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A formula for birational rowmotion on rectangles

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Abstract. We give a formula in terms of families of non-intersecting lattice paths for iterated actions of the *birational rowmotion* map on a product of two chains, equivalently a rectangle. This allows us to give a much simpler direct proof of the key fact that the period of this map on a product of chains of lengths r and s is $r+s+2$ (first proved by D. Grinberg and the second author) as well as other consequences, as explained in [8].

Keywords: birational rowmotion, dynamical algebraic combinatorics, periodicity

1 Introduction

Birational rowmotion is an action on the space of assignments of rational functions to the elements of a finite partially-ordered set (poset). It is lifted from the well-studied *rowmotion* map on order ideals (equivariantly on antichains) of a poset P [1, 2, 9, 10, 12, 15], which when iterated on special posets has unexpected nice properties in terms of periodicity, cyclic sieving, and homomesy. Rowmotion is first lifted to a piecewise-linear action of the *order polytope* [13] of P, i.e., $\{f : P \to [0,1] : f \text{ is order preserving}\}$. This is then detropicalized (a.k.a. "geometricized") to a birational map, as detailed in [3, 4]. Theorems proven at the birational level generally imply their corresponding theorems at the piecewise-linear, and then the combinatorial level, but not vice-versa. For example, the only proof available as of this writing that piecewise-linear rowmotion is periodic uses the corresponding result for birational rowmotion.

Rowmotion appears to behave particularly nicely for posets associated with representations of finite-dimensional Lie algebras, e.g., root posets and minuscule posets. Armstrong, Stump, and Thomas proved a conjecture of Panyushev [9, Conjecture 2.1(iii)] that antichain-cardinality is a homomesic with respect to rowmotion action on antichains of root posets, i.e., for every orbit $\mathcal{O}$, the value $\frac{1}{\#\mathcal{O}} \sum_{A\in\mathcal{O}} \#A$ is a constant, independent of $\mathcal{O}$. This was one of the first explicit statements of a type later isolated by Propp and the second author as the **homomesy phenomenon** [10].

Iterating birational rowmotion also displays interesting and unexpected dynamics for certain "nice" posets, particularly rectangular ones (i.e., $P = [0,r] \times [0,s]$, the product

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of two finite chains). The order of birational rowmotion is (unexpectedly) the same is that for ordinary rowmotion, namely $r + s + 2$, a fact whose first proof [7, Theorem 30] was nontrivial. Here ordinary rowmotion appears to be related to Auslander–Reiten translation on certain quivers [16], and birational rowmotion to Y-systems of type $A_m \times A_n$ described in Zamolodchikov periodicity [11, Section 4.4] and to the recent notion of R-*systems* [6]. The orbits of these actions have natural *homomesic* statistics [10, 3, 4]. In [8], we show an instance of birational file homomesy, generalizing a combinatorial result of [10].

Proofs of periodicity and homomesy tend to be rather indirect, and often involve finding an equivariant bijection between rowmotion and an action that is easier to understand, or at least already better understood. For birational rowmotion, Grinberg and the second author parameterize poset labelings by ratios of determinants, then show periodicity via certain Plücker relations (overcoming a number of technical hurdles) [7]. The formula via non-intersecting lattice paths in this paper gives a more direct proof.

The paper is organized as follows. In Section 2 we give basic definitions, state our main result (Theorem 2.5), and give an extended illustrative example; we also prove as corollaries the main results of [7], namely the periodicity and reciprocity of birational rowmotion. In Section 3, we outline the proof of our main theorem, leaving details for the arXiv preprint [8]. Further results, i.e., birational file homomesy, also appear therein.

The initial impetus for this paper came when the authors were both at a workshop on "Dynamical Algebraic Combinatorics" hosted by the American Institute of Mathematics (AIM) in March 2015. We are grateful for AIM's hospitality that week, as well as for helpful conversations with Max Glick, Darij Grinberg, Christian Krattenthaler, James Propp, Pasha Pylyavskyy, and Richard Stanley. Computations of birational rowmotion programmed by Grinberg in SageMath [14] were of enormous assistance in helping us discover the main formula.

2 Definitions and main result

Definition 2.1. *Let P be any finite poset, and let $\widehat{P}$ be P with an additional global maximum (denoted $\widehat{1}$) and an additional global minimum (denoted $\widehat{0}$) adjoined. Let $\mathbb{K}$ be any field, and $f \in \mathbb{K}^{\widehat{P}}$ be any $\mathbb{K}$-labeling of $\widehat{P}$. We define the* ***birational toggle*** $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ *at $v \in P$ by*

