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K-theoretic Pieri rule via iterated residues

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Abstract. We prove a new formulation of the *K*-theoretic Pieri rule regarding multiplication of stable Grothendieck polynomials using iterated residues. We also deploy our method to establish straightening laws to transform Grothendieck polynomials corresponding to general integer sequences to linear combinations of those corresponding to partitions. The technique of iterated residues appears at once similar to raising operators; however, the connection to path integrals in the complex plane provides a different perspective.

Keywords: Grothendieck polynomial, K-theory, Pieri rule, iterated residue

1 Introduction

To each permutation, the seminal work of Lascoux and Schützenberger associates a *(double) Grothendieck polynomial* to represent the (torus equivariant) *K*-class of the corresponding Schubert variety in the complete flag variety [8]. Analogous to the method by which the Stanley symmetric functions are obtained from the Schubert polynomials, the *stable* (double) Grothendieck polynomial is obtained as the limit of embedding permutations into larger symmetric groups [7]. Their many interesting properties have been studied via geometry, e.g. [3] and tableaux combinatorics, e.g. [4].

Motivated by the study of quiver polynomials Buch instituted a systematic combinatorial description of the bialgebra of stable Grothendieck polynomials [4]. Every element of this bialgebra can be expressed as a polynomial in stable Grothendieck polynomials corresponding to *Grassmannian permutations* and hence *partitions*. We apply the technique of *iterated residues* to study the structure of this bialgebra. In this note we focus on a special case of the multiplicative structure, the *K-theoretic Pieri rule*.

The (double) stable Grothendieck polynomials for partitions are representatives for the (torus equivariant) *K*-classes of Schubert varieties in *Grassmannians* and their multiplication gives the *K*-theoretic generalization of the *Littlewood–Richardson rule* [4]. In the setting of quiver polynomials, Buch [6] proved that the stable Grothendieck polynomials

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provide the proper basis in which to formulate conjectures regarding K-theoretic positivity and stability properties. The second author (joint with Szenes) formulated analogous results for K-theoretic Thom polynomials [11].

En route to an iterated residue attack on these problems, the first author proved a new residue formula for the K-theoretic quiver polynomials [1] and the paper [11] reports an iterated residue formula for K-theoretic Thom polynomials. Our hope is that the methods initiated in the present note will be applicable to these larger contexts.

1.1 Notation and terminology

Consider a sequence $I = (I_1, I_2, \dots)$ of integers. Each integer I_i is called a *part* of I. The sequence I is called *finite* if only finitely many of its parts are nonzero. For a finite integer sequence I, we define its *length* $\ell(I) = \max\{i : I_i \neq 0\}$ and its *weight* $|I| = \sum_i I_i$. A finite integer sequence λ is called a *partition* if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0$. Throughout, we will identify the partition λ with its Young diagram of boxes, e.g.

$$\lambda = (4,2,1) \leftrightsquigarrow \ydiagram{4,2,1}.$$

Given partitions $\mu \supset \lambda$, we let μ/λ denote the resulting skew diagram, which is a *horizontal n-strip* if $|\mu| - |\lambda| = n$ and no two boxes of μ/λ appear in the same column. Given two finite integer sequences I and J of respective lengths k and l we define the sequence I, J to be the concatenation $(I_1, \dots, I_k, J_1, \dots, J_l)$ and the sequence $I + J$ to be $(I_1 + J_1, \dots, I_m + J_m)$ where $m = \max\{k, l\}$ and it is understood that $I_i = 0$ for $i > k$ and likewise $J_j = 0$ for $j > l$. We analogously define $I - J$. For an integer a, we abuse notation by letting I, a, J denote the sequence $(I_1, \dots, I_k, a, J_1, \dots, J_l)$. For variables $\boldsymbol{u} = \{u_1, u_2, \dots\}$, we let $\mathbb{Z}[\boldsymbol{u}^{\pm 1}]$ denote the ring of Laurent polynomials. Finally, for any positive integer n, we set $[n] = \{1, 2, \dots, n\}$.

2 Residues and Grothendieck polynomials

2.1 Iterated residue operations

For a meromorphic function $\phi(z)$ in the single complex indeterminate z, define

$$\operatorname*{Res}_{z=0,\infty} (\phi(z)\, dz) := \operatorname*{Res}_{z=0}(\phi(z)\, dz) + \operatorname*{Res}_{z=\infty}(\phi(z)\, dz) \tag{2.1}$$

Now, for $f(\boldsymbol{z})$ a meromorphic function in $\boldsymbol{z} = (z_1, \dots, z_p)$ we define

$$\operatorname*{Res}_{\boldsymbol{z}=0,\infty} \left(f(\boldsymbol{z})\, dz_p \cdots dz_1\right) = \operatorname*{Res}_{z_1=0,\infty} \cdots \operatorname*{Res}_{z_p=0,\infty} \left(f(\boldsymbol{z})\, dz_p \cdots dz_1\right). \tag{2.2}$$

