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Total Nonnegativity and Evaluations of Hecke Algebra Characters

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Abstract. Let $\mathfrak{S}_{[i,j]}$ be the subgroup of the symmetric group $\mathfrak{S}_n$ generated by adjacent transpositions $(i, i+1), \ldots, (j-1, j)$. We give a combinatorial rule for evaluating induced sign characters of the type A Hecke algebra $H_n(q)$ at all elements of the form $\sum_{w \in \mathfrak{S}_{[i,j]}} T_w$ and at all products of such elements. By a result of Stembridge, our result connects $H_n(q)$ trace evaluation to immanants of totally nonnegative matrices.

Keywords: Hecke algebra trace, total nonnegativity

1 Introduction

Related to the study of totally nonnegative matrices, those matrices having only nonnegative minors, is the study of polynomials $p(x_{1,1}, \ldots, x_{n,n})$ satisfying $p(a_{1,1}, \ldots, a_{n,n}) \geq 0$ for every totally nonnegative $n \times n$ matrix $A = (a_{i,j})$. We call these *totally nonnegative (TNN)* polynomials. In particular, work of Lusztig [10] implies that if we view $\mathbb{Z}[x] := \mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ as a free $\mathbb{Z}$-module, then certain elements which are related to the dual canonical basis of the quantum group $\mathcal{O}_q(SL_n(\mathbb{C}))$ are TNN polynomials.

In practice, it is sometimes possible to use cluster algebras and a computer to demonstrate that a polynomial is TNN by expressing it as a subtraction-free rational expression in matrix minors [5]. On the other hand, no simple characterization of TNN polynomials is known. To improve our understanding of TNN polynomials, one might begin by investigating the *immanant subspace* $\mathrm{span}_{\mathbb{Z}}\{x_{1,w_1} \cdots x_{n,w_n} \,|\, w \in \mathfrak{S}_n\}$ of $\mathbb{Z}[x]$, where $\mathfrak{S}_n$ is the symmetric group, especially the generating functions

$$\mathrm{Imm}_\theta(x) := \sum_{w \in \mathfrak{S}_n} \theta(w) x_{1,w_1} \cdots x_{n,w_n} \tag{1.1}$$

for class functions $\theta : \mathfrak{S}_n \to \mathbb{Z}$. Or, since some published results concern Hecke algebra traces, linear functions $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ satisfying $\theta_q(gh) = \theta_q(hg)$, one might investigate these.

In particular, let $\{\widetilde{C}_w(q) \,|\, w \in \mathfrak{S}_n\}$ be the (modified, signless) *Kazhdan-Lusztig basis* of $H_n(q)$, defined by

$$\widetilde{C}_w(q) := q^{\frac{\ell(w)}{2}} C'_w(q) = \sum_{v \leq w} P_{v,w}(q) T_v,$$

where $\{T_w \,|\, w \in \mathfrak{S}_n\}$ is the natural basis of $H_n(q)$, $\{P_{v,w}(q) \,|\, v, w \in \mathfrak{S}_n\}$ are the Kazhdan-Lusztig polynomials [8], and $\leq$ denotes the Bruhat order. Specializing at $q^{\frac{1}{2}} = 1$ we have $T_v \mapsto v$ and $H_n(1) \cong \mathbb{Z}[\mathfrak{S}_n]$. For $1 \leq a < b \leq n$, let $s_{[a,b]} \in \mathfrak{S}_n$ be the *reversal* whose one-line notation is $1 \cdots (a-1)b(b-1) \cdots (a+1)a(a+2) \cdots n$. Stembridge [14] showed that for any linear function $\theta : \mathbb{Z}[\mathfrak{S}_n] \to \mathbb{Z}$, the immanant $\mathrm{Imm}_\theta(x)$ is TNN if for all sequences $J_1, \ldots, J_r$ of subintervals of $[1, n]$, we have

$$\theta(\widetilde{C}_{s_{J_1}}(1) \cdots \widetilde{C}_{s_{J_r}}(1)) \geq 0. \tag{1.2}$$

Furthermore, by Lindström's Lemma and its converse [2], a combinatorial interpretation of the above expression would immediately yield a combinatorial interpretation of the number $\mathrm{Imm}_\theta(A)$ for A a TNN matrix. Now if $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ specializes at $q^{\frac{1}{2}} = 1$ to θ, then (1.2) is clearly a consequence of the condition

$$\theta_q(\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_r}}(q)) \in \mathbb{N}[q], \tag{1.3}$$

and a combinatorial interpretation of the coefficients of the resulting polynomial would yield combinatorial interpretations of the earlier expressions. Haiman [6, Appendix] observed that (1.3) in turn follows from the condition that for all $w \in \mathfrak{S}_n$ we have

$$\theta_q(\widetilde{C}_w(q)) \in \mathbb{N}[q], \tag{1.4}$$

since products of Kazhdan-Lusztig basis elements belong to $\mathrm{span}_{\mathbb{N}[q]}\{\widetilde{C}_w(q) \,|\, w \in \mathfrak{S}_n\}$.