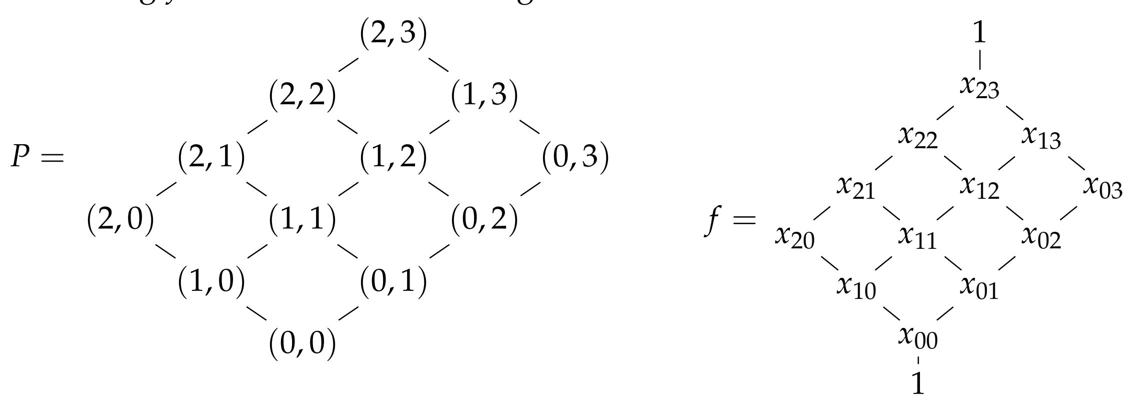
$$(T_v f)(y) = f(y) \quad \text{if} \quad y \neq v; \quad (T_v f)(y) = \frac{1}{f(v)} \cdot \frac{\sum\limits_{\substack{w \in \widehat{P};\\ w \lessdot v}} f(w)}{\sum\limits_{\substack{z \in \widehat{P};\\ z \gtrdot v}} \frac{1}{f(z)}} \quad \text{if} \quad y = v;$$

for all $y \in \widehat{P}$. Note that this rational map T_v is well-defined, because the right-hand side is well-defined on a Zariski-dense open subset of $\mathbb{K}^{\widehat{P}}$.

The toggle map T_v changes only the label of the poset at v, and does this by (a) *inverting* the label at v, (b) multiplying by the *sum* of the labels at vertices *covered by* v, and (c) multiplying by the *parallel sum* of the labels at vertices *covering* v. Because toggles in the tropical category involve both the operations of min and max, one of these is lifted to $+$ and the other to the (associative) parallel sum operation $\maltese$, defined by $a \maltese b := \frac{1}{\frac{1}{a}+\frac{1}{b}}$ (when $a, b \neq 0$ and $a \neq -b$). **Birational Rowmotion** ρ_B is then defined as $\rho_B := T_{v_1} T_{v_2} \dots T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ where $v_1, v_2, \dots, v_n$ is any linear extension of P, i.e., by togging at each element of P from top to bottom, as for ordinary rowmotion.

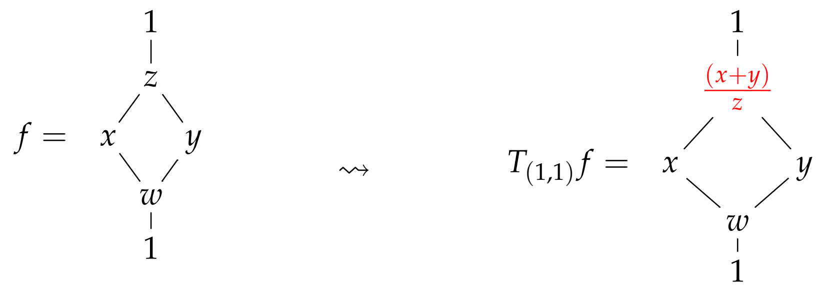
The main result of our paper is a formula in terms of families of non-intersecting lattice paths for the kth iteration, ρ_B^k of birational rowmotion on the product of two chains. For our purposes, it is more convenient to initially coordinatize our poset P as $[0, r] \times [0, s]$ (where $[0, n] = \{0, 1, 2, \dots, n\}$), with minimal element $(0, 0)$, maximal element (r, s) and covering relations: $(i, j) \lessdot (k, \ell)$ if and only if (1) $i = k$ and $\ell = j + 1$ or (2) $j = \ell$ and $k = i + 1$. We then initially assign the generic label x_{ij} (a.k.a. $x_{i,j}$) to the element (i, j), and the label 1 to the elements $\widehat{0}$ and $\widehat{1}$. No essential generality is lost by assigning 1 to the elements of $\widehat{P} - P$ (a "reduced labeling") [7, Section 4] or [3, Section 4], but it simplifies our formulae and figures, which will generally just display the labelings of P itself, not of $\widehat{P}$.

Example 2.2. The Hasse diagram of $P = [0, 2] \times [0, 3]$ is shown on the left, and the generic initial labeling f of $\widehat{P}$ is shown on the right.