2.2 Stable Grothendieck polynomials

Let $I = (I_1, \ldots, I_p)$ be a finite integer sequence and $\boldsymbol{z} = \{z_1, z_2, \ldots, z_p\}$ a set of complex-valued indeterminates; we take as many variables z_i as there are entries in I. Given sets of variables $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ and $\boldsymbol{\beta} = \{\beta_1, \beta_2, \ldots, \beta_l\}$ we define a (Laurent) polynomial $G_I(\boldsymbol{\alpha}; \boldsymbol{\beta})$, actually a polynomial in the variables α_i^{-1} and β_j, as follows. Set

$$\Delta(\boldsymbol{z}) = \prod_{1 \le i < j \le p} \left(1 - \frac{z_j}{z_i}\right), \qquad P(\boldsymbol{z}|\boldsymbol{\alpha}; \boldsymbol{\beta}) = \prod_{i=1}^{p} (1 - z_i)^{k-l} \frac{\prod_{b=1}^{l}(1 - z_i \beta_b)}{\prod_{a=1}^{k}(1 - z_i \alpha_a)},$$

and define

$$G_I(\boldsymbol{\alpha}; \boldsymbol{\beta}) := \operatorname*{Res}_{\boldsymbol{z}=0,\infty} \left(\frac{\prod_{i=1}^{p}(1 - z_i)^{I_i - i}}{z_1 \cdots z_p} P(\boldsymbol{z}|\boldsymbol{\alpha}; \boldsymbol{\beta})\, \Delta(\boldsymbol{z})\, dz_p \cdots dz_1 \right) \tag{2.3}$$

called the *double stable Grothendieck polynomial*. When a result is independent of the choice of variables $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we simply write G_I. Remarkably, the second author and Szenes have proven in [11] that when I is a partition, Equation (2.3) agrees with previous formulations of the polynomial G_I, e.g. [4, 8]. Often in the literature, for example when comparing to [4], we make the substitutions $1 - \alpha_i^{-1} = x_i$ and $1 - \beta_j = y_j$. Throughout the remainder of the note, let $\Gamma = \bigoplus_\lambda \mathbb{Z} G_\lambda$, a $\mathbb{Z}$-linear span over partitions λ.

Remark 2.1. Since the meromorphic function of which we compute residues in Equation (2.3) does not blow up along any hyperplane $z_i = z_j$, Fubini's Theorem implies that the residues can be taken in any order. As a corollary of this fact, we may actually permute the $\boldsymbol{z}$ variables in (2.3) and the residue operation evaluates to the same polynomial $G_I(\boldsymbol{\alpha}; \boldsymbol{\beta})$. We will use this observation in the sequel.

2.3 An application: straightening laws for G_I

In 2002, Buch also defined a stable Grothendieck polynomial associated to any finite integer sequence I (not necessarily a partition) [5, Section 3]. In that work, G_I is given by a determinant, whose size grows with the number of $\boldsymbol{\alpha}$ variables. We now show that Equation (2.3) implies the straightening laws obtained by Buch in [5] and hence our iterated residue description gives the value of G_I (with I a partition or not) by a single formula, independent of the size of the sets of variables $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The goal is to write G_I as a finite sum over partitions λ, i.e. $G_I = \sum_\lambda d_I^\lambda G_\lambda$; the coefficients $d_I^\lambda \in \mathbb{Z}$ will automatically be uniquely determined since the polynomials G_λ are $\mathbb{Z}$-linearly independent. The two "straightening laws" described in the following theorem are sufficient, c.f. [5, Equation (3.1)].

Theorem 2.2. *For any integer sequences I and J, and any $a, b \in \mathbb{Z}$, we have*

$$G_{I,a,b,J} - G_{I,a+1,b,J} = G_{I,b,a+1,J} - G_{I,b-1,a+1,J}. \tag{2.4}$$

Furthermore, if J has only non-positive parts, then we have

$$G_{I,J} = G_I. \tag{2.5}$$

Observe that a consequence of (2.4) *is that* $G_{I,b-1,b,J} = G_{I,b,b,J}$.

Proof. We prove (2.4) in the case that I and J are empty. The general case is analogous, only with more notation. For the right hand side of (2.4), we consider the result of applying $\mathrm{Res}_{z=0,\infty}$ to

$$\left[(1-z_1)^{b-1}(1-z_2)^{a-1} - (1-z_1)^{b-2}(1-z_2)^{a-1}\right]\left(1-\frac{z_2}{z_1}\right)\frac{P(\boldsymbol{z}|\boldsymbol{\alpha};\boldsymbol{\beta})}{z_1z_2}\,dz_2dz_1. \tag{2.6}$$

Using the observation of Remark 2.1 we apply the simple transposition $z_1 \leftrightarrow z_2$ and Fubini's theorem to obtain

$$\underbrace{\left[(1-z_1)^{a-1}(1-z_2)^{b-2} - (1-z_1)^{a-1}(1-z_2)^{b-1}\right]\left(\frac{z_1}{z_2}\right)}_{\clubsuit}\left(1-\frac{z_2}{z_1}\right)\frac{P(\boldsymbol{z}|\boldsymbol{\alpha};\boldsymbol{\beta})}{z_1z_2}\,dz_2dz_1. \tag{2.7}$$

