters.

However, although all of our present results except those of section 3 can be generalized to words, and this will be done in a subsequent publication, [4], we restrict our attention here to permutations.)

MacMahon showed, algebraically, that exc is *equidistributed* with des, and that INV is equidistributed with MAJ, over $\mathcal{S}_n$. That is to say,

$$\sum_{\pi\in\mathcal{S}_n} t^{\operatorname{exc}\pi} = \sum_{\pi\in\mathcal{S}_n} t^{\operatorname{des} z} \quad \text{and} \quad \sum_{\pi\in\mathcal{S}_n} q^{\mathrm{INV}\,\pi} = \sum_{\pi\in\mathcal{S}_n} q^{\mathrm{MAJ}\,\pi}.$$

The first combinatorial proof of these equidistribution results were given by Foata (see [10]).

Any permutation statistic that is equidistributed with "des" is said to be *Eulerian* and a permutation statistic that is equidistributed with INV is said to be *Mahonian* (see [9]). Most of the permutation statistics found in the literature fall into one of these two categories; they are either Eulerian or Mahonian.

Curiously, new Eulerian statistics have not become prominent since MacMahon's definition of des and exc, whereas new Mahonian statistics are constantly entering the scene. Proving directly that a statistic is Mahonian is by no means always trivial, and there are still many such statistics for which no direct proof exists. What is more interesting, however, is the study of *pairs* of statistics, usually an Eulerian one and a Mahonian one, and equidistribution of such *bistatistics*, first developed in [9].

The first pair of equidistributed Euler-Mahonian bistatistics to be discovered was that of (des, INV) and (des, IMAJ), where IMAJ π is the major index of the *inverse* of the permutation π (see [13]). Although instrumental in some of the analytic developments of the subject, this discovery cannot be extended to words with repetition of letters. In addition, the purists among us are reluctant to admit to the Euler-Mahonian club a pair of pairs that really is only a triple. Thus, they would recommend that (des, INV) be accompanied by exc and a suitable Mahonian partner.

The first discovery of a *proper* pair of equidistributed Euler-Mahonian bistatistics is only a few years old, and it came from a rather unexpected direction. Denert [6], in 1990, conjectured that the pair (des, MAJ) was equidistributed with the pair (exc, DEN), where DEN is a Mahonian statistic somewhat related to, but crucially different from, INV. Shortly afterwards, her conjecture was proved by Foata and Zeilberger [14], who named the new statistic "Denert's statistic". In the process, Foata and Zeilberger defined yet another Mahonian statistic on permutations, which they called MAK, and which, when taken together with des, they showed to be equidistributed with (exc, DEN).

It is fair to say that the discovery of Denert's statistic paved the way to the more esoteric reaches of Mahonian statistics, because it was the first such statistic to be composed of "smaller" partial statistics. Since then, many such composite Mahonian statistics have been discovered, and most of these are treated here.

The pairs of bistatistics (exc, DEN), (des, MAJ) vs. (exc, DEN), (des, MAK) were the first proper pairs of Euler-Mahonian statistics to be shown equidistributed over the symmetric group, and they are, to the best of our knowledge, the only ones preceding the present paper. It is possible to vary the definition of MAK slightly, as will be made clear later, to obtain a new statistic. However, the bistatistics obtained are equidistributed with each other, and this is easy to show.

In the present paper, we define some new Mahonian statistics and redefine many of the existing ones, with an eye to illuminating their common properties and thus also their differences. Doing this allows us to recover some of the known instances of equidistribution among Euler-Mahonian pairs, and to prove the equidistribution of two new pairs introduced, as well as that of some similiar, but not equal, pairs of bistatistics. We do this simultaneously for all the statistics involved, by means of a single, simply described bijection.

All of our constructions, and some of our statistics, have appeared previously, in the work of several authors and in many different guises. They have involved Motzkin paths, binary trees, and even more exotic structures. As we will show, the bijections in the literature pertaining to these statistics, those of Foata–Zeilberger, Françon–Viennot [15], de Médicis–Viennot [20], Simion–Stanton [24] and Biane [1], defined in different ways and for different purposes, are all essentially the same, or inverses of each other. These bijections are equivalent to the bijection of this paper, but their relationships with each other have not before been elucidated.

Perhaps the most interesting fact to emerge is the equidistribution of the two bistatistics (des, MAD) and (exc, INV), where MAD is one of our new statistics. The latter bistatistic, whose components are classical, is *not* equidistributed with (des, MAJ) and might therefore, together with its equidistributed mates, be classified as an "Euler-Mahonian pair of the second kind." In fact there exist at least three different *families* of Euler-Mahonian statistics. The first one, containing $(\mathrm{des}, \mathrm{MAJ})$, $(\mathrm{des}, \mathrm{MAK})$, and $(\mathrm{exc}, \mathrm{DEN})$, has been extensively studied, both analytically and bijectively. For the family containing $(\mathrm{des}, \mathrm{INV})$ and $(\mathrm{des}, \mathrm{IMAJ})$, only the analytic branch has seen substantial development (see [11]). The bijective theory of the family with $(\mathrm{des}, \mathrm{MAD})$ and $(\mathrm{exc}, \mathrm{INV})$ is thoroughly analyzed in the present paper, but its analytic properties remain to be further elicited.

It is, of course, possible to define scores of different families of Euler-Mahonian statistics by arbitrarily combining an Eulerian statistic and a Mahonian one. Although some needles are sure to be found in that haystack, most of the possible such statistics seem rather unattractive, and unlikely to possess particularly interesting properties.

An essential feature of our bijection is that it simultaneously preserves each of several building blocks of the statistics involved. This allows us to derive the equidistribution of the triples of statistics $(\mathrm{des}, \mathrm{MAK}, \mathrm{MAD})$ and $(\mathrm{exc}, \mathrm{DEN}, \mathrm{INV})$, involving Mahonian statistics of both the first and second kind.

In the rest of this section we will present the formal definitions of our statistics and state the main results and indicate precisely the relationship between our statistics and those previously defined. These results will be proved in sections 2 and 3. In section 4 we present some variations and generalizations on our statistics.

1.1 Definitions and main results

We consider the set $\mathcal{S}_{\mathcal{A}}$ of all permutations $\pi = a_1 a_2 \cdots a_n$ on a totally ordered alphabet $\mathcal{A}$. Although it is not necessary, we always take $\mathcal{A}$ to be the interval $[n] = \{1, 2, \ldots, n\}$. Thus, we consider permutations in $\mathcal{S}_n$.

The *biword associated to* a permutation $\pi = a_1 a_2 \ldots a_n$ is

$$\widetilde{\pi} = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

This notation will be adhered to throughout, that is, if π is a permutation, then $\widetilde{\pi}$ has the above meaning.

Definition 1 *Let $\pi \in \mathcal{S}_n$. A* descent *in π is an integer i with $1 \leq i < n$ such that $a_i > a_{i+1}$. Here a_i is called the* descent top *and a_{i+1} is called the* descent bottom. *An* excedance *in π is an integer i with $1 \leq i \leq n$ such that and $a_i > i$. Here a_i is called the* excedance top. *The* number of descents in π *is denoted by* $\operatorname{des}\pi$, *and the* number of excedances *is denoted by* $\operatorname{exc}\pi$.