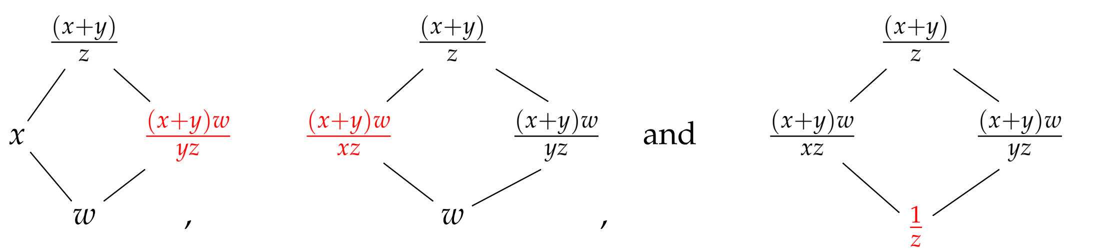


Example 2.3. Consider the 4-element poset $P := \{0, 1\} \times \{0, 1\}$, i.e., the product of two chains of length one, with the subscript-avoiding labeling shown below. Then f and the

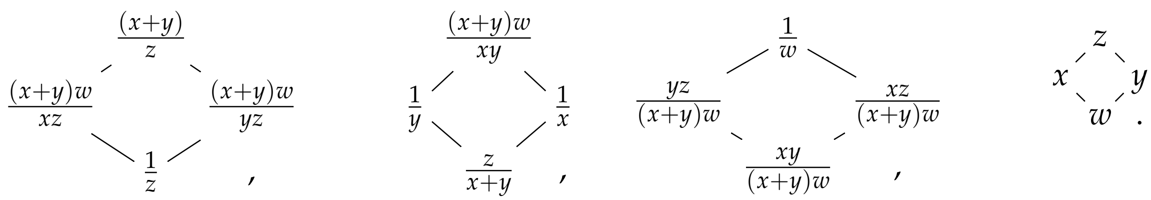
output of toggling f at the top element $(1,1)$ of P are as follows:



Since the labels at $\widehat{0}$ and $\widehat{1}$ never vary, we suppress displaying them in all future examples of birational rowmotion. (They are still involved in the computations.) Computing successively $T_{(0,1)}T_{(1,1)}f$, then $T_{(1,0)}T_{(0,1)}T_{(1,1)}f$, and $\rho_B f = T_{(0,0)}T_{(1,0)}T_{(0,1)}T_{(1,1)}f$ gives:



By repeating this procedure (or just substituting the labels of $\rho_B f$ obtained as variables), we can compute the iterated maps ρ_B, $\rho_B^2 f$, $\rho_B^3 f, \ldots$ obtaining



Notice that $\rho_B^4 f = f$, which generalizes to $\rho_B^{r+s+2} f = f$ for $P = [0,r] \times [0,s]$ (Corollary 2.8). More subtly, as one iterates ρ_B, the labels at certain poset elements are reciprocals of others occuring earlier at the antipodal position in the poset P. (See Corollary 2.9.)

2.1 Our Main Result: A lattice-path formula for birational rowmotion

A simple change of variables in the initial labeling greatly facilitates our ability to write our formula. Let

$$A_{ij} := \frac{\sum_{z \lessdot (i,j)} x_z}{x_{(i,j)}} = \frac{x_{i,j-1} + x_{i-1,j}}{x_{ij}}, \tag{2.1}$$

where we set $A_{i0} = \frac{x_{i-1,0}}{x_{i,0}}$, $A_{0j} = \frac{x_{0,j-1}}{x_{0,j}}$ and $A_{00} = \frac{1}{x_{00}}$ (working in $\widehat{P}$).

We define a **lattice path of length k** within $P = [0,r] \times [0,s]$ to be a sequence $v_1, v_2, \ldots, v_k$ of elements of P such that each difference of successive elements $v_i - v_{i-1}$ is either $(1,0)$ or $(0,1)$ for each $i \in [k]$. We call a collection of lattice paths **non-intersecting** if no two of them share a common vertex. We will frequently abbreviate **non-intersecting lattice paths** as **NILP**.

Definition 2.4. Given a triple $(k,m,n) \in \mathbb{N}^3$ (where $\mathbb{N}$ denotes the nonnegative integers $\{0,1,2,\ldots\}$) with $k \le \min\{r-m, s-n\}+1$, we define a polynomial $\varphi_k(m,n)$ in terms of the A_{ij}'s as follows:

1) Let $\bigvee_{(m,n)} := \{(u,v) : (u,v) \ge (m,n)\}$ be the *principal order filter at* (m,n) in P, which is isomorphic to $[r-m] \times [s-n]$. Set $\hexagon^k_{(m,n)} := \{(u,v) \in \bigvee_{(m,n)} : m+n+k-1 \le u+v \le r+s-k+1\}$, the *rank-selected subposet* consisting of all elements in $\bigvee_{(m,n)}$ whose rank (within $\bigvee_{(m,n)}$) is at least $(k-1)$ and whose corank is at least $(k-1)$.

2) More specifically, let $s_1, s_2, \ldots, s_k$ be the k minimal elements and $t_1, t_2, \ldots, t_k$ be the k maximal elements of $\hexagon^k_{(m,n)}$, i.e., $s_\ell = (m+k-\ell, n+\ell-1)$ and $t_\ell = (r-\ell+1, s-k+\ell)$ for $\ell \in [k]$. (When $k=0$, there are no s_ℓ's or t_ℓ's.) Our condition that $k \le \min\{r-m, s-n\}+1$ insures that these points all lie within $\hexagon^k_{(m,n)}$.