The portion of the above expression labeled $\clubsuit$ is equal to

$$(1-z_1)^{a-1}(1-z_2)^{b-2} - (1-z_1)^{a+1-1}(1-z_2)^{b-2}$$

and hence the result of applying $\mathrm{Res}_{z=0,\infty}$ to (2.7) is $G_{a,b} - G_{a+1,b}$ as desired. We prove (2.5) in the case that $\ell(J) = 1$; the general result follows inductively. Write $J = (j)$ with $j \leq 0$ and assume that $\ell(I) = p$. Set $\boldsymbol{z} = (z_1, \dots, z_p)$ and apply Fubini's theorem to get

$$\begin{aligned} G_{I,j}(\boldsymbol{\alpha};\boldsymbol{\beta}) = \operatorname*{Res}_{z=0,\infty}&\left(\frac{\prod_{i=1}^p(1-z_i)^{I_i-i}}{z_1\cdots z_p}P(\boldsymbol{z}|\boldsymbol{\alpha};\boldsymbol{\beta})\Delta(\boldsymbol{z})\,dz_p\cdots dz_1\right) \\ &\times \operatorname*{Res}_{\zeta=0,\infty}\left((1-\zeta)^{j-(p+1)}\cdot\frac{1}{\zeta}\cdot P(\zeta|\boldsymbol{\alpha};\boldsymbol{\beta})\cdot\prod_{i=1}^p\left(1-\frac{\zeta}{z_i}\right)\,d\zeta\right). \end{aligned} \tag{2.8}$$

We concentrate on the residues corresponding to the ζ variable. Set

$$g(\zeta) = (1-\zeta)^{j-(p+1)}\cdot\frac{1}{\zeta}\cdot P(\zeta|\boldsymbol{\alpha};\boldsymbol{\beta})\cdot\prod_{i=1}^p\left(1-\frac{\zeta}{z_i}\right)$$

and note $\zeta = 0$ is a simple pole of g. Thus, $\mathrm{Res}_{\zeta=0}(g\,d\zeta) = \lim_{\zeta\to 0}(\zeta\cdot g) = 1$. To compute $\mathrm{Res}_{\zeta=\infty}(g\,d\zeta)$ we consider

$$\widetilde{g}(\zeta) = -\frac{1}{\zeta^2}g(1/\zeta) = -\frac{1}{\zeta^2}\cdot(1-1/\zeta)^{j-p-1}\cdot\zeta\cdot P(1/\zeta|\boldsymbol{\alpha};\boldsymbol{\beta})\prod_{i=1}^p\left(1-\frac{1}{\zeta z_i}\cdot\right)$$

and use that $\operatorname{Res}_{\zeta=\infty}(g\,d\zeta) = \operatorname{Res}_{\zeta=0}(\widetilde{g}\,d\zeta)$. A calculation shows that

$$\widetilde{g}(\zeta) = -\frac{1}{\zeta^j}\cdot(\zeta-1)^{j-p-1+k-l}\cdot\frac{\prod_{b=1}^{l}(\zeta-\beta_b)}{\prod_{a=1}^{k}(\zeta-\alpha_a)}\cdot\prod_{i=1}^{p}(\zeta-1/z_i).$$

Since $j\le 0$, we see that $\widetilde{g}$ is holomorphic at $\zeta=0$ and hence $\operatorname{Res}_{\zeta=0}(\widetilde{g}\,d\zeta)=0$. This implies $\operatorname{Res}_{\zeta=0,\infty}(g\,d\zeta)=1$ and hence Equation (2.8) becomes

$$G_{I,j}(\boldsymbol{\alpha};\boldsymbol{\beta}) = \operatorname*{Res}_{z=0,\infty}\left(\frac{\prod_{i=1}^{p}(1-z_i)^{I_i-i}}{z_1\cdots z_p}P(\boldsymbol{z}|\boldsymbol{\alpha};\boldsymbol{\beta})\Delta(\boldsymbol{z})\,dz_p\cdots dz_1\right)\cdot 1 = G_I(\boldsymbol{\alpha};\boldsymbol{\beta})$$

as desired. Moreover, observe that the proof has been independent of k and l. □

Example 2.3. Using the straightening laws (2.4) and (2.5) we obtain that $G_{(-1,1,-2)} = G_{(-1,1)} = G_{(0,1)}+G_{(1,0)}-G_{(0,0)} = G_{(1,1)}+G_{(1)}-G_{\emptyset}$. Observe the coefficients do not satisfy a typical K-theory type of sign alternation, i.e. $(-1)^{|\lambda|-|I|}d_I^\lambda$ is not necessarily non-negative, a difficulty which we must confront in the sequel, e.g. Sections 3.2–3.4.