Stembridge [13] and Haiman [6] proved that for θ equal to any irreducible character χ^λ of $\mathfrak{S}_n$ (χ_q^λ of $H_n(q)$) the evaluations (1.2) and (1.4) belong to $\mathbb{N}$ and $\mathbb{N}[q]$, respectively. They conjectured the same [14], [6] for functions ϕ^λ (ϕ_q^λ, respectively), called *monomial traces*, related to irreducible characters by the inverse Kostka numbers. None of these results or conjectures included a combinatorial interpretation. To better understand TNN polynomials of the form (1.1), it would be desirable to solve the following problem.

Problem 1. *Give combinatorial interpretations of all of the expressions in (1.2) or (1.3) when θ_q varies over all elements of any basis of the $H_n(q)$ trace space.*

So far, only some special cases have such interpretations. In the case that w avoids the patterns 3412 and 4231, the Kazhdan-Lusztig basis element $\widetilde{C}_w(q)$ is closely related to a product of the form appearing in (1.3). Combinatorial interpretations of the corresponding expressions (1.3) and (1.4) were given in [3] for $\theta_q \in \{\chi_q^\lambda \,|\, \lambda \vdash n\}$, and for θ_q belonging to several other bases of the $H_n(q)$ trace space, including the basis $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$ of induced sign characters. Also in this case, combinatorial interpretations for $\theta_q = \phi_q^\lambda$ were given only when when λ has at most two parts, or when λ has rectangular shape and $q = 1$ [14, Thm. 2.8]. In the case that all permutations $s_{J_1}, \ldots, s_{J_r}$ in (1.3) are adjacent transpositions $(s_1, \ldots, s_{n-1})$, combinatorial interpretations of the corresponding expressions in (1.3) were given in [7] for $\theta_q = \epsilon_q^\lambda$.

We solve Problem 1 for the trace space basis $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$ and state our solution in Section 3. In Section 2 we introduce our computational tools: the quantum matrix bialgebra, combinatorial structures called star networks, and our general evaluation theorem which links the two.

2 The quantum matrix bialgebra and star networks

Define the *quantum matrix bialgebra* (See, e.g., [11]) $\mathcal{A} = \mathcal{A}(n, q)$ to be the associative algebra with unit 1 generated over $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ by n^2 variables $x = (x_{1,1}, \ldots, x_{n,n})$, subject to the relations

$$\begin{aligned} x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}} x_{i,k}x_{i,\ell}, & x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k}, \\ x_{j,k}x_{i,k} &= q^{\frac{1}{2}} x_{i,k}x_{j,k}, & x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{i,\ell}x_{j,k}, \end{aligned} \tag{2.1}$$

for all indices $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$. The counit map $\varepsilon(x_{i,j}) = \delta_{i,j}$, and coproduct map $\Delta(x_{i,j}) = \sum_{k=1}^n x_{i,k} \otimes x_{k,j}$ give $\mathcal{A}$ a bialgebra structure. While not a Hopf algebra, $\mathcal{A}$ is closely related to the quantum group $\mathcal{O}_q(SL_n(\mathbb{C})) \cong \mathbb{C} \otimes \mathcal{A}/(\det_q(x) - 1)$, where

$$\det_q(x) \underset{\text{def}}{=} \sum_{v \in \mathfrak{S}_n} (-q^{\frac{-1}{2}})^{\ell(v)} x_{1,v_1} \cdots x_{n,v_n} = \sum_{v \in \mathfrak{S}_n} (-q^{\frac{-1}{2}})^{\ell(v)} x_{v_1,1} \cdots x_{v_n,n} \tag{2.2}$$

is the $(n \times n)$ *quantum determinant* of the matrix $x = (x_{i,j})$. (The second equality holds in $\mathcal{A}$ but not in the noncommutative ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]\langle x_{1,1}, \ldots, x_{n,n} \rangle$.) The antipode map of this Hopf algebra is $\mathcal{S}(x_{i,j}) = (-q^{\frac{1}{2}})^{j-i} \det_q(x_{[n] \smallsetminus \{j\}, [n] \smallsetminus \{i\}})$, where

$$[n] \underset{\text{def}}{=} \{1, \ldots, n\}, \qquad x_{L,M} \underset{\text{def}}{=} (x_{\ell,m})_{\ell \in L, m \in M}, \tag{2.3}$$

and $\det_q(x_{L,M})$ is defined analogously to (2.2), assuming $|L| = |M|$. Specializing $\mathcal{A}$ at $q^{\frac{1}{2}} = 1$, we obtain the commutative ring $\mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$.

$\mathcal{A}$ has a natural $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-basis $\{x_{1,1}^{a_{1,1}} \cdots x_{n,n}^{a_{n,n}} \,|\, a_{1,1}, \ldots, a_{n,n} \in \mathbb{N}\}$ of monomials in which variables appear in lexicographic order, and the relations (2.1) provide an algorithm for expressing any other monomial in terms of this basis. The submodule $\mathcal{A}_{[n],[n]}$ spanned by the monomials $\{x^{u,v} \underset{\text{def}}{=} x_{u_1,v_1} \cdots x_{u_n,v_n} \mid u, v \in \mathfrak{S}_n\}$ has rank $n!$ and natural basis $\{x^{e,w} \,|\, w \in \mathfrak{S}_n\}$.