3) Let $S_k(m,n)$ be the set of families of NILPs in $\hexagon^k_{(m,n)}$ from $\{s_1, s_2, \ldots, s_k\}$ to $\{t_1, t_2, \ldots, t_k\}$. We let $\mathcal{L} = \{L_1, L_2, \ldots, L_k\} \in S_k(m,n)$ denote such a family.

4) Define

$$\varphi_k(m,n) := \sum_{\mathcal{L} \in S_k(m,n)} \prod_{\substack{(i,j) \in \hexagon^k_{(m,n)} \\ (i,j) \notin L_1 \cup L_2 \cup \cdots \cup L_k}} A_{ij}. \tag{2.2}$$

5) Finally, set $[\alpha]_+ := \max\{\alpha, 0\}$ and let $\mu^{(a,b)}$ be the transformation that takes a rational function in $\{A_{u,v}\}$ and simply shifts each index in each factor of each term: $A_{u,v} \mapsto A_{u-a,v-b}$.

We are now ready to state our main result.

Theorem 2.5. *Fix $k \in [0, r+s+1]$, and let $\rho_B^{k+1}(i,j)$ denote the rational function in $\mathbb{K}[x_{u,v}]$ associated to the poset element (i,j) after $(k+1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0,r] \times [0,s]$. Set $M = [k-i]_+ + [k-j]_+$. We obtain the following formula for $\rho_B^{k+1}(i,j)$:*

***(a)** When $M \le k$:*

$$\rho_B^{k+1}(i,j) = \mu^{([k-j]_+, [k-i]_+)} \left(\frac{\varphi_{k-M}(i-k+M, j-k+M)}{\varphi_{k-M+1}(i-k+M, j-k+M)} \right) \tag{2.3}$$

where $\varphi_t(v,w)$ and $\mu^{(a,b)}$ are as defined in 4) and 5) of Definition 2.4.

***(b)** When $M \geq k$:*

$$\rho_B^{k+1}(i,j) = 1/\rho_B^{k-i-j}(r-i, s-j)$$

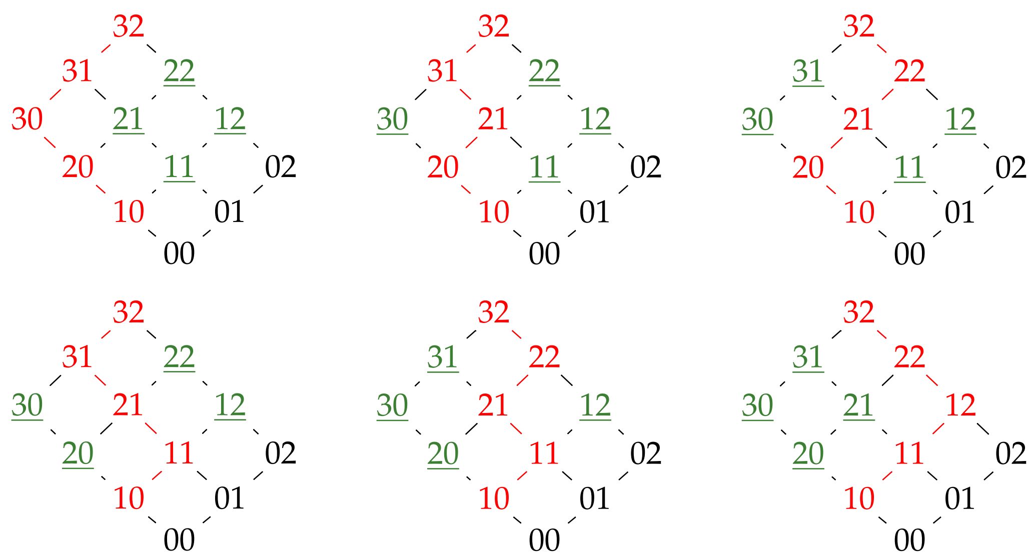
where $\rho_B^{k-i-j}(r-i, s-j)$ is well-defined by part (a).

Note that these formulas agree when $M = k$. Since on $P = [0,r] \times [0,s]$ we have $\rho_B^{r+s+2+d} = \rho_B^d$ by periodicity (Corollary 2.8), this gives a formula for **all** iterations of the birational rowmotion map on P. In the "generic" case where shifting $(i,j) \mapsto (i-k, j-k)$ (straight down by $2k$ ranks) still gives a point in P, we get the following simpler formula.

Corollary 2.6.

$$\text{For } k \leq \min\{i,j\},\ \rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)} \tag{2.4}$$

Figure 1: The six lattice paths (shown in red) involved in computing $\varphi_1(1,0)$ in $[0,3] \times [0,2]$. Corresponding A-variable subscripts are underlined in green.