The descent set of π, *$D(\pi)$, is the set of descents. The* excedance set of π, *$E(\pi)$, is the set of excedances.*

Given a permutation $\pi = a_1a_2\cdots a_n$, we separate π into its *descent blocks* by putting in dashes between a_i and a_{i+1} whenever $a_i \leq a_{i+1}$. A maximal contiguous subword of π which lies between two dashes is a descent block. A descent block is an *outsider* if it has only one letter, otherwise it is a *proper* descent block. The leftmost letter of a proper descent block is its *closer* and the rightmost letter is its *opener*. A letter which lies strictly inside a descent block is an *insider*. For example, the permutation 1 8 5 2 6 7 9 3 4 has descent block decomposition $1-8\ 5\ 2-6-7-9\ 3-4$, with closers 8, 9, corresponding openers 2, 3, outsiders 1, 6, 7, 4 and insider 5. We will frequently write a permutation π with its separating dashes to emphasize this structure.

Let B be a proper descent block of the permutation π and let $\mathrm{C}(B)$ and $\mathrm{O}(B)$ be the closer and opener, respectively, of B. If a is a letter of w, we say that *a is embraced by B* if $\mathrm{C}(B) > a > \mathrm{O}(B)$.

Definition 2 *Let $\pi = a_1a_2\cdots a_n$ be a permutation. The* (right) embracing numbers *of π are the numbers $e_1, e_2, \ldots, e_n$, where e_i is the number of descent blocks in π that are strictly to the right of a_i and that embrace a_i. The* right embracing sum *of π, denoted by $\operatorname{Res}\pi$, is defined by*

$$\operatorname{Res}\pi = e_1 + e_2 + \cdots + e_n.$$

For instance, the embracing numbers of $\pi = 4\ 1 - 7 - 8\ 2 - 5 - 6\ 3$ are $2\ 0 - 1 - 0\ 0 - 1 - 0\ 0$, so $\operatorname{Res}\pi = 4$.

One can obviously define $\operatorname{Les}\pi$ in an analogous way, by simply replacing "right" by "left" in the above definition. (See section 4.)

Definition 3 *The* descent bottoms sum *of a permutation $\pi = a_1a_2\cdots a_n$, denoted by $\operatorname{Dbot}\pi$, is the sum of the descent bottoms of π. The* descent tops sum *of π, denoted by $\operatorname{Dtop}\pi$, is the sum of the descent tops of π. The* descent difference *of π is*

$$\operatorname{Ddif}\pi = \operatorname{Dtop}\pi - \operatorname{Dbot}\pi.$$

Otherwise expressed, $\operatorname{Ddif}\pi$ is the sum of closers minus the sum of openers of descent blocks. As an example, for $\pi = 4\,1{-}2{-}6\,5\,3{-}7$, $\operatorname{Dbot}\pi = 1{+}5{+}3 = 9$, $\operatorname{Dtop}\pi = 4+6+5 = 15$ and $\operatorname{Ddif}\pi = 15-9 = (4+6)-(1+3) = 6$.

Definition 4 *The* excedance bottoms sum *of a permutation $\pi = a_1 a_2 \cdots a_n$, denoted by* $\operatorname{Ebot}\pi$*, is the sum of the excedances of π. The* excedance tops sum *of π, denoted by* $\operatorname{Etop}\pi$*, is the sum of the excedance tops of π. The* excedance difference *of π is*

$$\operatorname{Edif}\pi = \operatorname{Etop}\pi - \operatorname{Ebot}\pi.$$

The excedance subword of π*, denoted by π_{E}, is the permutation consisting of all the excedance tops of π, in the order induced by π. The* non-excedance subword of π*, denoted by π_{N}, consists of those letters of π that are not excedance tops.*

For example, let $\pi = 6\ 5\ 4\ 3\ 7\ 1\ 2$, so

$$\widetilde{\pi} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 5 & 4 & 3 & 7 & 1 & 2 \end{pmatrix}.$$

Then $\pi_{\mathrm{E}} = 6\ 5\ 4\ 7$ and $\pi_{\mathrm{N}} = 3\ 1\ 2$. Also, $\operatorname{Ebot}\pi = 1+2+3+5 = 11$, $\operatorname{Etop}\pi = 6+5+4+7 = 22$ and $\operatorname{Edif}\pi = 22-11 = 11$.

Definition 5 *An* inversion *in a permutation $\pi = a_1 a_2 \cdots a_n$ is a pair (i,j) such that $i < j$ and $a_i > a_j$. The* number of inversions in π *is denoted by* $\mathrm{INV}\,\pi$.

The reason we spell INV with all capital letters is that INV is a Mahonian statistic. We do this consistently throughout the paper, that is, all Mahonian statistics are spelled with uppercase letters. The two Eulerian statistics, exc and des, are spelled with lowercase letters, while "partial statistics" (such as Res), used in the definitions of Mahonian statistics, are merely capitalized.

Definition 6 *Let $\pi = a_1 a_2 \cdots a_n$ be a permutation and i an excedance in π. We say that a_i* is the bottom of d inversions *if there are exactly d letters in π to the left of a_i that are greater than a_i, and we call d the* inversion bottom number of i. *Similarly, if i is a non-excedance in π and there are exactly d letters smaller than a_i and to the right of a_i in π, then we say that d is the* inversion top number of i. *The* side number of i in π *is the inversion bottom number or the inversion top number of* i *in* π*, according as* i *is an excedance or not in* π*. The* sequence of side numbers of π *is the sequence* $s_1, s_2, \ldots, s_n$ *where* s_i *is the side number of* i*.*

For example, let $\pi = 6\ 5\ 4\ 3\ 7\ 1\ 2$ as before, with $\pi_{\mathrm{E}} = 6\ 5\ 4\ 7$ and $\pi_{\mathrm{N}} = 3\ 1\ 2$. Then the inversion bottom numbers of the excedances in π are 0, 1, 2, 0 and the inversion top numbers of the non-excedances in π are 2, 0, 0. Hence the sequence of side numbers of π is 0, 1, 2, 2, 0, 0, 0.

Note that if i is an excedance of the permutation π, then any letter in π that is to the left of a_i and greater than a_i must also be an excedance. Hence, the sum of the inversion bottom numbers of the letters in π_{E} equals the total number of inversions in π_{E}, that is, $\textsc{inv}\,\pi_{\mathrm{E}}$. Similarly, the sum of the inversion top numbers of the letters in π_{N} equals $\textsc{inv}\,\pi_{\mathrm{N}}$.

Definition 7 *Let* π *be a permutation. Then*

$$\operatorname{Ine}\pi = \textsc{inv}\,\pi_{\mathrm{E}} + \textsc{inv}\,\pi_{\mathrm{N}}.$$

Hence, from the remark preceding definition 7, we have

$$\operatorname{Ine}\pi = s_1 + \cdots + s_n. \tag{1}$$

We now define the four Mahonian statistics central to this paper. All these statistics have been more or less introduced in the litterature in different ways (see [14, 20, 24, 21] and section 1.2), but we redefine them here in a way suitable to generalize to words).

Definition 8 *Let* π *be a permutation. Then*

$$\begin{aligned}
\textsc{mak}\,\pi &= \operatorname{Dbot}\pi + \operatorname{Res}\pi. \\
\textsc{mad}\,\pi &= \operatorname{Ddif}\pi + \operatorname{Res}\pi. \\
\textsc{den}\,\pi &= \operatorname{Ebot}\pi + \operatorname{Ine}\pi. \\
\textsc{env}\,\pi &= \operatorname{Edif}\pi + \operatorname{Ine}\pi.
\end{aligned}$$

As it turns out, our statistic ENV equals the classical INV. It may seem superfluous to redefine INV in this way, but it turns out that ENV's similarity in definition to MAD is crucial in proving our main results.

