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Hook length property of d-complete posets via q-integrals

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Abstract. The hook length formula for d-complete posets states that the P-partition generating function for them is given by a product in terms of hook lengths. We give a new proof of the hook length formula of d-complete posets using q-integrals. Proctor proved that any connected d-complete poset can be uniquely decomposed into irreducible d-complete posets and classified all irreducible d-complete posets. In this work, we prove the hook length property of all the irreducible d-complete posets. The proof is done by a case-by-case analysis consisting of two steps. First, we express the P-partition generating function for each case as a q-integral and then we evaluate the q-integrals.

Keywords: Hook length formula, d-complete poset, P-partition, q-integral

1 Introduction

The classical hook length formula due to Frame, Robinson and Thrall [2] states that for a partition λ of n, the number f^λ of standard Young tableaux of shape λ is given by

$$f^\lambda = \frac{n!}{\prod_{x\in\lambda} h(x)},$$

where $h(x)$ is the hook length of the cell x in λ. One can naturally consider the shape λ as a poset P on the cells in λ. Then the P-partition generating function for the poset also has the following hook length formula:

$$\sum_{\sigma:P\to\mathbb{N}} q^{|\sigma|} = \prod_{x\in P} \frac{1}{1-q^{h(x)}},$$

where the sum is over all P-partitions σ. It is also well known that the P-partition generating functions for the posets coming from shifted shapes and forests satisfy the hook length property.

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Proctor [10] introduced d-complete posets, which include the posets of shapes, shifted shapes and forests, and with Peterson's help, he [8] proved that the d-complete posets have the hook length property:

Theorem 1.1 (Hook Length Formula for d-complete posets)**.** *For any d-complete poset P, we have*

$$\sum_{\sigma:P\to\mathbb{N}} q^{|\sigma|} = \prod_{x\in P} \frac{1}{1-q^{h(x)}},$$

where the sum is over all P-partitions σ.

We note that Theorem 1.1 was also proved by Nakada [7] and generalized by Ishikawa and Tagawa [4, 3] to "leaf posets". However, their proofs are only sketched in conference proceedings, and so a completely detailed proof of the hook length formula (Theorem 1.1) has not been available in the literature. In this work, we provide a new and complete proof of Theorem 1.1 using q-integrals. This is an extended abstract of [6].

2 Preliminaries

2.1 Basic definitions and notation

We will use the following notation for q-series:

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad (a_1,a_2,\dots,a_k;q)_n = (a_1;q)_n\cdots(a_k;q)_n.$$

Let δ_n denote the staircase partition $(n-1,n-2,\dots,1,0)$. For a partition $\lambda=(\lambda_1,\dots,\lambda_n)$, the *alternant* $a_\lambda(x_1,\dots,x_n)$ is defined by

$$a_\lambda(x_1,\dots,x_n) = \det(x_i^{\lambda_j})_{i,j=1}^n.$$

Given a partition λ, the *Young diagram* of λ is the left-justified array of squares in which there are λ_i squares in the ith row from the top and the *Young poset* of λ is the poset whose elements are the squares in the Young diagram of λ with relation $x\le y$ if x is weakly below and weakly to the right of y.

If λ has no nonzero identical parts, λ is called *strict*. For a strict partition λ, the *shifted Young diagram* of λ is the diagram obtained from the Young diagram of λ by shifting the ith row to the right by $i-1$ units. The *shifted Young poset* of λ is defined similarly. If there is no confusion, we identify a partition λ with its Young diagram and also with its Young poset. For a strict partition λ, the shifted Young diagram of λ is denoted by λ^*. Similarly, the shifted Young poset of λ will also be written as λ^*.

For a Young diagram or a shifted Young diagram λ, a *semistandard Young tableau* of shape λ is a filling of λ with nonnegative integers such that the integers are weakly increasing in each row and strictly increasing in each column. A *reverse plane partition* of shape λ is a filling of λ with nonnegative integers such that the integers are weakly increasing in each row and each column. We denote by $\mathrm{SSYT}(\lambda)$ and $\mathrm{RPP}(\lambda)$ the set of semistandard Young tableaux of shape λ and the set of reverse plane partitions of shape λ, respectively.

Let λ be a strict partition. For $T \in \mathrm{SSYT}(\lambda^*)$ or $T \in \mathrm{RPP}(\lambda^*)$, the leftmost entry in each row is called a *diagonal entry*. We define the *reverse diagonal sequence* $\mathrm{rdiag}(T)$ to be the sequence of diagonal entries in the non-increasing order.

