Séminaire Lotharingien de Combinatoire **82B** (2019)
Article #49, 12 pp.

Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics (Ljubljana)

On the Schur positivity of sums of power sums

Sheila Sundaram*

Pierrepont School, One Sylvan Road North, Westport, CT 06880

Abstract. Let T be a nonempty subset of positive integers and p_n the nth power sum symmetric function. Consider the multiplicity-free sum of power sums $F_n^T = \sum_{\lambda \vdash n} p_\lambda$, where the sum ranges over all partitions of n with parts in the set T. We define a new symmetric function f_T and give two descriptions of the (possibly virtual) symmetric group representation associated to the series $\prod_{n \in T}(1-p_n)^{-1} = \sum_{n \geq 0} F_n^T$: one in terms of the Lie representation, and another as the symmetric or exterior power of (again possibly virtual) modules induced from centralisers of the symmetric group.

When $T = \{1\}$, the degree n term of f_T reduces to the Frobenius characteristic of the Lie representation Lie_n. At the other extreme, when T is the set of all positive integers, it is the conjugacy action of S_n. The function f_T allows us to unify previous results on the Schur positivity of multiplicity-free sums of power sums, as well as investigate new ones. We also uncover some curious plethystic relationships between the conjugacy action and the Lie representation.

Finally we establish some special cases of an earlier conjecture of this author on the Schur positivity of sums of power sums in the intervals $[(1^n), \mu]$ in reverse lexicographic order.

Keywords: Schur positivity, plethysm, Lie representation, conjugacy action, symmetric and exterior powers

1 Introduction

In this paper we investigate the positivity of the row sums in the character table of S_n. For each irreducible character χ^λ indexed by a partition λ of n, and any subset T of the conjugacy classes, one can form the sum $\sum_{\mu \in T} \chi^\lambda(\mu)$, and ask when this sum is nonnegative. In the language of symmetric functions, one asks for what subsets T of partitions of n the sum of power sums $\sum_{\mu \in T} p_\mu$ is the Frobenius characteristic of a true representation of S_n, i.e. a symmetric function with nonnegative integer coefficients in the basis of Schur functions. A method for generating such classes of subsets T was presented in [7].

*shsund@comcast.net

We present a new approach to the Schur positivity problem for power sums. Specifically, we give a general formula which expresses the product $\prod_{n\in T}(1-p_n)^{-1}$ as a symmetrised module over a sequence of possibly virtual representations f_n^T, having the specific property that their characters vanish unless the conjugacy class has all cycles of equal length. The goal then is to determine for what choices of the set T the f_n^T are true S_n-modules, thereby establishing the Schur positivity of the product $\prod_{n\in T}(1-p_n)^{-1}$. The module Lie_n, the S_n-module afforded by the multilinear component of the free Lie algebra on n generators, plays a prominent role in the construction. In particular we describe simple relationships (see Theorem 3.3 and equations (3.6) and (3.9) with $q=2$) between Lie_n, the conjugacy action, and the variant $Lie_n^{(2)}$. This variant was the subject of [5], and was shown to have remarkable properties in [8].

In the course of these calculations many interesting plethystic identities emerge, as well as many new conjectures on Schur positivity.

2 Preliminaries

Recall [2] that the S_n-module Lie_n is the action of S_n on the multilinear component of the free Lie algebra, and coincides with the induced representation $\exp(\frac{2i\pi}{n})\uparrow_{C_n}^{S_n}$, where C_n is the cyclic group generated by an n-cycle in S_n.

Another module that will be of interest is the S_n-module $Conj_n$ afforded by the conjugacy action of S_n on the class of n-cycles. Clearly we have $Conj_n \simeq \mathbf{1}\uparrow_{C_n}^{S_n}$.

We follow [1] and [4] for notation regarding symmetric functions. In particular, h_n, e_n and p_n denote respectively the complete homogeneous, elementary and power sum symmetric functions. If ch is the Frobenius characteristic map from the representation ring of the symmetric group S_n to the ring of symmetric functions with real coefficients, then $h_n = \mathrm{ch}(1_{S_n})$ is the characteristic of the trivial representation, and $e_n = \mathrm{ch}(\mathrm{sgn}_{S_n})$ is the characteristic of the sign representation of S_n. If μ is a partition of n then define $p_\mu = \prod_i p_{\mu_i}$; h_μ and e_μ are defined multiplicatively in analogous fashion. As in [1], the Schur function s_μ indexed by the partition μ is the Frobenius characteristic of the S_n-irreducible indexed by μ. Finally, ω is the involution on the ring of symmetric functions which takes h_n to e_n, corresponding to tensoring with the sign representation.