We now describe the main results of this paper.

In section 1.2 we will prove the result referred to above.

Proposition 1 *For any permutation π we have* $\mathrm{ENV}\,\pi = \mathrm{INV}\,\pi = \mathrm{INV}_{\mathrm{MV}}\,\pi$*.*

Here $\mathrm{INV}_{\mathrm{MV}}$ is a statistic, first defined by de Médicis and Viennot, that will be defined below.

In section 2 we will define a mapping Φ on $\mathcal{S}_n$ and prove the following result.

Proposition 2 *For any permutation π, we have*

$$\begin{aligned}(\mathrm{des}, \mathrm{Dbot}, \mathrm{Ddif}, \mathrm{Res})\ \pi &= (\mathrm{exc}, \mathrm{Ebot}, \mathrm{Edif}, \mathrm{Ine})\ \Phi(\pi),\\ (\mathrm{des}, \mathrm{MAD}, \mathrm{MAK})\ \pi &= (\mathrm{exc}, \mathrm{INV}, \mathrm{DEN})\ \Phi(\pi).\end{aligned}$$

By showing that Φ is a bijection, we deduce the following theorem.

Theorem 3 *The quadristatistics*

$$(\mathrm{des}, \mathrm{Dbot}, \mathrm{Ddif}, \mathrm{Res}) \quad \textit{and} \quad (\mathrm{exc}, \mathrm{Ebot}, \mathrm{Edif}, \mathrm{Ine})$$

are equidistributed over the symmetric group $\mathcal{S}_n$. That is,

$$\sum_{\pi\in\mathcal{S}_n} t^{\mathrm{des}\,\pi} x^{\mathrm{Dbot}\,\pi} y^{\mathrm{Ddif}\,\pi} q^{\mathrm{Res}\,\pi} = \sum_{\pi\in\mathcal{S}_n} t^{\mathrm{exc}\,\pi} x^{\mathrm{Ebot}\,\pi} y^{\mathrm{Edif}\,\pi} q^{\mathrm{Ine}\,\pi}.$$

Hence the triple $(\mathrm{des}, \mathrm{MAD}, \mathrm{MAK})$ *is equidistributed with* $(\mathrm{exc}, \mathrm{INV}, \mathrm{DEN})$ *over* $\mathcal{S}_n$*.*

In section 3, we shall make evident the relation between our bijection Φ and some well-known bijections between the symmetric group $\mathcal{S}_n$ and weighted Motzkin paths. As a by-product, we will obtain a continued fraction expansion, equation (14), for the ordinary generating function of

$$D_n(x,q) = \sum_{\pi\in\mathcal{S}_n} x^{\mathrm{des}\,\pi} q^{\mathrm{MAD}\,\pi},$$

and then derive a symmetric property of $D_n(x,q)$, see Corollary 10.

1.2 Links to the past

Some Mahonian statistics on $\mathcal{S}_n$ equivalent or similar to our ENV and MAD have been given by Simion and Stanton [24] and by de Médicis and Viennot [20], and more recently by Randrianarivony [21].

More precisely, de Médicis and Viennot introduced a statistic which we denote by INV_{MV} and which can be defined in our notation by

$$\begin{aligned}\text{INV}_{\text{MV}}\,\pi = \text{Ine}\,\pi \;&+\; \#\{(i,j) \mid i \le j < \pi(i),\, \pi(j) > j\} \\ &+\; \#\{(i,j) \mid \pi(i) < \pi(j) \le i,\, \pi(j) \le j\}.\end{aligned} \tag{2}$$

However, their proof that INV equals INV_{MV} is fairly complicated, and can be compared to that of the equivalence of the two definitions of DEN given in [14]. Another proof that INV equals INV_{MV} was given by Randrianarivony [21], who used Motzkin paths and the bijection of Foata and Zeilberger. In [2], Clarke gave a short proof of the equivalence of the two definitions of DEN, using equation (7) below. Here we will give a short proof of the results of de Médicis and Viennot and of Clarke, as well as of Proposition 1.

Lemma 4 *For any permutation $\pi = a_1 a_2 \cdots a_n$ we have*

$$\sum_{i \in E(\pi)} (a_i - i) = \sum_{i \in E(\pi)} \#\{\, j \mid i < j, a_i > a_j, j \notin E(\pi)\,\}. \tag{3}$$

Proof: The right-hand side equals

$$\sum_{i \in E(\pi)} \left(\#\{\, j \mid i < j, a_i > a_j\,\} - \#\{\, j \mid i < j, a_i > a_j, j \in E(\pi)\,\}\right). \tag{4}$$

Now,

$$\begin{aligned} a_i - i &= \#\{\, j \mid a_j < a_i\,\} - \#\{\, j \mid j < i\} \\ &= \#\{\, j \mid j > i, a_j < a_i\,\} - \#\{\, j \mid j < i, a_j > a_i\,\}. \end{aligned} \tag{5}$$

Hence, comparing (4) and (5), we must show that

$$\sum_{i \in E(\pi)} \#\{\, j \mid i < j, a_i > a_j, j \in E(\pi)\,\} = \sum_{i \in E(\pi)} \#\{\, j \mid j < i, a_j > a_i\,\}. \tag{6}$$

Clearly, each of the sums in equation (6) is $\text{INV}\,\pi_{\text{E}}$. $\square$

Lemma 5 *For any permutation $\pi = a_1 a_2 \cdots a_n$ we have*

$$\#\{(i,j) \mid i \le j < a_i,\, a_j > j\} = \#\{(i,j) \mid a_i < a_j \le i,\, a_j > j\}, \tag{7}$$

$$\#\{(i,j) \mid i \le j < a_i,\, a_j \le j\} = \#\{(i,j) \mid a_i < a_j \le i,\, a_j \le j\}. \tag{8}$$

Proof: Notice that

$$\begin{aligned}\sum_{i\in E(\pi)}(a_i-i) &= \#\{(i,j)\mid i\le j<a_i\}\\ &= \#\{(i,j)\mid i<j<a_i,\, a_j\le j\} \\ &\quad + \#\{(i,j)\mid i\le j<a_i,\, a_j>j\},\end{aligned}\tag{9}$$

and the right-hand side of (3) equals

$$\#\{(i,j)\mid i<j<a_i,\, a_j\le j\}+\ \#\{(i,j)\mid i<a_i,\, a_j<a_i\le j\}.$$

Identity (7) follows then from Lemma 4. On the other hand, it is not hard to see that $\sum_{i\in E(\pi)}(a_i-i)$ can be written as

$$\#\{(i,j)\mid i\le j<a_i\}=\#\{(i,j)\mid a_i<j\le i\}=\#\{(i,j)\mid a_i<a_j\le i\}.$$

Identity (8) follows immediately from (7). □

Identity (7) is that of Clarke [2].