To evaluate induced sign characters at elements $\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)$ of $H_n(q)$, we will associate to each such element a graph called a *star network*, a related matrix B, and a map

$\sigma_B : \mathcal{A}_{[n],[n]} \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$. A generating function $\mathrm{Imm}_{\epsilon_q^\lambda}(x) \in \mathcal{A}_{[n],[n]}$ for $\{\epsilon_q^\lambda(T_w) \,|\, w \in \mathfrak{S}_n\}$ will then allow us to compute

$$\epsilon_q^\lambda(\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)) = \sigma_B(\mathrm{Imm}_{\epsilon_q^\lambda}(x)) \tag{2.4}$$

and to combinatorially interpret the resulting polynomial.

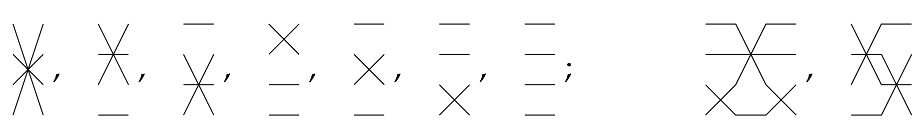
For $1 \leq a < b \leq n$, let $G_{[a,b]}$ be the directed planar graph on $2n+1$ vertices defined as follows.

1. In a column on the left, n vertices are labeled *source* 1 , . . . , *source* n, from bottom to top; in a column on the right, n more vertices are labeled *sink* 1 , . . . , *sink* n, from bottom to top.

2. For $i = 1, \ldots, a-1$ and $i = b+1, \ldots, n$ a directed edge begins at source i and terminates at sink i.

3. An interior vertex is placed between the sources and sinks. For $i = a, \ldots, b$, a directed edge begins at source i and terminates at the interior vertex, and another directed edge begins at the interior vertex and terminates at sink i.

For $a = 1, \ldots, n$ we define $G_{[a,a]}$ to be the similar directed planar graph on n sources and n sinks, with one edge from source i to sink i for $i = 1, \ldots, n$. Call each of the above graphs a *simple star network*. Define a *star network* to be the concatenation of finitely many simple star networks. We write $G \circ H$ for the network in which sink i of G is identified with source i of H, for $i = 1, \ldots, n$. In figures we will not explicitly draw vertices or show edge orientations (assumed to be from left to right). For $n = 4$, there are seven simple star networks: $G_{[1,4]}$, $G_{[2,4]}$, $G_{[1,3]}$, $G_{[3,4]}$, $G_{[2,3]}$, $G_{[1,2]}$, $G_{[1,1]} = \cdots = G_{[4,4]}$. Drawing these and two more star networks $G_{[1,2]} \circ G_{[2,4]} \circ G_{[1,2]}$ and $G_{[2,4]} \circ G_{[1,3]}$, we have



(2.5)

Let $\pi = (\pi_1, \ldots, \pi_n)$ be a sequence of source-to-sink paths in a star network G. We call π a *path family* if there exists a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$ such that π_i is a path from source i to sink w_i. In this case, we say more specifically that π has *type* w. We say that the path family *covers* G if it contains every edge exactly once.

One can enhance a star network by associating to each edge a *weight* belonging to some ring R, and by defining the *weight of a path* to be the product of its edge weights. If R is noncommutative, then one multiplies weights in the order that the corresponding edges appear in the path. For a *family* $\pi = (\pi_1, \ldots, \pi_n)$ of n paths in a planar network, one defines $\mathrm{wgt}(\pi) = \mathrm{wgt}(\pi_1) \cdots \mathrm{wgt}(\pi_n)$. The *(weighted) path matrix* $B = B(G) = (b_{i,j})$ of G is defined by letting $b_{i,j}$ be the sum of weights of all paths in G from source i to sink j. Thus the product $b_{1,w_1} \cdots b_{n,w_n}$ is equal to the sum of weights of all path families of type w in G (covering G or not).

Assigning weights to the edges of $G = G_{J_1} \circ \cdots \circ G_{J_m}$ can aid in the evaluation of a linear function $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ at $\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)$ by relating this evaluation to the generating function

$$\mathrm{Imm}_{\theta_q}(x) := \sum_{w \in \mathfrak{S}_n} q^{\frac{-\ell(w)}{2}} \theta_q(T_w) x_{1,w_1} \cdots x_{n,w_n} \in \mathcal{A}(n,q), \tag{2.6}$$

which specializes at $q^{\frac{1}{2}} = 1$ to the generating function (1.1) in $\mathbb{Z}[x]$. In particular, write $G_{J_p} = G_{[i_p,j_p]}$ and let $\{z_{h,p,k} \,|\, 1 \leq p \leq m; i_p \leq h \leq j_p; 1 \leq k \leq 2\}$ be indeterminate weights satisfying

$$z_{h_2,p_2,k_2} z_{h_1,p_1,k_1} = \begin{cases} z_{h_1,p_1,k_1} z_{h_2,p_2,k_2} & \text{if } p_1 \neq p_2, \text{ or } k_1 \neq k_2, \\ q^{\frac{1}{2}} z_{h_1,p_1,k_1} z_{h_2,p_2,k_2} & \text{if } p_1 = p_2,\ k_1 = k_2, \text{ and } h_1 < h_2. \end{cases} \tag{2.7}$$

We assign weights to the edges of G_{J_p} as follows.