By a slight abuse of notation we will also write Lie_n (resp. $Conj_n$) to mean the Frobenius characteristic of the representation Lie_n (resp. $Conj_n$). Let $\mu(d)$ denote the number-theoretic Möbius function, and $\phi(d)$ the Euler totient function. The following facts are well known (see [2]).

$$Lie_n = \text{ch}\exp(\frac{2i\pi}{n})\uparrow_{C_n}^{S_n} = \frac{1}{n}\sum_{d|n}\mu(d)p_d^{\frac{n}{d}}; \tag{2.1}$$

$$Conj_n = \text{ch}\,\mathbf{1}\uparrow_{C_n}^{S_n} = \frac{1}{n}\sum_{d|n}\phi(d)p_d^{\frac{n}{d}} \tag{2.2}$$

If q and r are characteristics of representations of S_m and S_n respectively, they yield a representation of the wreath product $S_m[S_n]$ in a natural way, with the property that when this representation is induced up to S_{mn}, its Frobenius characteristic is the plethysm $q[r]$. For more background about this operation, see [1]. We will make extensive use of the properties of this operation, in particular the fact that plethysm with a symmetric function r is an endomorphism on the ring of symmetric functions [1, (8.3)]. See also [4, Chapter 7, Appendix 2, A2.6]. Define the symmetric functions

$$H = \sum_{i\geq 0} h_i, \quad E = \sum_{i\geq 0} e_i; \tag{2.3}$$

$$Lie = \sum_{i\geq 1} Lie_i; \quad Conj = \sum_{i\geq 1} Conj_i. \tag{2.4}$$

We collect some of the key tools used to establish our results.

Let $\psi(n)$ be any real-valued function defined on the positive integers. Define symmetric functions f_n by $f_n = \frac{1}{n}\sum_{d|n}\psi(d)p_d^{\frac{n}{d}}$, and the associated polynomial in one variable, t, by $f_n(t) = \frac{1}{n}\sum_{d|n}\psi(d)t^{\frac{n}{d}}$.

Theorem 2.1. *[7, Theorem 3.2] Let $F = \sum_{n\geq 1} f_n$ where f_n is of the form described above, $H(v) = \sum_{n\geq 0} v^n h_n$ and $E(v) = \sum_{n\geq 0} v^n e_n$. We have the following plethystic generating functions:*

$$H(v)[F] = \prod_{m\geq 1}(1-p_m)^{-f_m(v)} \tag{2.5}$$

$$E(v)[F] = \prod_{m\geq 1}(1-p_m)^{f_m(-v)} \tag{2.6}$$

The plethystic formulas in this abstract are also consequences of the following propositions:

Proposition 2.1. *Let $F = \sum_{n\geq 1} f_n$, $G = 1 + \sum_{n\geq 1} g_n$ be formal series of symmetric functions, as usual with f_n, g_n being of homogeneous degree n. Then*

$$H[F] = G \iff E[F] = \frac{G}{G[p_2]} = \left(\frac{p_1}{p_2}\right)[G].$$

Hence the exterior power of the series F may also be obtained as the symmetric power of another series, namely, the plethysm

$$H[\frac{F}{F[p_2]}] = E[F] = \frac{H[F]}{H[F[p_2]]}$$

In particular if F is Schur-positive, then so is $\frac{G}{G[p_2]}$.

Proposition 2.2. *[8] The following pairs are plethystic inverses:*

$$\sum_{n \text{ odd}} g(n)p_n \text{ and } \sum_{n \text{ odd}} g(n)\mu(n)p_n, \tag{2.7}$$

for any function $g(n)$ defined on the positive integers, such that $g(mn) = g(m)g(n)$;

$$\sum_{n\geq 1} Lie_n \text{ and } \frac{H-1}{H} = \sum_{n\geq 1}(-1)^{n-1}e_n. \tag{2.8}$$

$$\sum_{n\geq 1}(-1)^{n-1}\omega(Lie_n) \text{ and } H-1 \tag{2.9}$$

3 A formula for $\prod_{n\in T}(1-p_n)^{-1}$

In this section we explore, for a fixed subset T of positive integers, the sum of power sums resulting from the product $\prod_{n\in T}(1-p_n)^{-1}$. (This product is 1 if T is the empty set.)