Proof of Proposition 1: The first equality follows immediately from the first part of Lemma 4. Comparing definition (2) with equation (9), it follows from (8) that $\mathrm{INV}_{\mathrm{MV}}\,\pi=\mathrm{ENV}\,\pi$. □

On the other hand, de Médecis and Viennot [20, Proposition 6.2] gave a Mahonian statistic they called "lag" (but which we call LAG, for the sake of consistency). It can be defined as follows: Given a permutation $\pi = a_1a_2\cdots a_n$, let

$$\pi'=(n+1)a_1a_2\cdots a_n0\in\mathcal{S}_{n+2}$$

and let $\mathrm{Run}\,\pi$ be the number of descent blocks in π. Then

$$\mathrm{LAG}\,\pi=\mathrm{Ddif}\,\pi'+\mathrm{Les}\,\pi'-\mathrm{Run}\,\pi-n.$$

It is not hard to see that $\mathrm{LAG}\,\pi=\mathrm{MAD}\,\pi^r$, where $\pi^r=a_na_{n-1}\cdots a_1$.

We define the *dual* of a permutation $\pi=a_1a_2\cdots a_n\in\mathcal{S}_n$ as the permutation $\pi^*=b_1b_2\cdots b_n$, where $b_i=n+1-a_i$ for $1\le i\le n$. Simion and Stanton [24] use notions dual to ours, with ascents instead of descents, and embracing by ascent blocks, which they call "runs". They also use the notion of left embracing. Their statistic, translated into our dual setting, is

$$\mathrm{SIST}\ \pi=n-\mathrm{Run}\,\pi+2\,\mathrm{Les}\,\pi+\mathrm{Res}\,\pi$$

(see Theorem 2 in [24]), so, since $n - \operatorname{Run}\pi = \operatorname{des}\pi$, we have

$$\textsc{sist}\ \pi = \operatorname{des}\pi + 2\operatorname{Les}\pi + \operatorname{Res}\pi.$$

A counterpart of this statistic, namely

$$\textsc{sist}'\ \pi = \operatorname{des}\pi + 2\operatorname{Res}\pi + \operatorname{Les}\pi,$$

whose dual was also defined by Simion and Stanton, is readily seen to equal MAD, because of the identity:

$$\operatorname{Ddif}\pi = \operatorname{des}\pi + \operatorname{Res}\pi + \operatorname{Les}\pi.$$

2 The bijection Φ

Before proving Proposition 2 and Theorem 3, we describe the construction of a bijection $\Phi : \mathcal{S}_n \to \mathcal{S}_n$ which takes a permutation π to a permutation τ such that the set of descent tops in π equals the set of excedance tops in τ and the set of descent bottoms in π equals the set of excedances in τ. Moreover, the embracing numbers of π are preserved in a way that we now describe.

Observe that, since the letters of a permutation are distinct, we can refer to the i-th embracing number e_i of the permutation π as the embracing number of the *letter* a_i in π, and we will then denote e_i by $e(a_i)$. Similarly, we may if we wish denote the i-th side number of π by $d(a_i)$.

We will construct $\tau = \Phi(\pi)$ in such a way that the embracing number of a letter a_i in π is the side number of a_i in τ.

Given a permutation π, we first construct two biwords, $\begin{pmatrix} f \\ f' \end{pmatrix}$ and $\begin{pmatrix} g \\ g' \end{pmatrix}$, and then form the biword $\tau' = \begin{pmatrix} f\ g \\ f'\ g' \end{pmatrix}$ by concatenating f and g, and f' and g', respectively. The permutation f is defined as the subword of descent bottoms in π, ordered increasingly, and g is defined as the subword of non-descent bottoms in π, also ordered increasingly. The permutation f' is the subword of descent tops in π, ordered so that the inversion bottom number of a letter a in f' is the embracing number of a in π, and g' is the subword of non-descent tops in π, ordered so that the inversion top number of a letter b in g' is the embracing number of b in π. Rearranging the columns of τ', so that the top row is in increasing order, we obtain the permutation $\tau = \Phi(\pi)$ as the bottom row of the rearranged biword.

Example 1 *Let* $\pi = 4\ 1 - 2 - 7 - 9\ 6\ 5 - 8\ 3$, with embracing numbers 1, 0, 0, 2, 0, 1, 1, 0, 0. Then

$$\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} 1\ 3\ 5\ 6 \\ 8\ 4\ 6\ 9 \end{pmatrix}, \quad \begin{pmatrix} g \\ g' \end{pmatrix} = \begin{pmatrix} 2\ 4\ 7\ 8\ 9 \\ 1\ 2\ 7\ 5\ 3 \end{pmatrix}, \quad \tau' = \begin{pmatrix} 1\ 3\ 5\ 6\ 2\ 4\ 7\ 8\ 9 \\ 8\ 4\ 6\ 9\ 1\ 2\ 7\ 5\ 3 \end{pmatrix}$$

and thus $\Phi(\pi) = \tau = 8\ 1\ 4\ 2\ 6\ 9\ 7\ 5\ 3$. It is easily checked that the descent tops and descent bottoms in π are the excedance tops and excedances in τ, respectively, and that the embracing number of each letter in π is the side number of the same letter in τ.

Proof of Proposition 2: Assuming that the construction of f' and g' can be carried out in the way described, and such that $f' = \tau_{\mathrm{E}}$ and $g' = \tau_{\mathrm{N}}$, it is clear that the excedance tops and excedances in τ are the descent tops and descent bottoms, respectively, in π, and that

$$\operatorname{Res}\pi = \textsc{inv}\,\tau_{\mathrm{E}} + \textsc{inv}\,\tau_{\mathrm{N}} = \operatorname{Ine}\tau.$$

As a consequence, we have

$$(\mathrm{des}, \mathrm{Dbot}, \mathrm{Ddif}, \mathrm{Res})\ \pi = (\mathrm{exc}, \mathrm{Ebot}, \mathrm{Edif}, \mathrm{Ine})\ \Phi(\pi).$$

To complete the proof, we need to show two things. Firstly, that f' and g' can be constructed so that the inversion bottom numbers and the inversion top numbers of f' and g' respectively are those claimed, and, secondly, that $f' = \tau_{\mathrm{E}}$ (and thus $g' = \tau_{\mathrm{N}}$).

Let a be the least descent top in π. Then, if the embracing number of a in π is k, there are k descent blocks in π to the right of a that embrace a. Thus, there are at least k descent tops in π that are larger than a, namely the closers of the descent blocks embracing a. Also, there are at least $k+1$ descent bottoms in π that are smaller than a, namely the openers of the descent blocks embracing a, together with the opener of the descent block containing a. If we put a in the $(k+1)$–st place in f' from the left, then the inversion bottom number of a in f' is k as desired, and the $(k+1)$–st place *does exist* in f', because, as pointed out, there are at least $(k+1)$ descent bottoms, and thus at least $(k+1)$ descent tops, in π if a has embracing number k.

Moreover, the same argument shows that a is larger than the $(k+1)$–st letter in f, because the first $(k+1)$ letters in f are descent bottoms in π that are smaller than a. Hence, a is an excedance in τ. If we now remove the letter a from f' and its corresponding descent bottom, the $(k+1)$–st letter of f, from f, then we can repeat the argument, appealing to induction, to show that f' can be constructed in the way described. That is, so that each letter x in f' is an excedance top in τ whose inversion bottom number equals the embracing number of x in π. An analogous argument shows that g' can be constructed as claimed, and so that each letter in g' is not an excedance top in τ. $\square$

In order to prove Theorem 3, we must show that Φ is a bijection. Since

Φ is a map from a finite set to itself, it suffices to show Φ injective, but in the process we will, in fact, construct an inverse to Φ.