Now we recall basic properties of P-partitions. Let P be a poset with n elements. A *P-partition* is a map $\sigma : P \to \mathbb{N}$ such that $x \le_P y$ implies $\sigma(x) \ge \sigma(y)$. In other words, a P-partition is just an order-reversing map from P to $\mathbb{N}$.

For an integer $m \ge 0$, we denote by $\mathcal{P}_{\ge m}(P)$ the set of all P-partitions σ with $\min(\sigma) \ge m$. We also define $\mathcal{P}(P) = \mathcal{P}_{\ge 0}(P)$. For a P-partition σ, the *size* $|\sigma|$ of σ is defined by

$$|\sigma| = \sum_{x \in P} \sigma(x).$$

For a poset P, we define $\mathrm{GF}_q(P)$ to be the P-partition generating function:

$$\mathrm{GF}_q(P) = \sum_{\sigma \in \mathcal{P}(P)} q^{|\sigma|}.$$

The following definitions allow us to build d-complete posets starting from a chain.

Definition 2.1. *Let P be a poset containing a chain $C = \{x_1 < x_2 < \cdots < x_n\}$. For $\lambda \in \mathrm{Par}_n$, we denote by $D(P, C, \lambda)$ the poset obtained by taking the disjoint union of P and $(\lambda + \delta_{n+1})^*$ and identifying $x_n, x_{n-1}, \ldots, x_1$ with the diagonal elements of $(\lambda + \delta_{n+1})^*$.*

Definition 2.2. *Let n and k be positive integers. Let*

$$X = \{(\lambda^{(i)}, n_i, s_i) : 1 \le i \le k\},$$

where n_i and s_i are positive integers with $s_i + n_i - 1 \le n$, $\lambda^{(i)} \in \mathrm{Par}_{n_i}$. We define $P_n(X)$ to be the poset constructed as follows. Let P_0 be a chain $x_1 < x_2 < \cdots < x_n$ with n elements, called diagonal entries*. For $1 \le i \le k$, we define $P_i = D(P_{i-1}, C_i, \lambda^{(i)})$ where $C_i = \{x_{s_i} < x_{s_i+1} < \cdots < x_{s_i+n_i-1}\}$. Finally we define $P_n(X) = P_k$. We also define $P_n^m(X)$ to be the poset obtained from $P_n(X)$ by attaching a chain with m elements above x_n. We say that an element $y \in P_n(X)$ is* of level i *if $y \le x_i$ and $y \not\le x_{i-1}$.*

Here, we do not provide the detailed definition of d-complete posets. In this paper, we basically follow the set up of [9].

2.2 Some properties of P-partitions

For a poset P, let P^+ be the poset obtained from P by adding a new element which is greater than all elements in P. If P has a unique maximal element, we define P^- to be the poset obtained from P by removing the maximal element. Note that $(P^+)^- = P$ for any poset P. If P has a unique maximal element, $(P^-)^+ = P$. There is a simple relation between $\mathrm{GF}_q(P^+)$ and $\mathrm{GF}_q(P)$.

Lemma 2.3. *For a poset P with p elements, we have*

$$\mathrm{GF}_q(P^+) = \frac{1}{1-q^{p+1}} \mathrm{GF}_q(P).$$

Let P be a poset in which there is a unique maximal element y_1 and a specified element y_2 covered by y_1. For integers $m, k \geq 1$, we define $D_{m,k}(P)$ to be the poset obtained from P by adding a disjoint chain $z_m > \cdots > z_1 > z_0 > z_{-1} > \cdots > z_{-k}$ and a new element y_0 with additional covering relations $z_1 > y_0, z_0 > y_1, z_{-1} > y_2$ and $y_0 > y_1$. See Figure 1. We also define $D_k(P)$ to be the poset obtained from $D_{m,k}(P)$ by removing the elements $z_m, \ldots, z_1$ and y_0.

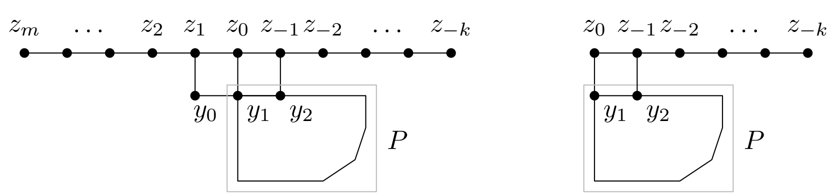

Figure 1: The posets $D_{m,k}(P)$ on the left and $D_k(P)$ on the right.