Definition 3.1. *Fix a nonempty subset T of the positive integers. Define, on the set of positive integers, a function ψ^T by $\psi^T(d) = \sum_{m|d,\, m\in T} m\, \mu\left(\frac{d}{m}\right)$.*

Definition 3.2. *For each nonempty subset T of positive integers, define a sequence of (possibly virtual) representations indexed by the subset T, with Frobenius characteristic*

$$f_n^T = \frac{1}{n}\sum_{d|n}\psi^T(d)p_d^{\frac{n}{d}}.$$

Set $F^T = \sum_{n\geq 1} f_n^T$. Finally let $p^T = \sum_{n\in T} p_n$.

These definitions imply:

Lemma 3.1. $f_n^T(1) = 1$ *if and only if $n \in T$, and* $f_n^T(1) = 0$ *otherwise.*

Proof. Let us write $\delta(m \in T)$ for the indicator function of the set T, so that $\delta(m \in T) = 1$ if $m \in T$, and is zero otherwise.

By definition of ψ^T, we have

$$\psi^T(n) = \sum_{d|n}\mu(\frac{n}{d})\, d\, \delta(d \in T).$$

Hence Möbius inversion gives

$$n\delta(n \in T) = \sum_{d|n} \psi^T(d) = n\, f_n^T(1),$$

i.e. $f_n^T(1) = \delta(n \in T)$ as claimed. □

With this lemma and Theorem 2.1 in Section 2, we can prove:

Theorem 3.1. *Let T be a nonempty subset of the positive integers. Then:*

$$H[F^T] = \prod_{n \in T} (1 - p_n)^{-1} \tag{3.1}$$

$$F^T = p^T[Lie] = \sum_{m \in T} Lie[p_m], \text{ or equivalently } f_n^T = \sum_{\substack{m \in T \\ m|n}} Lie_{\frac{n}{m}}[p_m]. \tag{3.2}$$

If $G^T = \sum_{k \geq 0} \sum_{m \in T} Lie[p_{m \cdot 2^k}]$, then

$$E[G^T] = \prod_{n \in T} (1 - p_n)^{-1} = H[F^T]. \tag{3.3}$$

Corollary 3.1. *If either F^T or G^T is Schur-positive, then so is*

$$\prod_{n \in T} (1 - p_n)^{-1} = \sum_{\substack{\lambda \in Par \\ \lambda_i \in T}} p_\lambda.$$

Recall from [5] the following definitions:

Definition 3.3. *Let $S = \{q_1, \ldots, q_k, \ldots\}$ be a set of distinct primes. Every positive integer n factors uniquely into $n = Q_n \ell_n$ where $Q_n = \prod_{q \in S} q^{a_q(n)}$ for nonnegative integers $a_q(n)$, and $(\ell_n, q) = 1$ for all $q \in S$. We associate to the set S two symmetric functions, defined as follows. For each $n \geq 1$:*

$$L_n^S = \frac{1}{n} \sum_{d|n} \psi(d) p_d^{\frac{n}{d}} \quad \textit{with } \psi(d) = \phi(Q_d)\mu(\ell_d), \textit{ and} \tag{3.4}$$

$$L_n^{\bar{S}} = \frac{1}{n} \sum_{d|n} \bar{\psi}(d) p_d^{\frac{n}{d}} \quad \textit{with } \bar{\psi}(d) = \phi(\ell_d)\mu(Q_d). \tag{3.5}$$

Note that $L_n^{\bar{S}} = L_n^T$ where T is the set of primes not in S.

Theorem 3.2. *[5, Theorem 3.2], [8, Theorem 3.1, Definition 3.2] The symmetric functions L_n^S and $L_n^{\bar{S}}$ are Frobenius characteristics of true S_n-modules, and are thus Schur positive.*

When S consists of a single prime $\{q\}$, we write $L_n^S = Lie_n^{(q)}$. The particular case $q = 2$ was investigated extensively in [5] and [8]. In that case, the S_n-module $Lie_n^{(2)}$ was shown to have remarkable properties closely parallelling those of the Lie representation.