1. Assign weight 1 to the $n - j_p + i_p - 1$ edges not incident upon the central vertex.

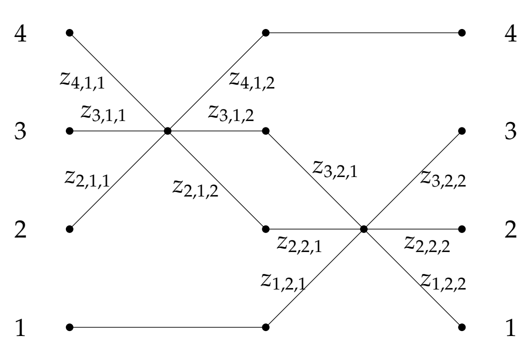
2. Assign weights $z_{i_p,p,1}, z_{i_p+1,p,1}, \ldots, z_{j_p,p,1}$, to the $j_p - i_p + 1$ edges entering the central vertex, from bottom to top.

3. Assign weights $z_{i_p,p,2}, z_{i_p+1,p,2}, \ldots, z_{j_p,p,2}$, to the $j_p - i_p + 1$ edges leaving the central vertex, from bottom to top.

Let Z_G be the quotient of the noncommutative ring

$$\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]\langle z_{h_p,p,k} \,|\, p = 1, \ldots, m; h_p = i_p, \ldots, j_p; k = 1, 2\rangle$$

modulo the ideal generated by the relations (2.7), and assume that $q^{\frac{1}{2}}$, $q^{\frac{-1}{2}}$ commute with all other indeterminates. Let z_G be the product of all indeterminates $z_{h_p,p,k}$, in lexicographic order, and for $f \in Z_G$, let $[z_G]f$ denote the coefficient of z_G in f. For example, the star network $G_{[2,4]} \circ G_{[1,3]}$ has weighting



(2.8)

and monomial $z_G = z_{1,2,1}z_{1,2,2}z_{2,1,1}z_{2,1,2}z_{2,2,1}z_{2,2,2}z_{3,1,1}z_{3,1,2}z_{3,2,1}z_{3,2,2}z_{4,1,1}z_{4,1,2}$.

To complete the description of (2.4), we define a map which allows us to evaluate a linear functional θ_q on certain $H_n(q)$ elements via the corresponding immanant $\mathrm{Imm}_{\theta_q}(x) \in \mathcal{A}(n,q)$. Given matrix $B \in \mathrm{Mat}_{n\times n}(Z_G)$, let σ_B be the $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-linear map

$$\begin{aligned} \sigma_B : \mathcal{A}_{[n],[n]} &\to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}] \\ x_{1,v_1}\cdots x_{n,v_n} &\mapsto [z_G]b_{1,v_1}\cdots b_{n,v_n}, \end{aligned} \tag{2.9}$$

where $[z_G]b_{1,v_1}\cdots b_{n,v_n}$ denotes the coefficient of z_G in $b_{1,v_1}\cdots b_{n,v_n}$, taken after $b_{1,v_1}\cdots b_{n,v_n}$ is expanded in the lexicographic basis of Z_G. Note that the "substitution" $x_{i,j} \mapsto b_{i,j}$ is performed only for monomials of the form $x^{e,v}$ in $\mathcal{A}_{[n],[n]}$: we define $\sigma_B(x^{u,w})$ by first expanding $x^{u,w}$ in the basis $\{x^{e,v} \,|\, v \in \mathfrak{S}_n\}$, and *then* performing the substitution. Now we have the following immanant evaluation identity for star networks (cf. [7, Thm. 3.7]).

Theorem 1. *Assign weights to the edges of $G = G_{J_1} \circ \cdots \circ G_{J_m}$ as above and let B be the resulting path matrix. Then for any linear function $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ we have*

$$\theta_q(\widetilde{C}_{s_{J_1}}(q)\cdots\widetilde{C}_{s_{J_m}}(q)) = [z_G]\sigma_B(\mathrm{Imm}_{\theta_q}(x)). \tag{2.10}$$

Proof. Omitted. □

To illustrate, we let $n = 4$ and consider the element

$$\widetilde{C}_{s_{[2,4]}}(q)\widetilde{C}_{s_{[1,3]}}(q) = (1+q)\sum_{w \leq 3421} T_w \tag{2.11}$$

of $H_4(q)$. Its star network (2.8) has weighted path matrix

$$B = \begin{bmatrix} z_{1,1,1}z_{1,2,2} & z_{1,2,1}z_{2,2,2} & z_{1,2,1}z_{3,2,2} & 0 \\ z_{2,1,1}(z_D+z_U)z_{1,2,2} & z_{2,1,1}(z_D+z_U)z_{2,2,2} & z_{2,1,1}(z_D+z_U)z_{3,2,2} & z_{2,1,1}z_{4,1,2} \\ z_{3,1,1}(z_D+z_U)z_{1,2,2} & z_{3,1,1}(z_D+z_U)z_{2,2,2} & z_{3,1,1}(z_D+z_U)z_{3,2,2} & z_{3,1,1}z_{4,1,2} \\ z_{4,1,1}(z_D+z_U)z_{1,2,2} & z_{4,1,1}(z_D+z_U)z_{2,2,2} & z_{4,1,1}(z_D+z_U)z_{3,2,2} & z_{4,1,1}z_{4,1,2} \end{bmatrix}, \tag{2.12}$$