Then the following lemma enables us to decompose the P-partition generating function of d-complete posets.

Lemma 2.4. *Let $P = \{y_1, y_2, \ldots, y_p\}$ be a poset in which y_1 is the unique maximal element and y_2 is covered by y_1. Then*

$$\mathrm{GF}_q(D_{m,k}(P)) = \frac{1}{(q^{p+k+1};q)_{m+2}} \left(\frac{q^{p+1}}{(q;q)_{k-1}} \mathrm{GF}_q(P^+) + (1-q^{2p+2k+2}) \mathrm{GF}_q(D_k(P)) \right).$$

2.3 Semi-irreducible d-complete posets

Definition 2.5. *A d-complete poset P is* semi-irreducible *if it is obtained from an irreducible d-complete poset by attaching a chain with arbitrary number of elements (possibly 0) below each acyclic element.*

The semi-irreducibility is a slight generalization of the irreducibility defined by Proctor [9].

Lemma 2.6. *Let P_0 be an irreducible d-complete poset with k acyclic elements $y_1, \dots, y_k$. Suppose that $P_1, \dots, P_k$ are (possibly empty) connected d-complete posets having the hook length property. Let P be the poset obtained from P_0 by attaching P_i below y_i for each $1 \le i \le k$, i.e.,*

$$P = (\cdots ({P_0}^{y_1} \backslash_{v_1} P_1)^{y_2} \backslash_{v_2} P_2) \cdots {}^{y_k} \backslash_{v_k} P_k),$$

where v_i is the unique maximal element of P_i. Then P also has the hook length property.

This lemma tells us that it suffices to prove the hook length property of the semi-irreducible posets to prove that every d-complete poset has the hook length property. Hence we prove:

Theorem 2.7. *Every semi-irreducible d-complete poset has the hook length property.*

3 q-integrals

In this section we express the P-partition generating function for $P_n(X)$ as a q-integral, where $P_n(X)$ is the poset defined in Definition 2.2.

The *q-integral* of a function $f(x)$ over $[a, b]$ is defined by

$$\int_a^b f(x) d_q x = (1-q) \sum_{i=0}^{\infty} \left(f(bq^i) bq^i - f(aq^i) aq^i \right),$$

where it is assumed that $0 < q < 1$ and the sum absolutely converges.

For a multivariable function $f(x_1, \dots, x_n)$ and a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we denote $f(q^\lambda) = f(q^{\lambda_1}, \dots, q^{\lambda_n})$.

We define the multivariate q-integral over the simplex $\{(x_1, \dots, x_n) : 0 \le x_1 \le \cdots \le x_n \le 1\}$ by

$$\int_{0 \le x_1 \le \cdots \le x_n \le 1} f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n = \int_0^1 \int_0^{x_n} \int_0^{x_{n-1}} \cdots \int_0^{x_2} f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n.$$

Lemma 3.1. *We have*

$$\int_{0 \le x_1 \le \cdots \le x_n \le 1} f(x_1, \dots, x_n) d_q x_1 \cdots d_q x_n = (1-q)^n \sum_{\mu \in \mathrm{Par}_n} q^{|\mu|} f(q^\mu),$$

where Par_n *denotes the set of partitions of length at most n.*

Note that every semi-irreducible d-complete poset can be written as $P_n^m(X)$, by its construction in Definition 2.5. By Lemma 2.3, we have

$$\mathrm{GF}_q(P_n^m(X)) = \frac{1}{(q^{|P_n(X)|+1}; q)_m} \mathrm{GF}_q(P_n(X)).$$

We introduce some lemmas which allow us to write $\mathrm{GF}_q(P_n(X))$ as a q-integral.