We illustrate Theorem 3.1 and the functions f_n^T by examining the following special cases:

1. If $T = \{1\}$, then $\psi^T(d) = \mu(d)$ and f_n^T corresponds to the representation Lie_n. In this case (3.1) of Theorem 3.1 gives the classical result of Thrall and Poincaré-Birkhoff-Witt [2]:
$$H[Lie] = (1 - p_1)^{-1},$$
and (3.3) gives the surprising exterior power analogue for the variant S_n-modules $Lie_n^{(2)}$ studied in [5] and [8]:
$$E[Lie^{(2)}] = (1 - p_1)^{-1}.$$

2. If T is the set of all positive integers, then $\psi^T(d) = \phi(d)$ by Möbius inversion of the well-known identity $m = \sum_{d|m} \phi(d)$. Thus f_n^T is the characteristic of the conjugacy action on the class of n-cycles $\mathbf{1} \uparrow_{C_n}^{S_n}$, i.e. $f_n^T = Conj_n$. Now (3.1) of Theorem 3.1 reduces to a theorem of Solomon [3]:
$$H[Conj] = \prod_{n \geq 1} (1 - p_n)^{-1} = \sum_{\lambda \vdash n, n \geq 0} p_\lambda.$$
Also (3.3) reduces to a result of this author, first proved in [7], implying Schur positivity of the right-hand side below:
$$E[Conj] = \prod_{n \geq 1, n \text{ odd}} (1 - p_n)^{-1} = \sum_{\lambda \vdash n, n \geq 0, \lambda_i \text{ odd}} p_\lambda.$$

3. Fix a set S of primes. Let T be the set of all integers whose prime factors are all in S. Then clearly if $d \in T$, $\psi^T(d) = \phi(d)$ by the identity used above. Otherwise $d = Q_d \ell_d$ with $Q_d \in T$ and ℓ_d relatively prime to Q_d and also relatively prime to all integers in T. Hence, since μ is multiplicative, Definition 3.1 gives
$$\psi^T(d) = \sum_{m \in T} m\mu(Q_d/m)\mu(\ell_d) = \mu(\ell_d) \cdot \sum_{m \in T} m\mu(Q_d/m) = \mu(\ell_d)\psi^T(Q_d),$$
Since $Q_d \in T$, we obtain $\psi^T(d) = \mu(\ell_d)\phi(Q_d)$, which is precisely the formula given by (3.4). Thus $f_n^T = L_n^S$. Setting $L^S = \sum_{n \geq 1} L_n^S$, and defining $P(S)$ to be the set of positive integers whose prime factors are a subset of S, (3.1) becomes the result of [8, Theorem 3.5], again establishing Schur positivity of the sum on the right. (Note that $1 \in P(S)$.)
$$H[L^S] = \prod_{n \in P(S)} (1 - p_n)^{-1} = \sum_{\lambda \in Par: \lambda_i \in P(S)} p_\lambda.$$

From Theorem 3.1 and the preceding observations, we have the following decompositions of the representations $Conj_n$, $Lie_n^{(q)}$:

Theorem 3.3.
$$\sum_{m\geq 1} p_m[Lie] = \sum_{n\geq 1} Conj_n; \tag{3.6}$$

$$\sum_{m\geq 1} p_m = \sum_{n\geq 1} Conj_n[\sum_{r\geq 1}(-1)^{r-1}e_r] \tag{3.7}$$

The plethystic inverse of Conj is

$$(\sum_{n\geq 1} Conj_n)^{\langle -1\rangle} = \sum_{r\geq 1}(-1)^{r-1}e_r[\sum_{n\geq 1}\mu(n)p_n]. \tag{3.8}$$

Let q be prime, and let $n = \ell q^k$ where $(\ell, q) = 1$. Then

$$Lie_n^{(q)} = \sum_{r=0}^{k} Lie_{\ell q^{k-r}}[p_{q^r}]. \tag{3.9}$$

The plethystic inverse of $Lie^{(q)}$ is

$$\sum_{n\geq 1}(Lie_n^{(q)})^{\langle -1\rangle} = Lie^{\langle -1\rangle}[p_1 - p_q] = (\sum_{r\geq 1}(-1)^{r-1}e_r)[p_1 - p_q]. \tag{3.10}$$

Computations for $q = 2, 3, 5$ (verified for $n \leq 32$, $n \leq 27$ and $n \leq 25$ respectively) support a curious conjecture on the partial sums in (3.9):