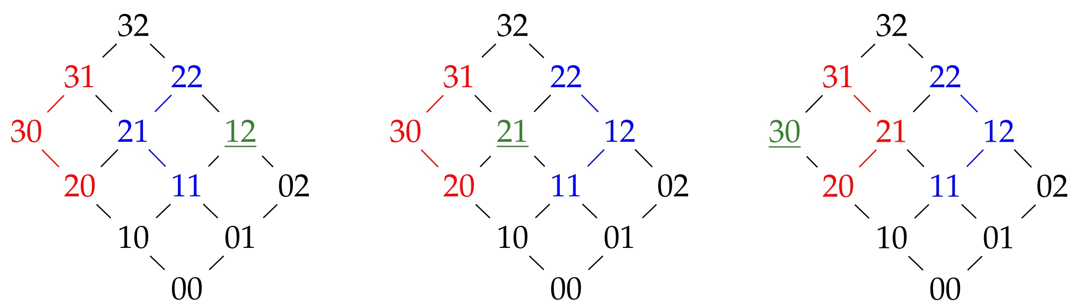


Example 2.7. We use Theorem 2.5 to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for $k = 0, 1, 2, \ldots, 6$. Periodicity kicks in for $k \geq 6$. Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.

- **When $\mathbf{k = 0}$**, $M = 0$ and we get $\rho_B^1(2,1) = \dfrac{\varphi_0(2,1)}{\varphi_1(2,1)} = \dfrac{A_{21}A_{22}A_{31}A_{32}}{A_{22}+A_{31}}$. In general $\varphi_0(i,j) = \prod_{(m,n)\geq(i,j)} A_{m,n}$, the product of all the A-variables in the order filter $\bigvee_{(i,j)}$.
- **When $\mathbf{k = 1}$**, we still have $M = 0$, and $\rho_B^2(2,1) = \dfrac{\varphi_1(1,0)}{\varphi_2(1,0)} =$

$$\frac{A_{11}A_{12}A_{21}A_{22}+A_{11}A_{12}A_{22}A_{30}+A_{11}A_{12}A_{30}A_{31}+A_{12}A_{20}A_{22}A_{30}+A_{12}A_{20}A_{30}A_{31}+A_{20}A_{21}A_{30}A_{31}}{A_{12}+A_{21}+A_{30}}.$$

Figure 2: The three pairs of lattice paths (shown in red and blue) used for computing $\varphi_2(1,0)$ in $[0,3]\times[0,2]$. A-variable subscripts are circled in green.



For the numerator, $s_1=(1,0)$, $t_1=(3,2)$, and there are six lattice paths from s_1 to t_1, each of which covers 5 elements and leaves 4 uncovered (Figure 1). For the denominator, $s_1=(2,0)$, $s_2=(1,1)$, $t_1=(3,1)$, and $t_2=(2,2)$, and each pair of lattice paths leaves exactly one element uncovered (Figure 2).

• **When k = 2**, we get $M=[2-2]_+ + [2-1]_+ = 1 \le 2 = k$. So by part (a) of the main theorem we have

$$\rho_B^3(2,1)=\mu^{(1,0)}\left[\frac{\varphi_1(1,0)}{\varphi_2(1,0)}\right]=(\text{just shifting indices in the } k=1 \text{ formula})$$

$$\frac{A_{01}A_{02}A_{11}A_{12}+A_{01}A_{02}A_{12}A_{20}+A_{01}A_{02}A_{20}A_{21}+A_{02}A_{10}A_{12}A_{20}+A_{02}A_{10}A_{20}A_{21}+A_{10}A_{11}A_{20}A_{21}}{A_{02}+A_{11}+A_{20}}.$$

• **When k = 3**, we get $M=[3-2]_+ + [3-1]_+ = 3 = k$. Therefore,

$$\rho_B^4(2,1)=\mu^{(2,1)}\left[\frac{\varphi_0(2,1)}{\varphi_1(2,1)}\right]=\mu^{(2,1)}\left[\frac{A_{21}A_{22}A_{31}A_{32}}{A_{22}+A_{31}}\right]=\frac{A_{00}A_{01}A_{10}A_{11}}{A_{01}+A_{10}}.$$

We can also use part (b) of the main theorem to get $\rho_B^4(2,1)=1/\rho_B^{3-2-1}(3-2,2-1)=1/\rho_B^0(1,1)=\dfrac{1}{x_{11}}$. The equality between these two expressions is easily checked.

• **When k = 4**, we get $M=[4-2]_+ + [4-1]_+ = 5 > k$. Therefore, by part (b) of the main theorem, then part (a),

$$\rho_B^5(2,1)=1/\rho_B^{4-2-1}(3-2,2-1)=1/\rho_B^1(1,1)=\frac{\varphi_1(1,1)}{\varphi_0(1,1)}=\frac{A_{12}A_{22}+A_{12}A_{31}+A_{21}A_{31}}{A_{11}A_{12}A_{21}A_{22}A_{31}A_{32}}.$$

- **When k = 5**, we get $M = [5-2]_+ + [5-1]_+ = 7 > k$. Therefore,

$$\rho_B^6(2,1) = 1/\rho_B^{5-2-1}(1,1) = 1/\rho_B^2(1,1) = \frac{\varphi_2(0,0)}{\varphi_1(0,0)} = \frac{A_{02}A_{12}+A_{02}A_{21}+A_{11}A_{21}+A_{30}A_{02}+A_{30}A_{11}+A_{30}A_{20}}{\text{A sum of 10 degree-6 monomials in } A_{ij}}.$$