Lemma 3.2. *Let n and k be positive integers and*

$$X = \{(\lambda^{(i)}, n_i, s_i) : 1 \le i \le k\},$$

where n_i and s_i are positive integers with $s_i + n_i - 1 \le n$ and $\lambda^{(i)} \in \mathrm{Par}_{n_i}$. For $\mu = (\mu_1, \dots, \mu_n) \in \mathrm{Par}_n$, let $\mu^{[i]} = (\mu_{s_i}, \mu_{s_i+1}, \dots, \mu_{s_i+n_i-1})$. Then we have

$$\mathrm{GF}_q(P_n(X)) = q^{-\sum_{i=1}^n (n-i)\ell_i} \sum_{\substack{\mu\in\mathrm{Par}_n \\ \mu:\text{strict}}} q^{|\mu|} \prod_{i=1}^{k} \sum_{\substack{T\in\mathrm{SSYT}((\delta_{n_i+1}+\lambda^{(i)})^*) \\ \mathrm{rdiag}(T)=\mu^{[i]}}} q^{|T|-|\mu^{[i]}|},$$

where ℓ_i is the number of elements of level i in $P_n(X)$.

Lemma 3.3. *For $\lambda, \mu \in \mathrm{Par}_n$ we have*

$$\sum_{\substack{T\in\mathrm{SSYT}((\delta_{n+1}+\lambda)^*) \\ \mathrm{rdiag}(T)=\mu}} q^{|T|-|\mu|} = \frac{(-1)^{\binom{n}{2}} a_{\lambda+\delta_n}(q^\mu)}{\prod_{j=1}^n (q;q)_{\lambda_j+n-j}}.$$

The following result is the key ingredient to express $\mathrm{GF}_q(P_n(X))$ as a q-integral.

Theorem 3.4. *Let n and k be positive integers and*

$$X = \{(\lambda^{(i)}, n_i, s_i) : 1 \le i \le k\},$$

where n_i and s_i are positive integers with $s_i + n_i - 1 \le n$, $\lambda^{(i)}$ is a partition with n_i parts. Suppose that for every $1 \le j \le n-1$, there is $1 \le i \le k-1$ such that $s_i \le j < j+1 \le s_i + n_i - 1$. Then

$$\mathrm{GF}_q(P_n(X)) = \frac{q^{-\sum_{i=1}^n (n-i)\ell_i}}{(1-q)^n} \int_{0\le x_1\le\cdots\le x_n\le 1} \prod_{i=1}^{k} \frac{(-1)^{\binom{n_i}{2}} a_{\lambda^{(i)}+\delta_{n_i}}(x_{s_i}, x_{s_i+1}, \dots, x_{s_i+n_i-1})}{\prod_{j=1}^{n_i} (q;q)_{\lambda_j^{(i)}+n_i-j}} d_q x_1 \cdots d_q x_n,$$

where ℓ_i is the number of elements of level i in $P_n(X)$.

4 Evaluation of the q-integrals

In [9, Table 1], Proctor classified all irreducible d-complete posets in 15 classes, and in [6] slightly generalized posets have been considered, namely, semi-irreducible d-complete posets. In a nutshell, the computation of q-integrals corresponding to the P-partition generating functions can be summarized as follows.

Classes	Diagnosis
1,2	shape and shifted shape; proofs are known
3,5,6,7,8′,9,13,14,15	finite type; can be verified by Sage [1]
8-(4), 10, 12	finite type; but modification is necessary to verify by Sage
4, 11	infinite type; proof is done by using partial fraction identities

In [6], the class 8 is divided into 4 subclasses and 8′ in the above table includes 8-(1), 8-(2) and 8-(3). Note that *finite (infinite, resp.) type* means that there are finite (infinite, resp.) number of integration variables in the q-integral.

Here, we demonstrate the computation of one class in each category.

4.1 Class 2: Shifted shapes

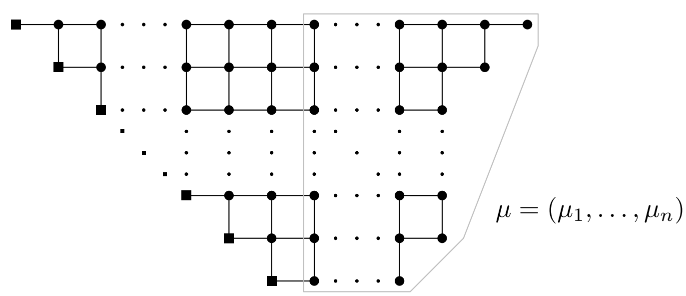

Figure 2: A semi-irreducible d-complete poset of class 2. This is irreducible if and only if $\mu_1 = \mu_2$.