Conjecture 1. *Fix a prime q and $n = \ell q^k$ where $(q, \ell) = 1$. Define $W_i = \sum_{r=0}^{i} Lie_{\ell q^{k-r}}[p_{q^r}]$. Then W_i is Schur-positive for all $i = 0, \ldots, k$. Note that the W_i are all modules of dimension $(n-1)!$, and $W_0 = Lie_n$, while $W_k = Lie_n^{(q)}$.*

The construct of Definition 3.2 allows us to remove the restriction that q be prime, as follows. Let $k \geq 2$ be any positive integer, and take T to be the set of all nonnegative powers of k. In this case Theorem 3.1 gives

$$H[\sum_{n\geq 1} f_n^T] = \prod_{r\geq 0}(1 - p_{k^r})^{-1}, \qquad \sum_{n\geq 1} f_n^T = \sum_{r\geq 0} p_{k^r}[Lie]. \tag{3.11}$$

By inverting this equation plethystically, we obtain the recurrence

$$\text{For } k \geq 2, \qquad f_n^T = \begin{cases} Lie_n + f_{\frac{n}{k}}^T[p_k], & k|n; \\ Lie_n, & \text{otherwise,} \end{cases} \tag{3.12}$$

However computations show that for $k = 4$, f_n^T is not Schur-positive when $n = 4, 16$, and the degree 16 term in the product $\prod_{r\geq 0}(1 - p_{4^r})^{-1}$ is not Schur-positive. In both cases it is the sign representation that appears with coefficient (-1).

Conjecture 2. *For any ODD positive integer k, f_n^T as defined above is Schur-positive.*

Conjecture 3. *The product $\prod_{r\geq 0}(1-p_{k^r})^{-1}$ is Schur-positive for any ODD positive integer k.*

Fix $k \geq 2$ and consider the subset $T=\{1,k\}$. It was shown in [7, Theorem 4.23] that the symmetric function

$$W_{n,k} = \sum_{\mu\vdash n, \mu_i=1\, or\, k} p_\mu$$

is Schur-positive. Define $W_{0,k}=1$. Then

$$\sum_{n\geq 0} W_{n,k} = \prod_{n\in T}(1-p_n)^{-1} = (1-p_1)^{-1}(1-p_k)^{-1}.$$

For $k=1$ we set $W_{n,1}=p_1^n$ for all $n\geq 0$, so that the preceding equation reduces, as expected, to

$$\textstyle\sum_{n\geq 0} W_{n,1} = \prod_{n\in T}(1-p_n)^{-1} = (1-p_1)^{-1}.$$

Proposition 3.1. *If $T=\{1,k\}$ and $k\geq 2$, then*

$$f_n^T = \begin{cases} Lie_n + Lie_{\frac{n}{k}}[p_k], & k|n; \\ Lie_n, & \text{otherwise}, \end{cases} \tag{3.13}$$

and hence $\sum_{n\geq 0} W_{n,k} = H[\sum_{n\geq 0} f_n^T]$.

If k is prime, then $f_n^{\{1,k\}} = \mathrm{ch}(\exp\frac{2ki\pi}{n})\uparrow_{C_n}^{S_n} = \ell_n^{(k)}$, and hence the symmetric function defined by (3.13) is Schur-positive.

Proof. Equation (3.13) is immediate from Theorem 3.1. Now let k be prime. It was shown in [7, Lemma 5.5, Theorem 5.6] (see also Theorem 3.6 below) that the following identity holds:

$$\sum_{\lambda\vdash n:\lambda_i=1,k} p_\lambda = H[\sum_{m\geq 1} \mathrm{ch}\,(\exp(\frac{2\pi ik}{n})\uparrow_{C_n}^{S_n})]|_{\deg n},$$

and thus the left-hand side is precisely p^T for k prime and $T=\{1,k\}$. But $H-1$ is invertible with respect to plethysm (see Proposition 2.2), so $H[F]=H[G]$ if and only if $F=G$. Hence f_n^T must coincide with $\ell_n^{(k)} = \mathrm{ch}(\exp\frac{2ki\pi}{n})\uparrow_{C_n}^{S_n}$. □

Computations indicate that

Conjecture 4. *$f_n^{\{1,k\}}$ is Schur-positive for $k=2$ and for all odd $k\geq 3$. (This is trivially true if $k=1$.)*

When k is even and not equal to 2, this fails. For instance, if $n = k = 4m$, it is easy to see that $Lie_{4m} + p_{4m}$ contains the sign representation with coefficient (-1). However we have $H[F^{\{1,k\}}] = (1-p_1)^{-1}(1-p_k)^{-1}$, which we know to be Schur-positive from [7, Proposition 4.23]. This example shows that it is not always possible to write a Schur positive sum of power sums as a symmetrised module over a sequence of true S_n-modules, since $Lie_k + p_k$ fails to be Schur-positive when k is even.