We introduce the idea of the *skeleton* of a permutation. This is closely related to the idea of "gravid permutation" introduced by Foata and Zeilberger [7]. We first adjoin to the positive integers a symbol ∞ such that $a < \infty$ for any positive integer a.

Definition 9 *Let n be a positive integer. A* block *is a subset B of $[n] \cup \infty$ such that $B \cap [n] \neq \emptyset$. The block B is called* open *if $\infty \in B$,* closed *if $\infty \notin B$ and* improper *if $|B| = 1$. The* opener*, $\mathrm{O}(B)$, of B is the smallest element of B, the* closer*, $\mathrm{C}(B)$, of B is the largest element of B.*

A skeleton *is a sequence $S = B_1 - B_2 - \ldots - B_r$ of blocks such that any pair of blocks intersect in at most $\{\infty\}$. The skeleton S is* valid *if for each i with $1 \leq i < n$ we have $\mathrm{O}(B_i) < \mathrm{C}(B_{i+1})$.*

Definition 10 *Let $\pi \in \mathcal{S}_n$ be a permutation with descent block decomposition $B_1 - B_2 - \cdots - B_r$. Let $1 \leq a \leq n$. The a-*skeleton *of π is the sequence of blocks obtained by*

- *deleting any descent block B of π for which $\mathrm{O}(B) > a$;*
- *replacing any remaining letter of π that is greater than a by ∞;*
- *replacing each remaining descent block (which is a* sequence *of elements) by its underlying set.*

For example, if $\pi = 3\ 1 - 6 - 7\ 5 - 9\ 8\ 4\ 2$, the 4-skeleton of π is the sequence $\{1, 3\} - \{2, 4, \infty\}$, which we will write as $3\ 1 - \infty\ 4\ 2$.

It is clear that one can recover π from its n-skeleton.

Lemma 6 *Let $\pi \in \mathcal{S}_n$. For any a with $1 \leq a \leq n$, the a-skeleton of π is valid.*

Proof: We use downwards induction on a, knowing that the result is true for the n-skeleton. Suppose it is true for a. To obtain the $(a-1)$-skeleton from the a-skeleton, we merely replace a by ∞ in the block B in which it occurs and delete that block if it now contains only ∞. But as the a-skeleton is valid and a is the largest finite element occuring in it, a block $\{a, \infty\}$ or $\{a\}$

can only occur as the last block in the a-skeleton, in which case its deletion will not cause invalidity. $\square$

The following result is easy to see.

Lemma 7 *The embracing number of the letter a in π equals the number of open blocks to the right of the block containing a in the a-skeleton of π.* $\square$

Proof of Theorem 3: We show that Φ is an injection by showing that, given a permutation τ in the image of Φ, there is at most one permutation π such that $\Phi(\pi) = \tau$. Given such a τ, it is clear what the associated biword $\begin{pmatrix} f & g \\ f' & g' \end{pmatrix}$ must be. Namely, $\begin{pmatrix} f \\ f' \end{pmatrix}$ consists of those columns of $\widetilde{\tau}$ that represent excedances in τ, and $\begin{pmatrix} g \\ g' \end{pmatrix}$ consists of the remaining columns of $\widetilde{\tau}$. Now denote by F, F', G and G' the sets whose elements are the letters of f, f', g and g' respectively. Then we can identify the openers of π as the letters in $F \cap G'$, the closers as the letters in $F' \cap G$, the insiders as those in $F \cap F'$ and the outsiders as those in $G \cap G'$. As we can calculate the side numbers of τ, we know the embracing numbers of π. We will show how to construct successively the 1-, 2-, $\ldots$, n-skeletons of π.

The 1-skeleton of π is either $\{1\}$ or $\{1, \infty\}$, according as 1 is an outsider or an opener. Suppose that the $(a-1)$-skeleton $S = B_1 - B_2 - \ldots - B_r$ of π has been constructed. To construct the a-skeleton of π we must insert a in the correct place in S. Let the embracing number of a in π be e. Let B_i be the $(e+1)$-st open block from the right in S. If a is an insider or a closer in π then, by Lemma 7, a must be inserted into B_i, and if a is a closer then B_i must be closed by the removal of its ∞. Suppose that a is an outsider. Then the improper block $B = \{a\}$ must be inserted immediately to the left of B_i. For if B is inserted to the left of B_{i-1} then either the resulting skeleton will be invalid or Lemma 7 will be violated. Similarly, if a is an opener, the open block $\{a, \infty\}$ must be inserted immediately before B_i.

After the n-skeleton of π has been constructed, we can immediately construct π. Hence there is at most one permutation π such that $\Phi(\pi) = \tau$. Hence, Φ is injective, and so a bijection, and its inverse is defined by the construction just described.

The equidistribution of (des, Dbot, Ddif, Res) and (exc, Ebot, Edif, Ine) now follows from Proposition 2 and the fact that Φ is a bijection. As

$\textsc{inv} = \textsc{env}$, the equidistribution of (des, $\textsc{mak}$, $\textsc{mad}$) with (exc, $\textsc{den}$, $\textsc{inv}$) follows from the definitions of the Mahonian statistics involved, since each is the sum of two of the partial statistics Dbot, Ddif, Res and Ebot, Edif, Ine, respectively. $\square$

Example 2 Let $\tau = 8\ 1\ 4\ 2\ 6\ 9\ 7\ 5\ 3$, so

$$\binom{f}{f'} = \binom{1\ 3\ 5\ 6}{8\ 4\ 6\ 9}, \quad \binom{g}{g'} = \binom{2\ 4\ 7\ 8\ 9}{1\ 2\ 7\ 5\ 3}, \quad \tau' = \binom{1\ 3\ 5\ 6\ 2\ 4\ 7\ 8\ 9}{8\ 4\ 6\ 9\ 1\ 2\ 7\ 5\ 3}.$$

For clarity, we now rewrite τ' with a bar separating f from g and f' from g', and we write the inversion bottom and inversion top numbers in f' and g' respectively as subscripts of their corresponding letters, omitting those that are zero.

$$\tau' = \left(\begin{array}{c|c} f & g \\ f' & g' \end{array}\right) = \left(\begin{array}{cccc|ccccc} 1 & 3 & 5 & 6 & 2 & 4 & 7 & 8 & 9 \\ 8 & 4_1 & 6_1 & 9 & 1 & 2 & 7_2 & 5_1 & 3 \end{array}\right).$$

The sequence of k-skeletons, for $k = 1, 2, \ldots, 9$, of our required permutation π is:

$$\begin{aligned}
&\infty\ 1;\\
&\infty\ 1-2;\\
&\infty\ 1-2-\infty\ 3;\\
&4\ 1-2-\infty\ 3;\\
&4\ 1-2-\infty\ 5-\infty\ 3;\\
&4\ 1-2-\infty\ 6\ 5-\infty\ 3;\\
&4\ 1-2-7-\infty\ 6\ 5-\infty\ 3;\\
&4\ 1-2-7-\infty\ 6\ 5-8\ 3;\\
&4\ 1-2-7-9\ 6\ 5-8\ 3.
\end{aligned}$$

Hence

$$\pi = 4\ 1-2-7-9\ 6\ 5-8\ 3.$$

3 Motzkin paths and a continued fraction expansion

A *Motzkin path*, informally, is a connected sequence of n line segments, or "steps," in the first quadrant of $\mathbf{R}^2$, starting out from the origin in $\mathbf{R}^2$ and ending at $(0, n)$. The steps are of four different types, northeast steps (N) going from (a, b) to $(a+1, b+1)$, southeast (S) going from (a, b) to $(a+1, b-1)$ and solid/dotted east steps (E,dE), from (a, b) to $(a + 1, b)$ (see Figure 1). Formally, a Motzkin path is defined as follows.