- **When k = 6**, we get $M = [6-2]_+ + [6-1]_+ = 9 > k$. Therefore,

$$\rho_B^7(2,1) = 1/\rho_B^{6-2-1}(3-2,2-1) = 1/\rho_B^3(1,1) = \mu^{(1,1)}\left[\frac{\varphi_1(1,1)}{\varphi_0(1,1)}\right]$$

$$= \mu^{(1,1)}\left[\frac{A_{12}A_{22}+A_{12}A_{31}+A_{21}A_{31}}{A_{11}A_{11}A_{21}A_{22}A_{31}A_{32}}\right] = \frac{A_{01}A_{11}+A_{01}A_{20}+A_{10}A_{20}}{A_{00}A_{01}A_{10}A_{11}A_{20}A_{21}} = x_{21}$$

Notice that periodicity also kicks in for this case and $\rho_B^7(2,1) = \rho_B^0(2,1) = x_{21}$.

As corollaries of Theorem 2.5, we get the main two theorems on birational rowmotion on a product of two chains (by applying part (b) once or twice).

Corollary 2.8 ([7, Theorem 30])**.** *The birational rowmotion map ρ_B on the product of two chains $P = [0,r] \times [0,s]$ is periodic, with period $r+s+2$.*

Corollary 2.9 ([7, Theorem 32])**.** *The birational rowmotion map ρ_B on the product of two chains $P = [0,r] \times [0,s]$ satisfies the following reciprocity: $\rho_B^{i+j+1} = 1/\rho_B^0(r-i,s-j) = \frac{1}{x_{r-i,s-j}}$.*

3 Proof of Main Theorem

There is only space here to outline the main ideas of the proof of Theorem 2.5, which is a complicated triple induction on i, j, and k. We start with $k=0$ and work our way down the poset from the upper boundary, then repeat with $k=1$, etc. The key ingredients are a Plücker-like recurrence relation for the φ_k's, Lemma 3.1, and a generalization thereof (omitted herein) that includes the shifting $\mu^{(i,j)}$'s, but whose proof can be reduced to it. We prove Lemma 3.1 using a colorful combinatorial bijection on pairs of k-tuples of NILPs, which we later learned is similar to those in [5] for Schur functions identities.

Lemma 3.1. *For $1 \leq k \leq \min\{i,j\}$ we have the Plücker-like relation*

$$\begin{aligned}\varphi_k(i-k,j-k)\varphi_{k-1}(i-k+1,j-k+1) = \varphi_k(i-k+1,j-k)\varphi_{k-1}(i-k,j-k+1)\\ + \varphi_k(i-k,j-k+1)\varphi_{k-1}(i-k+1,j-k).\end{aligned} \tag{3.1}$$

Proof. The definition of φ_k (Equation (2.2)) involves summing monomials in the A_{ij}'s, with each term corresponding to the elements left uncovered by a k-tuple of NILPs. So a term on the left-hand side of the Lemma is represented by a pair of NILPs $(\mathcal{B}, \mathcal{R})$ offset from one another by one rank. Example 3.2 gives an example to illustrate both this and the bijection below. Specifically, the lower NILPs $\mathcal{B}$, whose endpoints are marked with $\circ$, represents a monomial from $\varphi_k(i-k, j-k)$, and the upper NILPs $\mathcal{R}$, whose endpoints are marked by $\times$, represents one from $\varphi_{k-1}(i-k+1, j-k+1)$. Our goal is to transform this pair into a pair of NILPs counted by one of the terms on the right-hand-side of Lemma 3.1.

Starting at each bottom $\circ$ (lowest points in $\mathcal{B}$), we create two **bounce paths** and $(k-2)$ **twigs** as follows. From the leftmost $\circ$ on the bottom, move up blue edges until encountering a vertex with a downward red edge. Then move down red edges until encountering a vertex with an upward blue edge. Continue in this way, reversing directions whenever possible and only traversing unused edges, until a terminal vertex is reached. (No such path can terminate at an internal vertex, since any edge by which one enters must be paired with a possible exit.) Do the same procedure starting from the rightmost $\circ$ on the bottom. We refer to both of these paths as **bounce paths**.