where $z_D = z_{2,1,2}z_{2,2,1}$, $z_U = z_{3,1,2}z_{3,2,1}$. Now consider the linear function $\theta_q : H_4(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ defined by $\theta_q(T_{3412}) = 1$, $\theta_q(T_{4312}) = -1$, and $\theta_q(T_w) = 0$ otherwise. Computing the left-hand side of (2.10) we have

$$\theta_q(\widetilde{C}_{s_{[2,4]}}(q)\widetilde{C}_{s_{[1,3]}}(q)) = (1+q)(1) + (0)(-1) = 1+q, \tag{2.13}$$

since T_{3412} appears in (2.11) with coefficient $1+q$ and T_{4312} appears with coefficient 0. To compute the right-hand side of (2.10), we begin by writing

$$\mathrm{Imm}_{\theta_q}(x) = q^{-2}x_{1,3}x_{2,4}x_{3,1}x_{4,2} - q^{\frac{-5}{2}}x_{1,4}x_{2,3}x_{3,1}x_{4,2}.$$

Substituting $b_{i,j}$ for $x_{i,j}$, we have $q^{\frac{-5}{2}}b_{1,4}b_{2,3}b_{3,1}b_{4,2} = 0$ and

$$q^{-2}b_{1,3}b_{2,4}b_{3,1}b_{4,2} = q^{-2}z_{1,2,1}z_{3,2,2}z_{2,1,1}z_{4,1,2}z_{3,1,1}(z_D + z_U)z_{1,2,2}z_{4,1,1}(z_D + z_U)z_{2,2,2}. \qquad (2.14)$$

Now since z_D^2 and z_U^2 are not square-free, we ignore terms in the expansion containing these. Since we have

$$\begin{aligned} z_{3,2,2}z_{4,1,2}z_D &= q^{\frac{1}{2}}z_D z_{3,2,2}z_{4,1,2}, & z_{3,2,2}z_{1,2,2} &= q^{\frac{1}{2}}z_{1,2,2}z_{3,2,2}, & \\ z_{3,2,2}z_{4,1,2}z_U &= q^{\frac{1}{2}}z_U z_{3,2,2}z_{4,1,2}, & z_{3,2,2}z_{2,2,2} &= q^{\frac{1}{2}}z_{2,2,2}z_{3,2,2}, & z_U z_D = q z_D z_U, \end{aligned}$$

we express the nonzero square-free monomials of (2.14) in lexicographic order to obtain

$$q^{-2}(q^2 + q^3)z_{1,2,1}z_{1,2,2}z_{2,1,1}z_D z_{2,2,2}z_{3,1,1}z_U z_{3,2,2}z_{4,1,1}z_{4,1,2} = (1+q)z_G,$$

which matches (2.13).

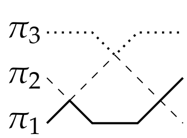
An important property of the map σ_B is that its evaluation at natural basis elements of $\mathcal{A}_{[n],[n]}$ is closely related to coefficients in the natural expansion of $\widetilde{C}_{s_{J_1}}(q)\cdots\widetilde{C}_{s_{J_m}}(q)$.

Proposition 1. *Let B be the weighted path matrix of star network $G = G_{J_1} \circ \cdots \circ G_{J_m}$, and fix $w \in \mathfrak{S}_n$. Then $\sigma_B(x^{e,w})$ is equal to $q^{\frac{\ell(w)}{2}}$ times the coefficient of T_w in the product $\widetilde{C}_{s_{J_1}}(q)\cdots\widetilde{C}_{s_{J_m}}(q)$.*

Proof. Omitted. □

3 G-tableaux and evaluation of induced sign characters

Theorem 1 provides half of the solution to the problem of evaluating the left-hand side of (2.4). The other half is a combinatorial interpretation of the right-hand-side of (2.10), which is a linear combination of expressions of the form $\sigma_B(x^{u,w}) \in \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$. To combinatorially interpret such evaluations, we arrange the paths of a path family π covering a star network G into a (French) Young diagram, using each path exactly once. We call the resulting structure a *G-tableau*, or more specifically a *π-tableau*. If type$(\pi) = w$, we say that the tableau has type w. For example, the following path family π covering the star network of $\widetilde{C}_{s_{[1,2]}}(q)\widetilde{C}_{s_{[2,3]}}(q)\widetilde{C}_{s_{[1,2]}}(q)$ yields six π-tableaux of shape 21 and type 213:



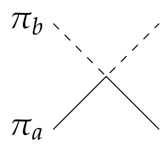
$$, \quad \begin{array}{|c|c}\hline \pi_3 & \\ \hline \pi_1 & \multicolumn{1}{|c|}{\pi_2} \\ \hline \end{array}, \quad \begin{array}{|c|c}\hline \pi_2 & \\ \hline \pi_1 & \multicolumn{1}{|c|}{\pi_3} \\ \hline \end{array}, \quad \begin{array}{|c|c}\hline \pi_3 & \\ \hline \pi_2 & \multicolumn{1}{|c|}{\pi_1} \\ \hline \end{array}, \quad \begin{array}{|c|c}\hline \pi_1 & \\ \hline \pi_2 & \multicolumn{1}{|c|}{\pi_3} \\ \hline \end{array}, \quad \begin{array}{|c|c}\hline \pi_2 & \\ \hline \pi_3 & \multicolumn{1}{|c|}{\pi_1} \\ \hline \end{array}, \quad \begin{array}{|c|c}\hline \pi_1 & \\ \hline \pi_3 & \multicolumn{1}{|c|}{\pi_2} \\ \hline \end{array}. \qquad (3.1)$$