A semi-irreducible d-complete poset of class 2 is $P_n(X_2)$, where $n \geq 4$ and $X_2 = \{(\mu, n, 1)\}$, with $\mu \in \mathrm{Par}_n$. For $1 \leq i \leq n$, we have $\ell_i = \mu_{n+1-i} + i$. By Theorem 3.4,

$$\mathrm{GF}_q(P_n(X_2)) = \frac{q^{-\binom{n+1}{3} - \sum_{i=1}^{n-1} i\mu_{i+1}}}{(1-q)^n} \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} \frac{a_{\mu+\delta_n}(x_1, \ldots, x_n)}{\prod_{i=1}^{n} (q;q)_{\mu_i+n-i}} d_q x_1 \cdots d_q x_n.$$

The hook length property for class 2 is equivalent to

$$\int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} a_{\mu+\delta_n}(x_1, \ldots, x_n) d_q x_1 \cdots d_q x_n$$
$$= q^{\binom{n+1}{3} + \sum_{i=1}^{n-1} i\mu_{i+1}} (1-q)^n \frac{\prod_{1 \leq i < j \leq n} (1 - q^{\mu_i - \mu_j + j - i})}{\prod_{1 \leq i \leq j \leq n} (1 - q^{2n+1-i-j+\mu_i+\mu_{j+1}})},$$

which is proved in [5, Theorem 8.16] using the connection between reverse plane partitions and q-integrals.

4.2 Class 5: Tailed insets

A semi-irreducible d-complete poset of class 5 is $P_3^{\lambda_1+1}(X_5)$ for $\lambda \in \mathrm{Par}_2$, $\mu \in \mathrm{Par}_3$ and

$$X_5 = \{(\lambda,2,1),(\mu,3,1),(\emptyset,2,2),((1),1,1)\},$$

with $\ell_1 = \lambda_2+\mu_3+2$, $\ell_2 = \lambda_1+\mu_2+3$ and $\ell_3 = \mu_1+4$.

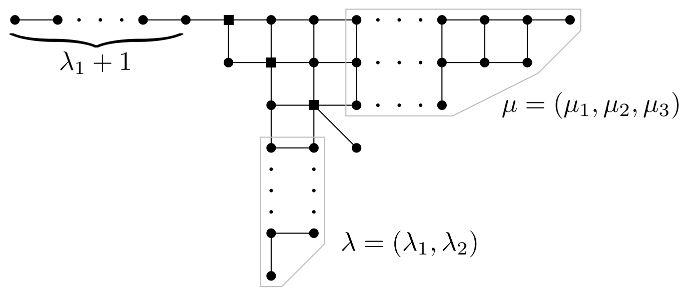

Figure 3: A semi-irreducible d-complete poset of class 5. This is irreducible if and only if $\mu_1 = \mu_2$.

By Lemma 2.3,

$$\mathrm{GF}_q(P_3^{\lambda_1+1}(X_5)) = \frac{1}{(q^{|\lambda|+|\mu|+10};q)_{\lambda_1+1}}\mathrm{GF}_q(P_3(X_5)),$$

where

$$\mathrm{GF}_q(P_3(X_5)) = \frac{q^{-(\sum_{i=1}^2 i(\lambda_i+\mu_{i+1})+7)}}{(1-q)^3}\int_{0\le x_1\le x_2\le x_3\le 1}\frac{-a_{\lambda+\delta_2}(x_1,x_2)}{(q;q)_{\lambda_1+1}(q;q)_{\lambda_2}} \times \frac{-a_{\mu+\delta_3}(x_1,x_2,x_3)}{\prod_{j=1}^3(q;q)_{\mu_j+3-j}}\cdot\frac{-a_{\delta_2}(x_2,x_3)}{1-q}\cdot\frac{a_{(1)+\delta_1}(x_1)}{1-q}d_qx_1d_qx_2d_qx_3.$$

Then the hook length property for class 5 is equivalent to the following identity

$$\int_{0\le x_1\le x_2\le x_3\le 1} x_1 a_{\delta_2}(x_2,x_3)a_{\lambda+\delta_2}(x_1,x_2)a_{\mu+\delta_3}(x_1,x_2,x_3)d_qx_1d_qx_2d_qx_3$$
$$= \frac{(-1)q^{\sum_{i=1}^2 i(\lambda_i+\mu_{i+1})+7}(1-q)^4(1-q^{\lambda_1-\lambda_2+1})(1-q^{|\lambda|+|\mu|+\lambda_1+10})(1-q^{|\lambda|+|\mu|+\lambda_2+9})}{1-q^{|\lambda|+|\mu|+9}}.$$
$$\times\frac{\prod_{1\le i<j\le 3}(1-q^{\mu_i-\mu_j+j-i})}{\prod_{i=1}^2\prod_{j=1}^3(1-q^{\lambda_i+\mu_j+7-i-j})\prod_{i=1}^3(1-q^{|\lambda|+|\mu|-\mu_i+4+i})}.$$

This formula has been verified by Sage[1].