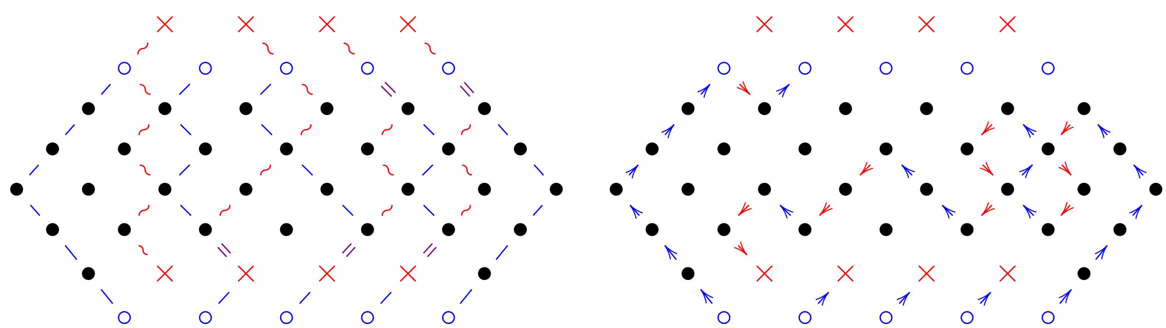
Since we reverse directions along bounce paths in a systematic way, we always follow blue edges upward and red edges downward. In addition to these two bounce paths, the $(k-2)$ $\circ$'s in the interior of the bottom immediately connect to a $\times$ in the rank second from the bottom. We refer to these blue edges as **twigs**. Since the twigs cover all but 1 of the $(k-2)$ $\times$'s, only one of the two bounce paths may return to the bottom of the poset ending with a segment of downward red edges. Furthermore the $(k-1)$ red paths, starting from the $\times$'s at the top, intersect $(k-1)$ of the k topmost $\circ$'s, leaving only 1 $\circ$ untouched. Notice It follows that one of these two bounce paths ends at the top of the interval, at the $\circ$ on top untouched by the red paths, and the other bounce path ends at the bottom of the interval, at the $\times$ on the bottom not covered by a twig. We call the former a **vertical bounce path** and the latter a **horizontal bounce path**.

We proceed by interchanging the colors of the edges along the horizontal bounce path, along all the twigs, and swap the $\times$ and $\circ$ endpoints at the bottom, while leaving the remaining edges of $\mathcal{B} \cup \mathcal{R}$ unchanged (also leaving the colors of the vertical bounce path unchanged). We then truncate the vertical bounce path by deleting the bottommost edge. These transformations result in a new pair of lattice path families which we denote as $(\mathcal{B}', \mathcal{R}')$. The bottom endpoints for $\mathcal{B}'$ will be one step either to the northeast or northwest of the original ones, indicating respectively whether it is contributing to the first or second summand on the right-hand-side of Lemma 3.1. The bottom endpoints of $\mathcal{R}'$ are skewed in the other direction, i.e., the southwest or the southeast, respectively.

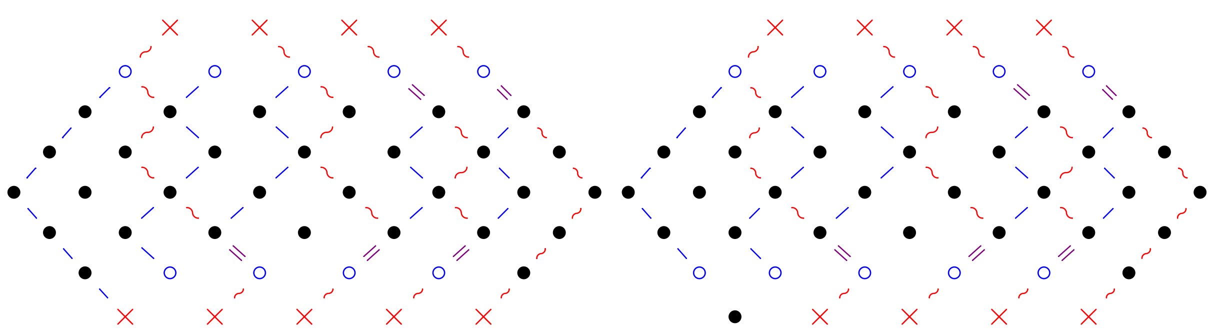
Furthermore, if lattice paths $L_B \in \mathcal{B}$ and $L_R \in \mathcal{R}$ did not originally intersect, then their edges would not lie along any bounce path. Consequently, L_B would be a lattice path again in $\mathcal{B}'$ unchanged, and the same is true for L_R in $\mathcal{R}'$. They would again not intersect. On the other hand, if L_B and L_R did originally intersect, then they could meet along a bounce path. Being part of larger NILPs, L_B would not intersect any path in $\mathcal{B}$ and L_R would not intersect any path in $\mathcal{R}$. Swapping colors of individual edges along L_B and L_R might break this intersection-free property, but since all colors of edges along a horizontal bounce path are swapped simultaneously, we ensure that each collection of paths, $\mathcal{B}'$ and $\mathcal{R}'$, is still intersection-free.