Given a π-tableau U, we define (integer) Young tableaux $L(U)$, $R(U)$ by replacing each path by its source index and sink index, respectively. For example, if U is the first π-tableau in (3.1), then we have

$$L(U) = \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline \end{array}, \quad R(U) = \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{|c|}{1} \\ \hline \end{array}.$$

It is easy to see that given two Young tableaux P, Q of the same shape, there is at most one π-tableau U satisfying $L(U) = P$, $R(U) = Q$.

We also define several statistics on G-tableaux. Suppose that two paths π_a, π_b in a star network $G = G_{J_1} \circ \cdots \circ G_{J_m}$ pass through the central vertex of some simple star network G_{J_p}. We call the triple (p, π_a, π_b) a *crossing* of π if the two paths cross there, and a *noncrossing* otherwise. Let U be any π-tableau. Define $\mathrm{C}(U) = \mathrm{C}(\pi)$ to be the number of crossings of π. Define $\mathrm{INVNC}(U)$, the number of *inverted noncrossings* of U, to be the number of noncrossings (p, π_a, π_b) of π such that π_a, π_b intersect at the central vertex of G_{J_p} with π_b above π_a,



, (3.2)

and π_b appearing in an earlier column of U than π_a (whether or not $b > a$). For example, each tableau U in (3.1) satisfies $\mathrm{C}(U) = 1$ because $\mathrm{C}(\pi) = 1$. The inverted noncrossings in these tableaux are triples $(1, \pi_1, \pi_2)$ with π_2 in an earlier column than π_1, or $(2, \pi_2, \pi_3)$ with π_3 in an earlier column than π_2. The numbers of inverted noncrossings for the six tableaux are $1, 0, 0, 0, 1, 1$, respectively.

Combining the above tableau statistics, we may combinatorially interpret $\sigma_B(x^{u,w})$ in terms of tableaux of shape (n) (i.e., consisting of a single row). A fixed path family π of type v and a permutation $u \in \mathfrak{S}_n$ determine a path tableau $U(\pi, u, uv) = \pi_{u_1} \cdots \pi_{u_n}$ which satisfies $L(U(\pi, u, uv)) = u_1 \cdots u_n$ and $R(U(\pi, u, uv)) = (uv)_1 \cdots (uv)_n$. The inclusion of uv in our notation $U(\pi, u, uv)$ is superfluous but makes clear the ordering of sinks as they appear in the tableau.

Proposition 2. *Let star network G have weighted path matrix B. For $u, w \in \mathfrak{S}_n$ we have*

$$\sigma_B(x^{u,w}) = \sum_{\pi} q^{\frac{\mathrm{C}(\pi)}{2}} q^{\mathrm{INVNC}(U)}, \tag{3.3}$$

where the sum is over path families π of type $u^{-1}w$ covering G, and $U = U(\pi, u, w)$ is the unique π-tableau of shape (n) satisfying $L(U) = u_1 \cdots u_n$, $R(U) = w_1 \cdots w_n$.

Proof. Omitted. □

The special case $u = e$ of Proposition 2 yields a proof of a generalization of Deodhar's defect formula [4, Prop. 3.5] for coefficients of the expression $(1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}})$. Let $\pi = (\pi_1, \ldots, \pi_n)$ be a path family covering a star network $G = G_{J_1} \circ \cdots \circ G_{J_m}$. If two paths π_i, π_j intersect at the central vertex of G_{J_p}, call the intersection *defective* if the paths have previously crossed an odd number of times (i.e., in $G_{J_1}, \ldots, G_{J_{p-1}}$). Define $\mathrm{D}(\pi)$, the number of *defects* of π, to be the number of triples (p, π_i, π_j) such that π_i and π_j intersect defectively at the central vertex of G_{J_p}.

Corollary 1. *The coefficients in the expansion* $\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q) = \sum_w a_w T_w$ *are*

$$a_w = \sum_{\pi} q^{\mathrm{D}(\pi)},$$

where the sum is over all path families of type w *which cover the star network* $G_{J_1} \circ \cdots \circ G_{I_m}$.

Proof. Omitted. □

By Theorem 1, the map σ_B (2.9) can be used to evaluate $\epsilon_q^{\lambda}(\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q))$ when one has a simple expression for the generating function $\mathrm{Imm}_{\epsilon_q^{\lambda}}(x)$. Such an expression was given by Konvalinka and the second author in [9, Thm. 5.4]:

$$\mathrm{Imm}_{\epsilon_q^{\lambda}}(x) = \sum_I \det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r}), \tag{3.4}$$

where $\det_q$ and $x_{L,M}$ are defined as in Section 2, and the sum is over all ordered set partitions $I = (I_1, \ldots, I_r)$ of $[n]$ satisfying $|I_j| = \lambda_j$. We will say that such an ordered set partition has *type* λ.