4.3 Class 10: Tagged Swivels

A semi-irreducible d-complete poset of class 10 is $P_6^{\lambda_1+4}(X_{10})$ with

$$X_{10} = \{(\lambda, 5, 1), (\emptyset, 2, 1), ((1), 2, 2), (\emptyset, 2, 3), (\emptyset, 3, 4), (\emptyset, 2, 5)\},$$

where $\lambda \in \mathrm{Par}_5$ and $\ell_1 = \lambda_5 + 1$, $\ell_2 = \lambda_4 + 3$, $\ell_3 = \lambda_3 + 5$, $\ell_4 = \lambda_2 + 5$, $\ell_5 = \lambda_1 + 6$, $\ell_6 = 4$.

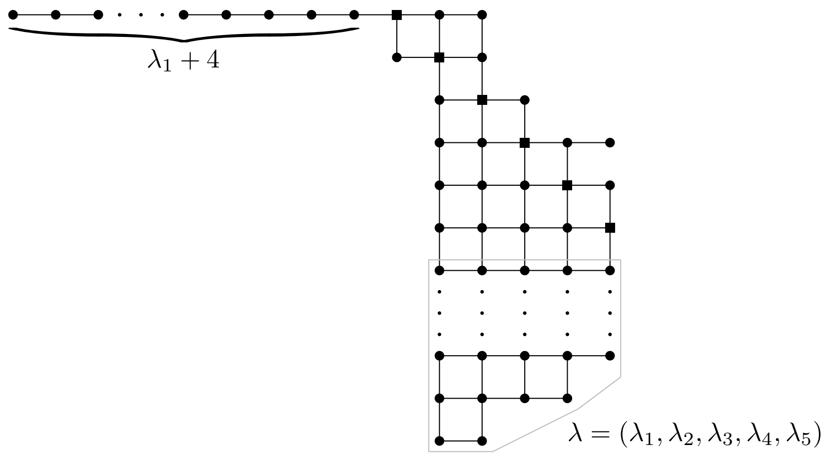

Figure 4: A semi-irreducible d-complete poset of class 10. This poset is always irreducible.

To evaluate the q-integral, for the sake of the simplicity of the computation, we decompose the poset $P_6^{\lambda_1+4}(X_{10})$ using Lemma 2.4.

Let $Q = P_5(X)^-$ for $X = \{(\mu, 5, 1), ((1), 1, 2)\}$ and $\mu = \lambda + (1^5)$. The poset $P_6^{\lambda_1+4}(X_{10})$ can be also expressed as $D_{\mu_1+4,1}(Q)$ and, by Lemma 2.4, the P-partition generating function satisfies the relation

$$\mathrm{GF}_q(D_{\mu_1+4,1}(Q)) = \frac{1}{(q^{|\mu|+17}; q)_{\mu_1+6}} (q^{|\mu|+16}\,\mathrm{GF}_q(Q^+) + (1 - q^{2|\mu|+34})\,\mathrm{GF}_q(D_1(Q))).$$

Note that $Q^+ = P_5(X)$ and $D_1(Q) = P_5(X')$ where $X' = \{(\mu, 5, 1), ((1), 1, 2), (\emptyset, 2, 4)\}$. By Theorem 3.4,

$$\mathrm{GF}_q(Q^+) = \frac{q^{-\sum_{i=1}^5 (i-1)\mu_i - 23}}{(1-q)^6 \prod_{i=1}^5 (q;q)_{\mu_i+5-i}} \int_{0 \le x_1 \le \cdots \le x_5 \le 1} x_2 a_{\mu+\delta_5}(x_1, \ldots, x_5) d_q x_1 \cdots d_q x_5$$

and

$$\mathrm{GF}_q(D_1(Q)) = \frac{(-1)q^{-\sum_{i=1}^5 (i-1)\mu_i - 23}}{(1-q)^7 \prod_{i=1}^5 (q;q)_{\mu_i+5-i}} \int_{0 \le x_1 \le \cdots \le x_5 \le 1} x_2(x_4 - x_5) a_{\mu+\delta_5}(x_1, \ldots, x_5) d_q x_1 \cdots d_q x_5.$$