Definition 11 *A* Motzkin path *is a word $w = c_1c_2 \cdots c_n$ on the alphabet $\{\mathrm{N}, \mathrm{S}, \mathrm{E}, \mathrm{dE}\}$ such that for each i the* level h_i of the i-th step*, defined by*

$$h_i = \#\{j|j < i, c_j = \mathrm{N}\} - \#\{j|j < i, c_j = \mathrm{S}\},$$

is non-negative, and equal to zero if $i = n$.

Definition 12 *A* weighted Motzkin path of length n *is a pair (c, d), where $c = c_1 \cdots c_n$ is a Motzkin path of length n, and $d = (d_1, \ldots, d_n)$ is a sequence of integers such that*

$$0 \leq d_i \leq \begin{cases} h_i & \text{if } c_i \in \{\mathrm{N}, \mathrm{E}\}, \\ h_i - 1 & \text{if } c_i \in \{\mathrm{S}, \mathrm{dE}\}. \end{cases}$$

The set of weighted Motzkin paths of length n is denoted by Γ_n.

Françon and Viennot [15] gave the first bijection Ψ_{FV} between $\mathcal{S}_n$ and Γ_n. Here we describe one *variant* of this bijection.

Definition 13 *Let $\pi = a_1 \cdots a_n \in \mathcal{S}_n$ and set $a_0 = 0$ and $a_{n+1} = n + 1$. For $1 \leq i \leq n$ we say that a_i is a*

- linear double ascent *(outsider) if $a_{i-1} < a_i < a_{i+1}$;*
- linear double descent *(insider) if $a_{i-1} > a_i > a_{i+1}$;*
- linear peak *(closer) if $a_{i-1} < a_i > a_{i+1}$;*
- linear valley *(opener) if $a_{i-1} > a_i < a_{i+1}$.*

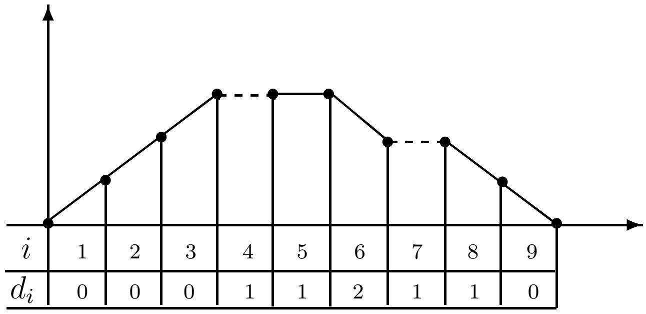

Figure 1

The bijection Ψ_{FV} of Françon and Viennot

Given a permutation $\pi \in \mathcal{S}_n$, determine the right embracing number e_i for each $i \in [n]$. Form the weighted Motzkin path $(c, d) = \Psi_{\mathrm{FV}}(\pi)$ by setting $d_{\pi(i)} = e_i$ and by defining c_i as follows:

- if i is a linear double descent, then $c_i = \mathrm{dE}$;
- if i is a linear double ascent then $c_i = \mathrm{E}$;
- if i is a linear peak then $c_i = \mathrm{S}$;
- if i is a linear valley then $c_i = \mathrm{N}$.

For example, if $\pi = 6\ 1 - 8\ 7\ 4\ 2 - 5 - 9\ 3$, then the corresponding weighted Motzkin path $\Psi_{\mathrm{FV}}(\pi) = (c, d)$ is shown in Figure 1.

The bijection Ψ_{FZ} of Foata and Zeilberger

In [14] Foata and Zeilberger gave another bijection from $\mathcal{S}_n$ to Γ_n, which can be described by the following example.

Start with a permutation π, say, $\pi = 9\ 4\ 7\ 6\ 1\ 2\ 8\ 5\ 3$, so

$$\widetilde{\pi} = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ 9\,4\,7\,6\,1\,2\,8\,5\,3 \end{pmatrix}.$$

As in section 2, separate $\widetilde{\pi}$ into two biwords corresponding to π_E and π_N to get

$$\begin{pmatrix} f \\ f' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 7 \\ 9 & 4 & 7 & 6 & 8 \end{pmatrix}, \quad \begin{pmatrix} g \\ g' \end{pmatrix} = \begin{pmatrix} 5 & 6 & 8 & 9 \\ 1 & 2 & 5 & 3 \end{pmatrix}.$$

Form the weighted Motzkin path $(c, d) = \Psi_{\mathrm{FZ}}(\pi)$ as follows: Let $s_1, s_2, \ldots, s_n$ be the sequence of side numbers of π (see Definition 6) and put

$$d_{\pi(i)} = s_i \text{ for } i = 1, 2, \ldots, n. \tag{10}$$

Let

$$c_i = \begin{cases} \mathrm{dE}, & \text{if } i \in F \cap F', \\ \mathrm{E}, & \text{if } i \in G \cap G', \\ \mathrm{S}, & \text{if } i \in F' \cap G, \\ \mathrm{N}, & \text{if } i \in F \cap G'. \end{cases}$$

Here we have $d = (0,\ 0,\ 0,\ 1,\ 1,\ 2,\ 1,\ 1,\ 0)$ and

$$F \cap F' = \{4,\ 7\}, \quad G \cap G' = \{5\}, \quad F' \cap G = \{6,\ 8,\ 9\}, \quad F \cap G' = \{1,\ 2,\ 3\}.$$

Definition 14 *For $\pi \in \mathcal{S}_n$ and $i \in [n]$, we say that i is a*

- cyclic double ascent *if $\pi^{-1}(i) < i < \pi(i)$;*
- cyclic double descent *if $\pi^{-1}(i) \geq i \geq \pi(i)$;*
- cyclic peak *if $\pi^{-1}(i) < i > \pi(i)$;*
- cyclic valley *if $\pi^{-1}(i) > i < \pi(i)$.*

Note that the four sets $F \cap F'$, $G \cap G'$, $F' \cap G$ and $F \cap G'$ correspond respectively to cyclic double ascents, cyclic double descents, cyclic peaks and cyclic valleys of π. The corresponding weighted Motzkin path is the same as in Figure 1. We note that $\Psi_{\mathrm{FV}} = \Psi_{\mathrm{FZ}} \circ \Phi$. In other words, we have the commutative diagram in Figure 2.

Biane's bijection

In [1], Biane gave a bijection similar to Ψ_{FZ} which we now describe.