To evaluate $\sigma_B(\mathrm{Imm}_{\epsilon_q^{\lambda}}(x))$, we expand each term on the right-hand side of (3.4) in a monomial basis $\{x^{u,v} \,|\, v \in \mathfrak{S}_n\}$ of $\mathcal{A}_{[n],[n]}$, where $u = u(I)$ is the concatenation of the r strictly increasing subwords

$$u_1 \cdots u_{\lambda_1}, \quad u_{\lambda_1+1} \cdots u_{\lambda_1+\lambda_2}, \quad u_{\lambda_1+\lambda_2+1} \cdots u_{\lambda_1+\lambda_2+\lambda_3}, \quad \ldots, \quad u_{n-\lambda_r+1} \cdots u_n \tag{3.5}$$

formed by listing the elements of each block $I_1, \ldots, I_r$ in increasing order. As I varies over all ordered set partitions of $[n]$ of type λ, the permutations $u(I)$ vary over the Bruhat-minimal representatives $\mathfrak{S}_{\lambda}^{-}$ of cosets $\mathfrak{S}_{\lambda} u$, where $\mathfrak{S}_{\lambda}$ is the *Young subgroup* of $\mathfrak{S}_n$ generated by

$$\{s_1, \ldots, s_{n-1}\} \smallsetminus \{s_{\lambda_1}, s_{\lambda_1+\lambda_2}, s_{\lambda_1+\lambda_2+\lambda_3}, \ldots, s_{n-\lambda_r}\}.$$

Expanding each term on the right-hand side of (3.4) and applying σ_B we have

$$\sigma_B(\det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r})) = \sum_{y \in \mathfrak{S}_{\lambda}} (-1)^{\ell(y)} q^{\frac{-\ell(y)}{2}} \sigma_B(x^{u(I), yu(I)}). \tag{3.6}$$

To combinatorially interpret the sum in (3.6) we may apply Proposition 2 and compute certain statistics for tableaux belonging to the set

$$\mathcal{U}_I = \mathcal{U}_I(G) \underset{\mathrm{def}}{=} \{U(\pi, u, yu) \,|\, \pi \text{ covers } G, u = u(I), y \in \mathfrak{S}_{\lambda}\}.$$

Note that our restriction on y forces the sink indices of paths in components

$$(\lambda_1 + \cdots + \lambda_{k-1} + 1), \ldots, (\lambda_1 + \cdots + \lambda_k) \tag{3.7}$$

of $U(\pi, u, yu)$ to be a permutation of the source indices of the same paths.

On the other hand, the sum in (3.6) has both positive and negative signs. We obtain a subtraction-free expression for the sum by applying a sign-reversing involution to the tableaux in each set $\mathcal{U}_I$. Tableaux which remain after cancellation are in bijection with G-tableaux in which paths in a single column have increasing indices and do not intersect. We call such tableaux *column-strict*.

Theorem 2. *Let $G = G_{J_1} \circ \cdots \circ G_{I_m}$. Then for $\lambda \vdash n$ we have*

$$\epsilon_q^\lambda(\widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q)) = \sum_W q^{\textsc{invnc}(W) + \textsc{c}(W)/2}, \tag{3.8}$$

where the sum is over all column-strict G-tableaux of type e and shape $\lambda^\top$.

Proof. (Idea) Let B be the path matrix of G. Combining the Theorems 1 and [9, Thm. 5.4] (i.e., (3.4)) with the identity (3.6), we see that the left-hand side of (3.8) is

$$\begin{aligned} \sigma_B(\mathrm{Imm}_{\epsilon_q^\lambda}(x)) &= \sum_I \sigma_B(\det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r})) \\ &= \sum_I \sum_{y \in \mathfrak{S}_\lambda} (-1)^{\ell(y)} q^{\frac{-\ell(y)}{2}} \sigma_B(x^{u(I), yu(I)}), \end{aligned} \tag{3.9}$$

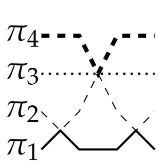
where the first two sums are over ordered set partitions $I = (I_1, \ldots, I_r)$ of $[n]$ of type λ. Fixing one such partition I and writing $u = u(I)$, we may use Proposition 2 and other lemmas to express the sum over elements of $\mathfrak{S}_\lambda$ as

$$\sum_{y \in \mathfrak{S}_\lambda} \sum_\pi (-1)^{\ell(y)} q^{\frac{-\ell(y)}{2}} q^{\frac{\textsc{c}(\pi)}{2}} q^{\textsc{invnc}(U(\pi,u,yu))} = \sum_{y \in \mathfrak{S}_\lambda} \sum_\pi (-1)^{\ell(y)} q^{\frac{-\ell(y)}{2}} q^{\frac{\textsc{c}(\pi)}{2}} q^{\textsc{invnc}(W) + \textsc{cdnc}(W)}, \tag{3.10}$$

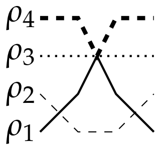
where the inner sums are over path families π of type $u^{-1}yu$ which cover G, and where $W = W(\pi, u, yu)$ is a related tableau of shape $\lambda^\top$, and $\textsc{cdnc}$ is a statistic related to defects and noncrossings.