The above q-integrals with 4 variables can be explicitly computed by computer and the hook lengths of the elements in $P_6^{\lambda_1+4}(X_{10}) = D_{\mu_1+4,1}(Q)$ can be explicitly computed. By combining the aforementioned observations, we obtain that the hook length property for class 10 is equivalent to

$$\prod_{i=1}^{5} \frac{1-q^{|\mu|+\mu_i-i+23}}{(1-q^{\mu_i+6-i})(1-q^{|\mu|-\mu_i+10+i})} \prod_{1\le i<j\le 5} \frac{1-q^{\mu_i-\mu_j+j-i}}{1-q^{\mu_i+\mu_j-i-j+13}}$$
$$= \sum_{\ell=1}^{5} \frac{(-1)^{5-\ell} q^{-\sum_{i=1}^{5}(i-1)\mu_i-23}}{(1-q)^5(1-q^{|\mu|+16})} \left((1-q^{2|\mu|+23}) g(\widehat{\mu}^{(\ell)},0) - (q^{|\mu|+16};q)_2 \cdot g(\widehat{\mu}^{(\ell)},1) \right),$$

where

$$g(\nu,m) := \int_{0\le x_1\le\cdots\le x_4\le 1} x_2 x_4^m a_{\mu+\delta_4}(x_1,x_2,x_3,x_4) d_q x_1 \cdots d_q x_4$$
$$= \frac{q^{12+\sum_{i=1}^{4} i\mu_{i+1}}(1-q)^4(1-q^{|\mu|+12}) \prod_{1\le i<j\le 4}(1-q^{\mu_i-\mu_j+j-i})}{(1-q^{|\mu|+11+m}) \prod_{1\le i<j\le 4}(1-q^{\mu_i+\mu_j+11-i-j}) \prod_{i=1}^{4}(1-q^{\mu_i+5-i})},$$

for $\nu \in \mathrm{Par}_5$ and an integer $m \ge 0$. We have verified this identity by computer.

4.4 Class 4: Insets

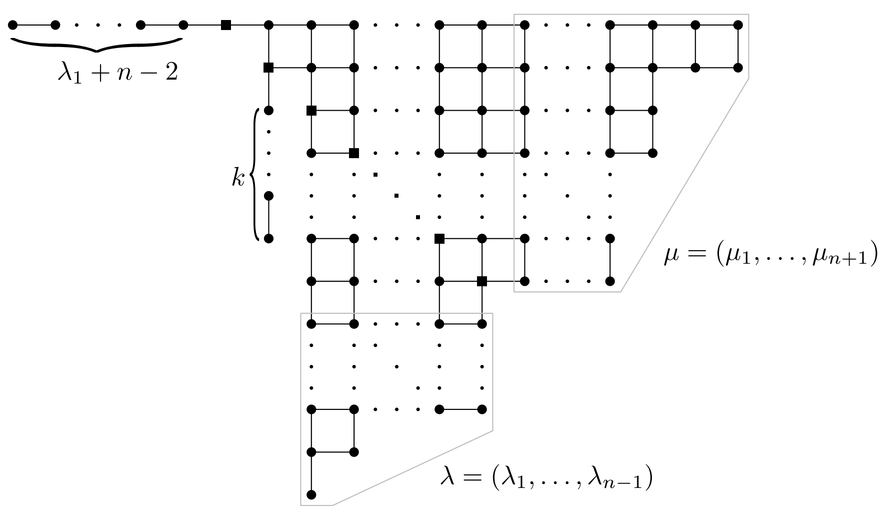

Figure 5: A semi-irreducible d-complete poset of class 4. This is irreducible if and only if $k = 0$ and $\mu_1 = \mu_2$.

A semi-irreducible d-complete poset of class 4 is $P_{n+1}^m(X_4)$, where $n \ge 2$, $k \ge 0$ and

$$X_4 = \{(\lambda, n-1, 1), (\mu, n+1, 1), ((k), 1, n)\},$$

for $\lambda \in \mathrm{Par}_{n-1}$ and $\mu \in \mathrm{Par}_{n+1}$. In this poset, $\ell_j = \lambda_{n-j} + \mu_{n-j+2} + 2j - 1$ for $1 \leq j \leq n-1$, $\ell_n = \mu_2 + n + k$ and $\ell_{n+1} = \mu_1 + n + 1$.