Theorem 3.4. *Let $k \geq 2$ and $T = \{n : n \leq k\}$. Then*

$$f_n^T = \sum_{\substack{m=1 \\ m|n}}^{k} Lie_{\frac{n}{m}}[p_m] \quad \text{and} \quad \prod_{n=1}^{k}(1-p_n)^{-1} = H[\sum_n f_n^T]. \tag{3.14}$$

Corollary 3.2. *Let $T = \{n : n \leq k\}, k \geq 2$. If n is prime, or $n = k$, or $n > k$ and n is such that its greatest proper divisor is at most k, then f_n^T is Schur-positive.*

Conjecture 5. *(See also [7, Conjecture 1].) $f_n^{\{1,\ldots,k\}}$ is Schur positive for all n and k, and hence so is $\prod_{n=1}^{k}(1-p_n)^{-1}$.*

Theorem 3.5. *Let $k \geq 2$ and $T = \{n : n|k\}$. Then*

$$f_n^T = \sum_{m|(k,n)} Lie_{\frac{n}{m}}[p_m] \quad \text{and} \quad \prod_{n|k}(1-p_n)^{-1} = H[\sum_n f_n^T].$$

For $k \geq 2$ recall from Proposition 3.1 that $\ell_n^{(k)}$ denotes the characteristic of the Foulkes character $\mathrm{ch}\,(\exp(\frac{2\pi i k}{n}) \uparrow_{C_n}^{S_n})$.

Theorem 3.6. *[7, Lemma 5.5, Theorem 5.6] The following sum is Schur-positive:*

$$\sum_{\lambda \vdash n : \lambda_i | k} p_\lambda = H[\sum_{m \geq 1} \ell_m^{(k)}]|_{\deg n}.$$

In particular we immediately have

Corollary 3.3.

$$\ell_n^{(k)} = \mathrm{ch} \exp(2i\pi \cdot k/n) \uparrow_{C_n}^{S_n} = \sum_{m|(k,n)} Lie_{\frac{n}{m}}[p_m].$$

Hence we have the following curious decomposition of the regular representation into virtual representations:

Corollary 3.4.

$$p_1^n = \sum_{k=1}^{n} \sum_{m|(k,n)} Lie_{\frac{n}{m}}[p_m] = \sum_{d|n} d\, Lie_d[p_{\frac{n}{d}}]. \tag{3.15}$$

Proof. The first sum can be rewritten as

$$\sum_{m|n} \sum_{\substack{r=1 \\ k=rm\leq n}}^{\frac{m}{n}} Lie_{\frac{m}{n}}[p_m] = \sum_{m|n} \frac{m}{n}\, Lie_{\frac{m}{n}}[p_m].$$

The result now follows from the well-known decomposition [2, Theorem 8.8] $p_1^n = \sum_{k=1}^{n} \ell_n^{(k)}$ of the regular representation, and the preceding corollary. □

Theorem 3.7. *Let* $T = \{n : n \equiv 1 \bmod k\}$. *Then*

$$f_n^T = \sum_{\substack{m\equiv 1 \bmod k \\ m|n}} Lie_{\frac{n}{m}}[p_m] \quad \textit{and} \quad \prod_{n\equiv 1 \bmod k} (1-p_n)^{-1} = H[\sum_n f_n^T].$$

After seeing an early version of [7], Richard Stanley made the following conjecture, verifying it for $n \leq 24$ and $k \leq 6$.

Conjecture 6. *(R. Stanley, 2015)* $\prod_{n\equiv 1 \bmod k}(1-p_n)^{-1}$ *is Schur-positive for all* k.

Note that Conjecture 6 holds for $k = 2$. We have two different ways of identifying the associated S_n-module.