2.4 Related operations

Equation (2.3) motivates the following operation on a rational function $f(z_1,\dots,z_p)$,

$$\mathcal{R}_{(\boldsymbol{z}|\boldsymbol{\alpha};\boldsymbol{\beta})}\left(f(z_1,\dots,z_p)\right) := \operatorname*{Res}_{z=0,\infty}\left(f(z_1,\dots,z_p)\,\frac{\prod_{i=1}^{p}(1-z_i)^{-i}}{z_1\cdots z_p}\,P(\boldsymbol{z}|\boldsymbol{\alpha};\boldsymbol{\beta})\,\Delta(\boldsymbol{z})\,dz_p\cdots dz_1\right).$$

Moreover, we define the $\mathbb{Z}$-linear mapping $\mathcal{G}_{\boldsymbol{t}}:\mathbb{Z}[\boldsymbol{t}^{\pm1}]\to\Gamma$ by setting

$$\mathcal{G}_{\boldsymbol{t}}\left(t^I\right) = G_I \tag{2.9}$$

for monomials $t^I = \prod_{i=1}^{p} t_i^{I_i}$. The following is a restatement of the content of (2.3).

Theorem 2.4. *For any Laurent polynomial $g(t_1,\dots,t_p)$ we have*

$$\mathcal{G}_{\boldsymbol{t}}\left(g(t_1,\dots,t_p)\right)(\boldsymbol{\alpha};\boldsymbol{\beta}) = \mathcal{R}_{(\boldsymbol{z}|\boldsymbol{\alpha};\boldsymbol{\beta})}\left(g(1-z_1,\dots,1-z_p)\right).$$ □

In this language, we obtain the following "straightening laws".

Theorem 2.5. *With I, J, a, b as in Theorem 2.2, if $\ell(I)=k-1$ then we have*

$$\mathcal{G}_{\boldsymbol{t}}\left(t^{I,a,b,J}(1-t_k)\right) = \mathcal{G}_{\boldsymbol{t}}\left(-t^{I,b-1,a+1,J}(1-t_k)\right). \tag{2.10}$$

Furthermore, if $f(t_1,\dots,t_k)$ is a Laurent polynomial such that the exponent of t_k is non-positive in every monomial, then

$$\mathcal{G}_{\boldsymbol{t}}\left(f(t_1,\dots,t_k)\right) = \mathcal{G}_{\boldsymbol{t}}\left(f(t_1,\dots,t_{k-1},1)\right). \tag{2.11}$$

Proof. Equations (2.10) and (2.11) respectively encode Equations (2.4) and (2.5). □

3 K-theoretic Pieri rule

We would like to extend the result of Theorem 2.4 to the case that g is a rational function. We do not expect this to be possible for *any* rational function, but one class of interest are functions of the form

$$f_{I,J}(\boldsymbol{t}) = t^{I,J} \prod_{i=1}^{p} \prod_{j=1}^{q} \frac{1-t_i}{1-t_i/t_{p+j}} \tag{3.1}$$

where $I \in \mathbb{Z}^p$ and $J \in \mathbb{Z}^q$. The reason is that a naive calculation shows

$$\begin{aligned} &\mathcal{R}_{(\boldsymbol{z},\boldsymbol{\zeta}|\boldsymbol{\alpha};\boldsymbol{\beta})} \left(f_{I,J}(1-z_1,\ldots,1-z_p,1-\zeta_1,\ldots,1-\zeta_q) \right) \\ &\qquad = \mathcal{R}_{(\boldsymbol{z}|\boldsymbol{\alpha};\boldsymbol{\beta})} \left(\prod_{i=1}^{p} (1-z_i)^{I_i} \right) \mathcal{R}_{(\boldsymbol{\zeta}|\boldsymbol{\alpha};\boldsymbol{\beta})} \left(\prod_{j=1}^{q} (1-\zeta_j)^{J_j} \right) = G_I(\boldsymbol{\alpha};\boldsymbol{\beta})\, G_J(\boldsymbol{\alpha};\boldsymbol{\beta}). \end{aligned} \tag{3.2}$$

Hence, one hopes to understand multiplication of Grothendieck polynomials in terms of the $\mathcal{G}_{\boldsymbol{t}}$ operation. The goal then, is to make $\mathcal{G}_{\boldsymbol{t}}\left(f_{I,J}(\boldsymbol{t})\right)$ a well-defined operation with image in Γ.

3.1 The case $q = 1$

Observe that the function $\frac{1-t_i}{1-t_i/t_{p+1}}$ has Laurent expansion $\sum_{u=0}^{\infty}(t_i^u/t_{p+1}^u) - \sum_{r=1}^{\infty}(t_i^r/t_{p+1}^{r-1})$. Hence, we define

$$Q_i^N = \sum_{u=0}^{N} \frac{t_i^u}{t_{p+1}^u} - \sum_{r=1}^{N} \frac{t_i^r}{t_{p+1}^{r-1}}.$$

We need the following technical vanishing lemma.