A sign reversing involution eliminates those tableaux W which are not column-strict, and another lemma allows us to interpret the given powers of q in terms of crossings and inverted noncrossings in the remaining column-strict tableaux. Thus the three expressions in (3.9) are equal to the right-hand side of (3.8). □

To illustrate the theorem, we compute $\epsilon_q^{211}(\widetilde{C}_{s_{[1,2]}}(q)\widetilde{C}_{s_{[2,4]}}(q)\widetilde{C}_{s_{[1,2]}}(q))$ using the star network $G = G_{[1,2]} \circ G_{[2,4]} \circ G_{[1,2]}$ pictured in (2.5). There are two path families of type e which cover G, and four column-strict G-tableau of shape $211^\top = 31$ for each:



$$, \quad U_\pi^{(1)} = \begin{array}{|c|c|c|}\hline \pi_3 \\ \hline \pi_1 & \pi_2 & \pi_4 \\ \hline \end{array}, \quad U_\pi^{(2)} = \begin{array}{|c|c|c|}\hline \pi_3 \\ \hline \pi_1 & \pi_4 & \pi_2 \\ \hline \end{array}, \quad U_\pi^{(3)} = \begin{array}{|c|c|c|}\hline \pi_4 \\ \hline \pi_1 & \pi_2 & \pi_3 \\ \hline \end{array}, \quad U_\pi^{(4)} = \begin{array}{|c|c|c|}\hline \pi_4 \\ \hline \pi_1 & \pi_3 & \pi_2 \\ \hline \end{array};$$



$$U_\rho^{(1)} = \begin{array}{|c|cc}\hline \rho_3 \\ \hline \rho_2 & \rho_1 & \rho_4 \\ \hline \end{array}, \quad U_\rho^{(2)} = \begin{array}{|c|cc}\hline \rho_3 \\ \hline \rho_2 & \rho_4 & \rho_1 \\ \hline \end{array}, \quad U_\rho^{(3)} = \begin{array}{|c|cc}\hline \rho_4 \\ \hline \rho_2 & \rho_1 & \rho_3 \\ \hline \end{array}, \quad U_\rho^{(4)} = \begin{array}{|c|cc}\hline \rho_4 \\ \hline \rho_2 & \rho_3 & \rho_1 \\ \hline \end{array}.$$

The path family π has no crossings, so tableau $U_\pi^{(i)}$ contributes $q^{\text{INVNC}(U_\pi^{(i)})} q^{\text{C}(U_\pi^{(i)})/2} = q^{\text{INVNC}(U_\pi^{(i)})}$ for all i. We have one noncrossing for each of the pairs (π_2, π_3), (π_2, π_4) and (π_3, π_4) and two for the pair (π_1, π_2). Counting only the inverted noncrossings, such as π_2 and π_3 in $U_\pi^{(1)}$, we find the contributions from $U_\pi^{(1)}, \dots, U_\pi^{(4)}$ are q, q^2, q^2, q^3, respectively. The tableaux for the path family ρ each have two crossings, and one noncrossing for each of the pairs (ρ_1, ρ_3), (ρ_1, ρ_4) and (ρ_3, ρ_4). Adding the contributions together we find the contributions for $U_\rho^{(1)}, \dots, U_\rho^{(4)}$ are $q^1 q^{2/2} = q^2$, $q^2 q^{2/2} = q^3$, $q^2 q^{2/2} = q^3$ and $q^3 q^{2/2} = q^4$ respectively. Hence we have $\epsilon_q^{211}(\widetilde{C}_{s_{[1,2]}}(q)\widetilde{C}_{s_{[2,4]}}(q)\widetilde{C}_{s_{[1,2]}}(q)) = q + 3q^2 + 3q^3 + q^4$.

Theorem 2 allows one to combinatorially interpret evaluations of ϵ_q^λ at (multiples of) certain elements $\widetilde{C}_w(q)$ of the Kazhdan-Lusztig basis of $H_n(q)$. In particular, for some elements $\widetilde{C}_w(q)$ there exists a polynomial $g_w(q)$ such that we have

$$g_w(q)\widetilde{C}_w(q) = \widetilde{C}_{s_{J_1}}(q) \cdots \widetilde{C}_{s_{J_m}}(q) \tag{3.11}$$

for some sequence $s_{J_1}, \dots, s_{J_m}$ of reversals. Such permutations include all 3412-avoiding, 4231-avoiding permutations, all of $\mathfrak{S}_4$ (even 4231 and 3412), all of $\mathfrak{S}_5$ except 45312, and all 321-*hexagon-avoiding* permutations. (See [1].)

Corollary 2. *Suppose that $\widetilde{C}_w(q)$ satisfies a factorization of the form (3.11) and define $G = G_{J_1} \circ \cdots \circ G_{J_m}$. Then we have*

$$\epsilon_q^\lambda(\widetilde{C}_w(q)) = \frac{1}{g_w(q)} \sum_U q^{\text{INVNC}(U) + \text{C}(U)/2}, \tag{3.12}$$

where the sum is over all column-strict G-tableaux of type e and shape $\lambda^\top$.

It would be interesting to characterize the factorizations (3.11) [12, Quest. 4.5].

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