Hence, the result is a new pair of NILPs $(\mathcal{B}', \mathcal{R}')$ with the lower endpoints of $\mathcal{B}'$ on the second rank from the bottom of the interval skewed left (resp. right) while the lower endpoints of $\mathcal{R}'$ are on the bottom rank of the interval and skewed right (resp. left). By construction, this map is well-defined, and $\mathcal{B}'$ is a collection of k lattice paths from $\circ$'s to $\circ$'s, and $\mathcal{R}'$ is a collection of $(k-1)$ lattice paths from $\times$'s to $\times$'s. Thus the new pair represents a pair of monomials "counted" by $\varphi_k(i-k, j-k+1)\varphi_{k-1}(i-k+1, j-k)$ in the former case, and "counted" by $\varphi_k(i-k+1, j-k)\varphi_{k-1}(i-k, j-k+1)$ in the latter. Finally this procedure is clearly reversible, giving the desired bijection. ∎

Example 3.2. Let $k = 5$, and consider the following pair of tuples of lattice paths, $(\mathcal{B}, \mathcal{R})$ (shown below left in (blue, red) in $\hexagon^5_{(i-5,j-5)} \cup \hexagon^4_{(i-4,j-4)}$, with $r - i = s - j = 2$. (Due to limitations of our drawing package, red edges are drawn as squiggles and violet double edges represent one edge of each color.)



We create bounce paths and twigs as shown above on the right. Notice the leftmost bounce path is vertical, i.e., it ends at the top, so its colors remain the same. We interchange the colors along the twigs and the rightmost bounce path, which is horizontal. We then fill in the original edges (with their original colors) and swap $\times$ and $\circ$ at the bottom.



The last step simply shortens the vertical bounce path by one edge, replacing $\times \mapsto \bullet$ with $\bullet \mapsto \circ$. The result is a new pair of NILPs $(\mathcal{B}', \mathcal{R}')$. The lower endpoints of $\mathcal{B}'$ are now "skewed left", representing a monomial in $\varphi_5(i-4, j-5)$, while those of $\mathcal{R}'$ are "skewed right", representing a monomial in $\varphi_4(i-5, j-4)$.

Just as Lemma 3.1 is used to prove Theorem 2.5 (a), covering the cases when $0 \le k \le M$, we reduce the proof of Theorem 2.5 (b), i.e., $M \le k \le r+s+1$, to the following claim.

Claim 3.3. *Under the hypotheses of Theorem 2.5, if $M = k$ (i.e., $i + j = k$) then*

$$\rho_B^{k+1}(i,j) = \mu^{(i,j)}\left(\frac{\varphi_0(i,j)}{\varphi_1(i,j)}\right) = \mu^{(i,j)}\rho_B^1(i,j) = \frac{1}{x_{r-i,s-j}}.$$

Proof. The first two equalities follow from Theorem 2.5 (a), while we prove the last equality as follows. Since the principal order filter $\bigvee_{(i,j)}$ is isomorphic to the product of chains $[0, r-i] \times [0, r-j]$, we easily reduce the claim to the case $i = j = k = 0$, i.e.,

$$\rho_B^1(0,0) = \frac{\varphi_0(0,0)}{\varphi_1(0,0)} = \frac{\prod_{p=0}^{r}\prod_{q=0}^{s} A_{pq}}{\sum_{\mathcal{L}\in S_1(0,0)} \prod_{\substack{(i,j)\in \varhexagon^1_{(0,0)} \\ (i,j)\notin L_1}} A_{ij}.} = \frac{1}{x_{r,s}}.$$

In this situation, our family of lattice paths reduces to a single lattice path L_1, the numerator can be thought of as $\prod\limits_{(p,q)\in P} A_{pq}$, and $\varhexagon^1_{(0,0)} = P$ as well. By clearing denominators and dividing through by the double-product) we equivalently need to show $\sum_{\mathcal{L}\in S_1(0,0)} \prod_{(i,j)\in L_1} A_{ij}^{-1} = x_{r,s}$. For the base case $s = 0$, we get P is a chain of length r and the only lattice path consists of every element of P. In this case $A_{i0} = \frac{x_{i,0}}{x_{i-1,0}}$ for $i \in [r]$, with $A_{00} = x_{00}$, so the single summand is the telescoping product $\frac{x_{0,0}}{1} \cdot \frac{x_{1,0}}{x_{0,0}} \cdot \frac{x_{2,0}}{x_{1,0}} \cdot \dots \cdot \frac{x_{r,0}}{x_{r-1,0}} = x_{r,0}$ as required. Symmetrically, the claim also holds for $r = 0$ and any s. Now suppose that $rs > 0$ and the claim holds for every rectangular posets whose dimensions are strictly smaller than $[0,r] \times [0,s]$. Set $\mathfrak{L}(p,q) := \{\text{lattice paths from } (0,0) \text{ to } (p,q)\}$. Any lattice path from $(0,0)$ to (r,s) must go through either $(r-1,s)$ or $(r,s-1)$. Thus,

$$\begin{aligned}\sum_{\mathcal{L}\in S_1(0,0)} \prod_{(i,j)\in L_1} A_{ij}^{-1} &= A_{r,s}^{-1} \sum_{L\in\mathfrak{L}(r-1,s)} \prod_{(i,j)\in L} A_{ij}^{-1} + A_{r,s}^{-1} \sum_{L\in\mathfrak{L}(r,s-1)} \prod_{(i,j)\in L} A_{ij}^{-1} \\ &= A_{r,s}^{-1}(x_{r-1,s} + x_{r,s-1}) = x_{r,s},\end{aligned}$$

using the induction hypothesis and the definition of $A_{i,j}$. ∎

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