Lemma 3.1. *Let $f \in \mathbb{Z}[t_1^{\pm 1},\ldots,t_{p+1}^{\pm 1}]$ be a Laurent polynomial and set*

$$g(t_1,\ldots,t_{p+1}) = f(t_1,\ldots,t_{p+1}) \prod_{i=1}^{p} \frac{1-t_i}{1-t_i/t_{p+1}},$$

$$\widetilde{g}_N(t_1,\ldots,t_{p+1}) = f(t_1,\ldots,t_{p+1}) \prod_{i=1}^{p} Q_i^N.$$

Then for large enough N and any choice of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we have

$$\mathcal{R}_{(\boldsymbol{z}|\boldsymbol{\alpha};\boldsymbol{\beta})}(g(1-z_1,\ldots,1-z_{p+1})) = \mathcal{R}_{(\boldsymbol{z}|\boldsymbol{\alpha};\boldsymbol{\beta})}(\widetilde{g}_N(1-z_1,\ldots,1-z_{p+1})). \tag{3.3}$$

Proof. Let $t_i = 1 - z_i$ for $i \in [p+1]$. The difference of the two sides of (3.3) is

$$\mathcal{R}_{(\boldsymbol{z}|\boldsymbol{\alpha};\boldsymbol{\beta})} \left(f(1-z_1,\ldots,1-z_{p+1}) \left(\prod_{i=1}^{p} \frac{1-t_i}{1-t_i/t_{p+1}} - \prod_{i=1}^{p} Q_i^N \right) \right) \tag{3.4}$$

Calculation shows that

$$\frac{1-t_i}{1-t_i/t_{p+1}} - Q_i^N = \frac{1-t_i}{1-t_i/t_{p+1}} \cdot \frac{t_i^{N+1}}{t_{p+1}^{N+1}} \cdot \frac{1-t_{p+1}}{1-t_i}.$$

Using this we obtain that

$$\prod_{i=1}^{p} \frac{1-t_i}{1-t_i/t_{p+1}} - \prod_{i=1}^{p} Q_i^N = \prod_{i=1}^{p} \frac{1-z_{p+1}}{1-z_{p+1}/z_i} \left(\underbrace{1 - \prod_{i=1}^{p} \left(1 - \frac{(1-z_i)^{N+1}}{(1-z_{p+1})^{N+1}} \frac{z_{p+1}}{z_i}\right)}_{\spadesuit} \right). \quad (3.5)$$

The expression $\spadesuit$ expands to $2^p - 1$ terms which can each be written as

$$\left(\frac{z_{p+1}}{(1-z_{p+1})^{N+1}}\right)^r \psi(z_1, \dots, z_p)$$

for some function ψ and some integer $r \geq 1$. We obtain that (3.4) is the sum of $2^p - 1$ terms, each of the form

$$\mathcal{R}_{(z|\boldsymbol{\alpha};\boldsymbol{\beta})} \left(f \cdot \left(\prod_{i=1}^{p} \frac{(1-z_{p+1})}{1-z_{p+1}/z_i} \right) \left(\frac{z_{p+1}}{(1-z_{p+1})^{N+1}} \right)^r \psi(z_1, \dots, z_p) \right). \quad (3.6)$$

Tracing back the definition of $\mathcal{R}_{(z|\boldsymbol{\alpha};\boldsymbol{\beta})}$ we find that (3.6) is equal to

$$\operatorname*{Res}_{z_1=0,\infty} \cdots \operatorname*{Res}_{z_p=0,\infty} \left[\operatorname*{Res}_{z_{p+1}=0,\infty} \left(f \cdot \frac{z_{p+1}^{r-1} P(z_{p+1}|\boldsymbol{\alpha};\boldsymbol{\beta})}{(1-z_{p+1})^{rN+r+1}} \, dz_{p+1} \right) \cdot \phi(z_1, \dots, z_p) \, dz_p \cdots dz_1 \right]$$

for some function ϕ independent of z_{p+1}. Now by counting degrees of z_{p+1}, we observe that both $\operatorname{Res}_{z_{p+1}=0}$ and $\operatorname{Res}_{z_{p+1}=\infty}$ evaluate to zero for $N \gg 0$. In the case that f is an honest *polynomial* in t_{p+1}, we remark that $N > \deg(f; t_{p+1})$ suffices. $\square$

Definition 3.2. For f, g, $\widetilde{g}_N$, and N large enough such that Lemma 3.1 holds, we define

$$\mathcal{G}_{\boldsymbol{t}} \left(f(t_1, \dots, t_{p+1}) \prod_{i=1}^{p} \frac{1-t_i}{1-t_i/t_{p+1}} \right) = \mathcal{G}_{\boldsymbol{t}} \left(g(t_1, \dots, t_{p+1}) \right) := \mathcal{G}_{\boldsymbol{t}} \left(\widetilde{g}_N(t_1, \dots, t_{p+1}) \right).$$