Definition 15 *A* labeled path of length n *is a pair (c, ξ), where $c = c_1 \cdots c_n$ is a Motzkin path of length n, and $\xi = (\xi_1, \ldots, \xi_n)$ is a sequence such that*

$$\xi_i \in \begin{cases} \{\Delta\}, & \textit{if } c_i = \mathrm{N}, \\ \{0, \ldots, h_i\}, & \textit{if } c_i = \mathrm{dE} \textit{ or } \mathrm{E}, \\ \{0, \ldots, h_i - 1\}^2, & \textit{if } c_i = \mathrm{S}. \end{cases}$$

$$\begin{tikzcd}
\mathcal{S}_n \arrow[rr, "\Phi"] \arrow[dr, "\Psi_{\mathrm{FV}}"'] & & \mathcal{S}_n \arrow[dl, "\Psi_{\mathrm{FZ}}"] \\
& \Gamma_n &
\end{tikzcd}$$

Figure 2

Biane's bijection is from the labeled paths of length n to $\mathcal{S}_n$. Using the same notation as in the description of Ψ_{FZ}, the inverse of Biane's bijection can be summarized as follows. Let $d_1, d_2, \ldots, d_n$ be the sequence of numbers calculated using equation (10) from the side numbers of π. Note that Biane gave a recursive algorithm to compute these numbers but did not point out that they are actually the side numbers of π, that is the inversion bottom and inversion top numbers in f' and g' respectively. Form the labeled path (c, ξ) thus:

- if $i \in F \cap G'$ (valley), let $c_i = \mathrm{N}$ and $\xi_i = \Delta$;
- if $i \in F \cap F'$ (double ascent), let $c_i = \mathrm{dE}$ and $\xi_i = d_i$;
- if $i \in G \cap G'$ (double descent), let $c_i = \mathrm{E}$ and $\xi_i = d_{\pi(i)}$;
- if $i \in F' \cap G$ (peak), let $c_i = \mathrm{S}$ and $\xi_i = (d_{\pi(i)}, d_i)$.

The path is the same as for Ψ_{FZ}, the only difference being the distribution of the side numbers associated to each step of the path. If we apply Biane's bijection to the permutation π above, we get the labeled path in Figure 3.

In [14], Foata and Zeilberger's purpose with the bijection Ψ_{FZ} was to code the DEN statistic by weighted Motzkin paths, in order to show that (exc, DEN) was equidistributed with (des, MAJ). That Ψ_{FZ} also keeps track of the INV statistic was first remarked by de Médicis and Viennot [20, Proposition 5.2]. They proved that

$$\mathrm{INV}\,\pi = \sum_{i=1}^{n} h_i + \sum_{i=1}^{n} d_i. \tag{11}$$

In Biane's bijection, on the other hand, the INV statistic is seen to equal

$$\mathrm{INV}\,\pi = \sum_{i=1}^{n} (h_i + |\xi_i|),$$

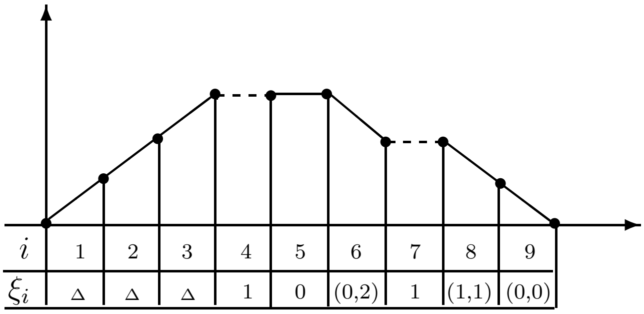

Figure 3

where $|\xi| = x + y$ if $\xi = (x, y)$ and $|\xi| = 0$ if $\xi = \Delta$. This is obviously equivalent to (11).

Using the connections between Motzkin paths and permutations we have described, we now give a continued fraction expansion for the generating function $D_n(x, q) = \sum_{\pi \in \mathcal{S}_n} x^{\operatorname{des} \pi} q^{\operatorname{MAD} \pi}$.

For $n \geq 0$ let $[n]_q = 1 + q + \cdots + q^{n-1}$ and let

$$f_n(x, p, q) = \sum_{\pi \in \mathcal{S}_n} x^{\operatorname{exc} \pi} q^{\operatorname{Edif} \pi} p^{\operatorname{Ine} \pi}.$$

Then, by Theorem 3, we also have

$$f_n(x, p, q) = \sum_{\pi \in \mathcal{S}_n} x^{\operatorname{des} \pi} q^{\operatorname{Ddif} \pi} p^{\operatorname{Res} \pi}.$$

Theorem 8 *The ordinary generating function of $f_n(x, p, q)$ has the following Jacobi continued fraction expansion:*

$$\sum_{n \geq 0} f_n(x, p, q) t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\ddots}{1 - b_n t - \cfrac{\lambda_{n+1} t^2}{\ddots}}}}},$$

where $b_n = q^n(x[n]_p + [n+1]_p)$ and $\lambda_{n+1} = x q^{2n+1}([n+1]_p)^2$ for $n \geq 0$.

Proof: For $\pi \in \mathcal{S}_n$, if $\Psi_{\mathrm{FZ}}(\pi) = (c, d)$, then it is easy to see that

$$\begin{aligned}
\operatorname{exc} \pi &= \sum_{c_i = \mathrm{N}} 1 + \sum_{c_i = \mathrm{dE}} 1, \\
\operatorname{Edif} \pi &= \sum_{c_i = \mathrm{S}} i - \sum_{c_i = \mathrm{N}} i = \sum_{i=1}^{n} h_i, \\
\operatorname{Ine} \pi &= \sum_{i=1}^{n} d_i.
\end{aligned}$$

It follows that

$$\begin{aligned}
f_n(x,p,q) &= \sum_{(c,d)\in\Gamma_n} \prod_{c_i=\mathrm{N}} xq^{h_i}p^{d_i} \prod_{c_i=\mathrm{S}} q^{h_i}p^{d_i} \prod_{c_i=\mathrm{dE}} xq^{h_i}p^{d_i} \prod_{c_i=\mathrm{E}} q^{h_i}p^{d_i} \\
&= \sum_{c\in M_n} \prod_{c_i=\mathrm{N}} xq^{h_i}[h_i+1]_p \prod_{c_i=\mathrm{S}} q^{h_i}[h_i]_p \prod_{c_i=\mathrm{dE}} xq^{h_i}[h_i]_p \prod_{c_i=\mathrm{E}} q^{h_i}[h_i+1]_p,
\end{aligned}$$

where M_n is the set of all Motzkin paths of length n. The theorem then follows by applying a result of Flajolet [8, Theorem 1]. ∎

Using the *contraction* formula

$$\cfrac{c_0}{1-\cfrac{c_1 t}{1-\cfrac{c_2 t}{\ddots}}} = c_0 + \cfrac{c_0 c_1 t}{1-(c_1+c_2)t-\cfrac{c_2 c_3 t^2}{1-(c_3+c_4)t-\cfrac{c_4 c_5 t^2}{\ddots}}}, \tag{12}$$

we immediately get the following Stieltjes continued fraction expansion for the same generating function.

Corollary 9 *We have*

$$\sum_{n\geq 0} f_n(x,p,q)t^n = \cfrac{1}{1-\cfrac{t}{1-\cfrac{xqt}{\cfrac{\ddots}{1-\cfrac{q^{n-1}[n]_p t}{1-\cfrac{xq^n[n]_p t}{\ddots}}}}}}. \tag{13}$$

$k\backslash m$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1															
1		5	4	7	8	10	10	8	4	1						
2			10	12	24	32	41	44	43	36	27	18	10	4	1	
3				10	12	24	32	41	44	43	36	27	18	10	4	1
4					5	4	7	8	10	10	8	4	1			
5						1										

Table 1: $[x^k q^m]D_n(x,q)$ for $n=6$.