By applying Lemma 2.3 and Theorem 3.4, we obtain

$$\mathrm{GF}_q(P_{n+1}^{\lambda_1+n-2}(X_4)) = \frac{1}{(q^{|\lambda|+|\mu|+n^2+k+3};q)_{\lambda_1+n-2}} \mathrm{GF}_q(P_{n+1}(X_4)),$$

where

$$\mathrm{GF}_q(P_{n+1}(X_4)) = \frac{q^{-(\sum_{i=1}^{n}((i+1)\lambda_i+i\mu_{i+1})+\frac{1}{6}n(n-1)(2n+5)+1+k)}}{(1-q)^{n+1}} \int_{0\leq x_1\leq\cdots\leq x_{n+1}\leq 1} \frac{a_{(k)}(x_n)}{(q;q)_k}$$
$$\times \frac{(-1)^{\binom{n-1}{2}} a_{\lambda+\delta_{n-1}}(x_1,\dots,x_{n-1})}{\prod_{i=1}^{n-1}(q;q)_{\lambda_i+n-1-i}} \cdot \frac{(-1)^{\binom{n+1}{2}} a_{\mu+\delta_{n+1}}(x_1,\dots,x_{n+1})}{\prod_{i=1}^{n+1}(q;q)_{\mu_i+n+1-i}} d_qx_1\cdots d_qx_{n+1}.$$

Taking the explicit hook lengths of the elements in the poset $P_{n+1}^{\lambda_1+n-2}(X_4)$ into consideration, the hook length property for class 4 can be written as the following identity

$$\int_{0\leq x_1\leq\cdots\leq x_{n+1}\leq 1} x_n^k a_{\lambda+\delta_{n-1}}(x_1,\dots,x_{n-1}) a_{\mu+\delta_{n+1}}(x_1,\dots,x_{n+1}) d_qx_1\cdots d_qx_{n+1}$$
$$= \frac{(-1)q^{\sum_{i=1}^{n}((i+1)\lambda_i+i\mu_{i+1})+\frac{1}{6}n(n-1)(2n+5)+1+k}(1-q)^{n+1}}{\prod_{i=1}^{n+1}(1-q^{|\lambda|+|\mu|-\mu_i+n(n-1)+k+i})} \cdot \frac{\prod_{j=1}^{n-1}(1-q^{|\lambda|+|\mu|+\lambda_j+n^2+n-j+k+1})}{1-q^{|\lambda|+|\mu|+n^2+k+2}}$$
$$\times \frac{\prod_{1\leq i<j\leq n+1}(1-q^{\mu_i-\mu_j+j-i})\prod_{1\leq i<j\leq n-1}(1-q^{\lambda_i-\lambda_j+j-i})}{\prod_{\substack{1\leq i\leq n+1\\1\leq j\leq n-1}}(1-q^{\mu_i+\lambda_j+2n-i-j+1})},$$

or

$$\frac{\prod_{j=1}^{n-1}(1-q^{|\lambda|+|\mu|+\lambda_j+n^2+n-j+k+1})}{\prod_{i=1}^{n+1}(1-q^{|\lambda|+|\mu|-\mu_i+n^2-n+k+i})}$$
$$= \sum_{\ell=1}^{n+1} \frac{q^{-|\lambda|-|\mu|+\mu_\ell-n^2+n-k-\ell}}{1-q^{|\lambda|+|\mu|-\mu_\ell+n^2-n+k+\ell}} \cdot \frac{\prod_{j=1}^{n-1}(1-q^{\mu_\ell+\lambda_j+2n-\ell-j+1})}{\prod_{j=1,j\neq\ell}^{n+1}(1-q^{\mu_\ell-\mu_j+j-\ell})}.$$

This identity can be proved by applying a partial fraction expansion identity [11, p. 451]

$$\frac{\prod_{j=1}^{n+1}(1-b_j/t)}{\prod_{j=1}^{n}(1-a_j/t)} = \sum_{\ell=1}^{n} \frac{\prod_{j=1}^{n+1}(1-a_\ell/b_j)}{(1-a_\ell/t)\prod_{j=1,j\neq\ell}^{n}(1-a_\ell/a_j)}, \quad \text{for } b_1\cdots b_{n+1} = a_1\cdots a_n t,$$

by making appropriate substitutions for a_i's, b_i's and t. We omit the details.

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