3.2 K-theoretic Pieri rule as an iterated residue

Theorem 3.3. *For any partition λ of length p and any $n \in \mathbb{Z}_{>0}$ we have*

$$G_\lambda \, G_{(n)} = \mathcal{G}_{\boldsymbol{t}} \left(t^{\lambda,n} \prod_{i=1}^{p} \frac{1-t_i}{1-t_i/t_{p+1}} \right). \quad (3.7)$$

Proof. Set g to be the argument of the right hand side of (3.7) and choose $N > n$. Lemma 3.1 holds for this g and N, and so the right hand side is given by Definition 3.2. We combine this with Equation (3.2) using $I = \lambda$ and $J = (n)$ to see the desired product. □

Fix λ and n throughout the remainder of the section. We wish to use the formula (3.7) to study the *K-theoretic Pieri rule*, which we state below. The result is originally due to Lenart [9, Theorem 3.2]

Theorem 3.4 (Lenart)**.** *There exist unique positive integers $c^{\mu}_{\lambda,n}$ (only finitely many of which are nonzero) such that*

$$G_\lambda \cdot G_{(n)} = \sum_{\mu} (-1)^{|\mu|-|\lambda|-n}\, c^{\mu}_{\lambda,n} G_\mu$$

where the sum is over partitions μ satisfying certain combinatorial conditions.

We choose to emphasize the *positivity* and *finiteness* aspects of Lenart's original paper. We remark that in that work, a complete combinatorial description of the coefficients is given. In short, the number $c^{\mu}_{\lambda,n}$ is nonzero only if μ can be obtained from λ by adding a horizontal strip, and is equal to an explicitly described binomial coefficient. We will give a combinatorial translation for our iterated residue formula below, which (to our knowledge) is a new formulation of the K-theoretic Pieri rule.

Notice that Lemma 3.1 and Definition 3.2 already imply the finiteness result. Moreover, by expanding the Laurent polynomial $\widetilde{g}_{n+1} = t^{\lambda,n} \prod_{i=1}^{p} Q_i^{n+1}$ and making careful cancellations (see [2] for details) we obtain the following manifestly positive formula.

Theorem 3.5. *Summing over partitions $\mu \supset \lambda$ such that μ/λ is a horizontal n-strip gives*

$$G_\lambda \cdot G_{(n)} = \mathcal{G}_{\boldsymbol{t}} \left(\sum_{\mu} t^{\mu} \prod_{i=1}^{p} (1 - t_i)^{\epsilon_i(\mu)} \right). \tag{3.8}$$

The numbers $\epsilon_i(\mu) \in \{0, 1\}$ are determined by the following rules:

(a) $\epsilon_1(\mu) = 1$ *if and only if* $\ell(\mu - \lambda) > 1$;

(b) *when* $i \geq \ell(\mu - \lambda)$ *then* $\epsilon_i(\mu) = 0$;

(c) *when* $1 < i < \ell(\mu - \lambda)$ *then* $\epsilon_i(\mu) = 0$ *if and only if* $\lambda_{i-1} = \mu_i$.

Moreover, each of the polynomials $t^{\mu} \prod_{i=1}^{p} (1 - t_i)^{\epsilon_i(\mu)}$ is "sorted" in the sense that every monomial in its expansion has the form $a_\nu\, t^\nu$ for a partition ν. Observe the sign of the integer coefficient a_ν must necessarily be $(-1)^{|\nu|-|\mu|} = (-1)^{|\nu|-|\lambda|-n}$. □

Theorem 3.5 can be restated in the more pleasing form below.

Theorem 3.6. *For λ and n as above, we have*

$$G_\lambda \cdot G_{(n)} = \mathcal{G}_{\boldsymbol{t}} \left(\sum_\mu t^\mu \prod_{i \in A_{\lambda,\mu}} (1 - t_i) \right) \tag{3.9}$$

where the sum ranges over partitions μ such that μ/λ is a horizontal n-strip and moreover, $t^\mu \prod_{i \in A_{\lambda,\mu}} (1 - t_i)$ is "sorted" as above. The sets $A_{\lambda,\mu}$ are described in the sequel. □

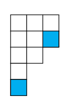
3.3 A combinatorial description of the sets $A_{\lambda,\mu}$

The skew diagram μ/λ is a horizontal strip if and only if μ and λ satisfy the following "interlacing" property, see e.g. [10, Section I.1],

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \tag{3.10}$$

Thus every horizontal strip μ/λ corresponds to a sequence of strict inequalities and equalities (with only finitely many inequalities). We will call this sequence the *code* of the horizontal strip, and denote it by $\mathrm{Code}(\mu/\lambda)$. From this, we define the *odd code* (respectively *even code*) which is the subsequence comprised of the terms of $\mathrm{Code}(\mu/\lambda)$ indexed by odd (resp. even) numbers; by convention, the first term of a sequence is indexed by 1. These will be respectively denoted by $\mathrm{OddCode}(\mu/\lambda)$ and $\mathrm{EvenCode}(\mu/\lambda)$.