In particular, if $D_n(x,q)=\sum_{\pi\in\mathcal{S}_n} x^{\operatorname{des}\pi}q^{\operatorname{MAD}\pi}$*, then it follows from Corollary 9, by putting* $p=q$ *in the above equation, that*

$$\sum_{n\geq 0} D_n(x,q)t^n = \cfrac{1}{1-\cfrac{t}{1-\cfrac{xqt}{\cfrac{\ddots}{1-\cfrac{q^{n-1}[n]_q t}{1-\cfrac{xq^n[n]_q t}{\ddots}}}}}}. \quad \square \tag{14}$$

Note that the continued fraction expansion of the generating function of

$$D_n(x,q)=\sum_{\pi\in\mathcal{S}_n} x^{\operatorname{des}\pi}q^{\operatorname{MAD}\pi}=\sum_{\pi\in\mathcal{S}_n} x^{\operatorname{exc}\pi}q^{\operatorname{INV}\pi}$$

can also be derived from [20, Theorem 6.5].

Corollary 10 *For* $0\leq k\leq n-1$ *and* $0\leq m\leq \frac{n(n-1)}{2}$ *we have*

$$[x^k q^{k+m}]D_n(x,q)=[x^{n-1-k}q^{n-1-k+m}]D_n(x,q), \tag{15}$$

where $[x^k q^m]D_n(x,q)$ *is the coefficient of* $x^k q^m$ *in the polynomial* $D_n(x,q)$*.*

Proof: Let $B_n(x,q)=D_n(xq^{-1},q)$. Then (15) is equivalent to

$$[x^k q^m]B_n(x,q)=[x^{n-1-k}q^m]B_n(x,q). \tag{16}$$

From (14) and (12) we derive

$$\sum_{n\geq 1} B_n(x,q)t^n = \cfrac{t}{1-(c_1+c_2)t-\cfrac{c_2c_3t^2}{1-(c_3+c_4)t-\cfrac{c_4c_5t^2}{\ddots}}}, \tag{17}$$

where $c_{2n-1} = q^{n-1}[n]_q$ and $c_{2n} = xq^{n-1}[n]_q$ for $n \geq 0$. Replacing x by $1/x$ and t by xt in (17) we get

$$\sum_{n\geq 1} x^{n-1}B_n(x^{-1},q)t^n = \cfrac{t}{1-(c_1+c_2)t-\cfrac{c_2c_3t^2}{1-(c_3+c_4)t-\cfrac{c_4c_5t^2}{\ddots}}}. \tag{18}$$

Comparing (17) and (18) then yields $B_n(x,q) = x^{n-1}B_n(x^{-1},q)$, which is clearly equivalent to (16). $\square$

To illustrate the above corollary, we give in Table 1 the number of permutations corresponding to the values of (des, MAD) when $n = 6$ and for clarity we omit writing zeros.

4 Generalizations on our statistics

4.1 Left and right variations

Two new statistics MADL and MAKL are defined for a permutation π as follows.

Definition 16

$$\begin{aligned} \text{MADL}\,\pi &= \text{Ddif}\,\pi + \text{Les}\,\pi. \\ \text{MAKL}\,\pi &= \text{Dbot}\,\pi + \text{Les}\,\pi. \end{aligned}$$

Recall that $\text{Les}\,\pi$ is the sum of the left embracing numbers of π, defined by replacing "right" by "left" in the definition of $\text{Res}\,\pi$. (See Definition 2.)

The relationship between these statistics and MAD and MAK is based on the following result, for the proof of which we refer the reader to [4].

Proposition 11 *There is an involution ϵ on $\mathcal{S}_n$ such that, for each $\pi \in \mathcal{S}_n$,*

$$(\mathrm{des}, \mathrm{Dbot}, \mathrm{Dtop}, \mathrm{Res}, \mathrm{Les})\ \pi = (\mathrm{des}, \mathrm{Dbot}, \mathrm{Dtop}, \mathrm{Les}, \mathrm{Res})\ \epsilon(\pi).$$

In particular,

$$\begin{aligned} \textsc{mad}\,\pi &= \textsc{madl}\,\epsilon(\pi), \\ \textsc{mak}\,\pi &= \textsc{makl}\,\epsilon(\pi). \end{aligned}$$

The involution ϵ can be informally described as follows. Let π be a permutation with descent block decompostion $B_1 - B_2 - \cdots - B_k$. Write the descent blocks of π down in reverse order to give a permutation $\pi' = B_k - B_{k-1} - \cdots - B_1$. Now this may not be the descent block decomposition of π', as new descents may have been introduced between adjacent descent blocks. If $\textsc{o}(B_2) > \textsc{c}(B_1)$, that is, if a new descent has been introduced between B_2 and B_1, then move block B_2 to the right of B_1 to get $B_k - B_{k-1} - \cdots - B_1 - B_2$, otherwise leave block B_2 where it is. Now consider block B_3. If there is a descent between B_3 and the block to its right, move B_3 past that block. Continue moving B_3 to the right until there is no descent between it and the block to its right. Continuing in this way we arrive at the permutation $\epsilon(\pi)$.

Example 3 If $\pi = 3\ 1 - 5\ 4\ 2 - 7\ 6$ with descent blocks $B_1 - B_2 - B_3$ then $\pi' = B_3 - B_2 - B_1 = 7\ 6 - 5\ 4\ 2 - 3\ 1$ and, as there is a descent between B_3 and B_2, we get successively $5\ 4\ 2 - 7\ 6 - 3\ 1$ and $\epsilon(\pi) = 5\ 4\ 2 - 3\ 1 - 7\ 6$.

It follows from Proposition 11 that the triple (des, $\textsc{madl}$, $\textsc{makl}$) is equidistributed with the triple (des, $\textsc{mad}$, $\textsc{mak}$) on $\mathcal{S}_n$.

4.2 Extensions to words

The two Mahonian statistics $\textsc{inv}$ and $\textsc{den}$ have already been generalized to words [19, 17]. Indeed, all of the results in this paper except those in section 3 can be nicely extended to the case of words with repeated letters. (Although we can *code* words with repeated letters to weighted Motzkin paths, we cannot (yet) use this coding to obtain results analogous to Theorem 8.) The definitions of our partial statistics Res, Dbot, etc., become more complicated, but no essential difficulties ensue. This theory will be presented in [4].

4.3 Large and small letters

Various authors, for example [3, 18, 27], have considered statistics on words and permutations in the context of an alphabet $\mathcal{A} = [m]$ in which the letters are divided into two classes, *large* and *small*. Namely, for some k with $0 \le k \le m$ and for $\ell = m - k$, the letters $1, 2, \ldots, \ell$ are designated small and the letters $\ell+1, \ldots, m$ are designated large. Then, for a word $w = a_1 a_2 \cdots a_m$, a *k-descent* is an integer i such that one of the following conditions holds:

- $1 \le i < m$ and $a_i > a_{i+1}$;
- $1 \le i < m$ and $a_i = a_{i+1} > \ell$;
- $i = m$ and $a_i > \ell$.

Then $\operatorname{des}_k w$ equals the number of k-descents of w. One can similarly define k-extensions of the other Eulerian and Mahonian statistics. The results of the present paper can all be k-extended, as will be presented by the present authors in a subsequent note [5].

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