Example 3.7. For example, with $\mu = (3, 3, 2, 1, 1)$ and $\lambda = (3, 2, 2, 1)$ the skew diagram μ/λ is the horizontal 2-strip (shaded)

 .

The interlacing property of Equation (3.10) takes the form

$$3 = 3 = 3 > 2 = 2 = 2 > 1 = 1 = 1 > 0 = \cdots$$

and the corresponding code is $\mathrm{Code}\,(\mu/\lambda) = \{=, =, >, =, =, >, =, =, >, =, \ldots\}$. Hence,

$$\mathrm{OddCode}\,(\mu/\lambda) = \{=, >, =, =, >, =, \ldots\} \qquad \mathrm{EvenCode}\,(\mu/\lambda) = \{=, =, >, =, \ldots\}.$$

Proposition 3.8. *If the last non-equality appearing in* $\mathrm{OddCode}(\mu/\lambda)$ *occurs in its k-th term, then $\ell(\mu - \lambda) = k$. Furthermore*

(a) $1 \in A_{\lambda,\mu}$ *if and only if $k > 1$;*

(b) $i \notin A_{\lambda,\mu}$ *for all $i \geq k$;*

(c) *for $1 < i < k$, $i \in A_{\lambda,\mu}$ if and only if the $(i-1)^{st}$ entry of* $\mathrm{EvenCode}(\mu/\lambda)$ *is a strict inequality.*

Proof. The i^{th} entry of the odd code compares μ_i to λ_i. For large enough i, both of these numbers are zero and are therefore equal. Hence, the last occurrence of a "$>$" sign in the odd code must be the length of the sequence $\mu - \lambda$. Properties (a), (b), and (c) are the straightforward translations of items (a), (b), and (c) from Theorem 3.5. □

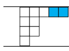
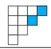
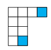
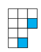
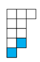
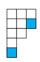
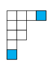
μ	$\ell(\mu-\lambda)$	$\epsilon(\mu)$	$A_{\lambda,\mu}$
 $=(5,2,2,1)$	1	$\epsilon_i=0 \quad \forall i\geq 1$	$\emptyset$
 $=(4,3,2,1)$	2	$\epsilon_1=1$ $\epsilon_i=0 \quad \forall i\geq 2$	$\{1\}$
 $=(4,2,2,2)$	4	$\epsilon_1=1$ $\mu_2\neq\lambda_1 \implies \epsilon_2=1$ $\mu_3=\lambda_2 \implies \epsilon_3=0$ $\epsilon_i=0 \quad \forall i\geq 4$	$\{1,2\}$
 $=(3,3,2,2)$	4	$\epsilon_1=1$ $\mu_2=\lambda_1 \implies \epsilon_2=0$ $\mu_3=\lambda_2 \implies \epsilon_3=0$ $\epsilon_i=0 \quad \forall i\geq 4$	$\{1\}$
 $=(3,2,2,2,1)$	5	$\epsilon_1=1$ $\mu_2\neq\lambda_1 \implies \epsilon_2=1$ $\mu_3=\lambda_2 \implies \epsilon_3=0$ $\mu_4=\lambda_3 \implies \epsilon_4=0$ $\epsilon_i=0 \quad \forall i\geq 5$	$\{1,2\}$
 $=(3,3,2,1,1)$	5	$\epsilon_1=1$ $\mu_2=\lambda_1 \implies \epsilon_2=0$ $\mu_3=\lambda_2 \implies \epsilon_3=0$ $\mu_4\neq\lambda_3 \implies \epsilon_4=1$ $\epsilon_i=0 \quad \forall i\geq 5$	$\{1,4\}$
 $=(4,2,2,1,1)$	5	$\epsilon_1=1$ $\mu_2\neq\lambda_1 \implies \epsilon_2=1$ $\mu_3=\lambda_2 \implies \epsilon_3=0$ $\mu_4\neq\lambda_3 \implies \epsilon_4=1$ $\epsilon_i=0 \quad \forall i\geq 5$	$\{1,2,4\}$

Table 1: The left hand column depicts all possible horizontal 2-strips added to $(3,2,2,1)$. The length of the sequence $\mu-\lambda$ is the row number of the southernmost added box.

3.4 An example computation

Let $\lambda=(3,2,2,1)$ and $n=2$. The results of computing $G_\lambda G_{(2)}$ by applying Theorems 3.5 and Theorem 3.6 are compiled in Table 1. In the end, we are left with the following sum of sorted polynomials

$$\begin{aligned} t^{(5,2,2,1)} &+ t^{(4,3,2,1)}(1-t_1)+t^{(3,3,2,2)}(1-t_1) \\ &+ t^{(4,2,2,2)}(1-t_1)(1-t_2)+t^{(3,2,2,2,1)}(1-t_1)(1-t_2) \\ &+ t^{(3,3,2,1,1)}(1-t_1)(1-t_4)+t^{(4,2,2,1,1)}(1-t_1)(1-t_2)(1-t_4) \end{aligned} \tag{3.11}$$

which, after applying the $\mathcal{G}_t$ operator, becomes

$$\begin{aligned} &G_{(5,2,2,1)}+G_{(4,3,2,1)}+G_{(3,3,2,2)}+G_{(4,2,2,2)}+G_{(3,2,2,2,1)}+G_{(3,3,2,1,1)}+G_{(4,2,2,1,1)} \\ &-G_{(5,3,2,1)}-2G_{(4,3,2,2)}-G_{(5,2,2,2)}-2G_{(4,2,2,2,1)}-2G_{(3,3,2,2,1)}-2G_{(4,3,2,1,1)}-G_{(5,2,2,1,1)} \\ &\qquad +G_{(5,3,2,2)}+3G_{(4,3,2,2,1)}+G_{(5,3,2,1,1)}+G_{(5,2,2,2,1)}-G_{(5,3,2,2,1)}. \end{aligned} \tag{3.12}$$

4 The case $q > 1$

Consider the rational function $f_{I,J}(t_1, \dots, t_{p+q})$ of Equation (3.1). We wish to define $\mathcal{G}_{\boldsymbol{t}}(f_{I,J})$ in such a way that the result of the operation is equal to the product $G_I\, G_J \in \Gamma$. When $q = \ell(J) = 1$, we have solved the problem in Section 3; here we consider $q > 1$. The main obstruction is the analogue of the vanishing result, Lemma 3.1. We propose the following approach. For any $k \in [p]$ consider the mapping

$$\mathcal{E}_k^d : \mathbb{Z}(t_1, \dots, t_{p+q}) \longrightarrow \mathbb{Z}(t_1, \dots, \widehat{t_k}, \dots, t_p, t_{p+1}, \dots, t_{p+q})[t_k^{\pm 1}]$$

which is the composition of sending $f_{I,J}$ to its expansion as a Laurent series about $t_k = 0$, followed by truncating the series to have only degree less than or equal to d. Observe that every resulting term in the Laurent series will have degree exceeding I_k. Thus, truncating to terms with degree *less than* d indeed results in a Laurent *polynomial* in $\mathbb{Z}(t_1, \dots, \widehat{t_k}, \dots, t_p, t_{p+1}, \dots, t_{p+q})[t_k^{\pm 1}]$.

For $d > I_k + N$, a computation shows that the coefficient of $t_k^{I_k+N}$ in $\mathcal{E}_k^d(f_{I,J})$ is

$$\begin{aligned} &\widetilde{h}_k^N(t_1, \dots, \widehat{t_k}, \dots, t_p, t_{p+1}, \dots, t_{p+q}) \\ &\quad = \left(t^{\hat{I},J} \prod_{i \in [p] \setminus \{k\}} \prod_{j \in q} \frac{1 - t_i}{1 - t_i / t_{p+j}} \right) \cdot \sum_{(N_1, \dots, N_q) \vdash N} \left(\prod_{j=1}^{q} \frac{(1 - t_{p+j})^{1 - \delta(N_j, 0)}}{t_j^{N_j}} \right) \end{aligned} \tag{4.1}$$

where $\hat{I}$ is the integer sequence with $\hat{I}_i = I_i$ for all $i \neq k$ but $\hat{I}_k = 0$, $(N_1, \dots, N_q) \vdash N$ denotes that $\sum_{j=1}^{q} N_j = N$ with $N_j \geq 0$, and δ is the Kronecker delta function. The following conjecture has been confirmed with many computer experiments.

Conjecture 4.1. *Let $N > \max(J)$ and $\boldsymbol{z} = (z_1, \dots, z_{p+q})$. For every choice of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we have*

$$\mathcal{R}_{(\boldsymbol{z}|\boldsymbol{\alpha};\boldsymbol{\beta})} \left(\widetilde{h}_k^N(1 - z_1, \dots, \widehat{1 - z_k}, \dots, 1 - z_p, 1 - z_{p+1}, \dots, 1 - z_{p+q}) \right) = 0.$$

Iterating (4.1) leads to the realization that the composition $\mathcal{E}_1^{d_1} \mathcal{E}_2^{d_2} \cdots \mathcal{E}_p^{d_p}$ results in a Laurent *polynomial* in $\mathbb{Z}[t_1^{\pm 1}, \dots, t_{p+q}^{\pm 1}]$ for any sequence of integers (d_i); i.e. the first product in (4.1) will be empty. Hence, supposing the truth of Conjecture 4.1 we define

$$\mathcal{G}_{\boldsymbol{t}}(f_{I,J}) := \mathcal{G}_{\boldsymbol{t}} \left(\mathcal{E}_1^{d_1} \mathcal{E}_2^{d_2} \cdots \mathcal{E}_p^{d_p}(f_{I,J}) \right)$$

provided that each d_i exceeds $I_i + \max(J)$.

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