Theorem 3.8. *[7], [5] and [8, Corollary 3.11]*

$$\prod_{n\equiv 1 \bmod 2} (1-p_n)^{-1} = E[Conj] = H[\overline{L^{(2)}}].$$

The equivalence of the two plethystic expressions is a consequence of Proposition 2.1 in Section 2. In this case, writing p^{odd} for $\sum_{n \text{ odd}} p_n$, we have the identity $p^{\text{odd}}[Lie] = \overline{L^{(2)}}$, and hence:

Theorem 3.9. $\sum_{\substack{m \text{ odd} \\ m|n}} Lie_{\frac{n}{m}}[p_m] = p^{\text{odd}}[Lie]|_{\deg n}$ *is Schur-positive; it is the Frobenius characteristic* $\overline{L_n^{(2)}}$ *of the representation* $\exp(2i\pi\ell/n)\uparrow_{C_n}^{S_n}$, *where* $n = 2^k \cdot \ell$ *and* ℓ *is odd.*

Proof. This is clear by using (2.8) of Proposition 2.2 since the symmetric powers of the two modules coincide, both being equal to $\prod_{n \text{ odd}}(1-p_n)^{-1}$. □

The module $Lie_n^{(2)}$ whose many intriguing properties are described in [5] and [8], makes an appearance in the decomposition of the module $Conj_n$ of the conjugacy action on the class of n-cycles as well. In fact Theorem 3.3 leads to several different decompositions of $Conj_n$, some of which we collect in the following:

Theorem 3.10. *For any prime q, we have*

$$\sum_n Conj_n = \sum_{\substack{n \\ q \text{ does not divide } n}} p_n[Lie^{(q)}] = \sum_n p_n[Lie], \tag{3.16}$$

and hence the two sums on the right are Schur-positive.

In fact for any positive integer q we have

$$\sum_n Conj_n = \sum_{\substack{n \\ q \text{ does not divide } n}} p_n[\sum_{k\geq 0} Lie[p_{q^k}]]. \tag{3.17}$$

4 Reverse Lexicographic Order

The previous sections focused on sums of power sums for partitions with restricted parts. In our original paper [7], however, other families of sums were considered, and shown to be Schur positive by identifying the sum as the characteristic of actual S_n-modules. In this section we describe partial progress on a conjecture of [7]. More details and a numerical analysis of the number of such Schur positive families can be found in [9].

Recall that the reverse lexicographic order on partitions is defined as follows [1, p. 6]. For partitions λ, μ of the same integer n, we say a partition λ is preceded by a partition μ in reverse lexicographic order if $\lambda_1 > \mu_1$ or there is an index $j \geq 2$ such that $\lambda_i = \mu_i$ for $i < j$ and $\lambda_j > \mu_j$. Thus for $n = 4$ we have the total order $(1^4) < (2, 1^2) < (2^2) < (3, 1) < (4)$. In particular our convention is that the minimal and maximal elements in this total order are (1^n) and (n) respectively.

Conjecture 7. *[7, 9, Conjecture 1] Let L_n denote the reverse lexicographic ordering on the set of partitions of n. Then the sum of power sum symmetric functions $\sum p_\lambda$, taken over any initial segment of the total order L_n, i.e. any interval of the form $[(1^n), \mu]$ for fixed μ, (and thus necessarily including the partition (1^n)), is Schur-positive.*

We are able to prove the following special cases of this conjecture.

Theorem 4.1. *[9, Theorem 18, Theorem 23, Proposition 36] The symmetric function $\psi_\mu = \sum_{(1^n)\leq\lambda\leq\mu} p_\lambda$ is Schur-positive if $\mu \leq (3, 1^{n-3})$ or $\mu \geq (n-4, 1^4)$ in reverse lexicographic order, and also if $\mu = (3, 2^k, 1^r)$ for $k \geq 1$ and $0 \leq r \leq 2$.*

By a theorem of Solomon [3], $\psi_{(n)} = \psi_n$ is the Frobenius characteristic of S_n acting on itself by conjugation. The proof for the Schur positivity of ψ_μ when μ lies in the interval $[(1^n), (3, 1^{n-3})]$ relies on some interesting symmetric function identities that we believe are new. The other half of this theorem is established by using bounds on the multiplicity of the irreducibles occurring in ψ_n. Specifically, we use

Theorem 4.2. *[6, Theorem 5.1] Let $n \neq 4, 8$. Then the conjugacy class indexed by a partition λ contains all irreducibles if and only if λ has at least two parts, and all its parts are distinct and odd. If $n = 8$, the conjugacy class indexed by $(7, 1)$ contains all irreducibles, while the class of the partition $(5, 3)$ contains all irreducibles except those indexed by (4^2) and (2^4).*

This in turn allows us to establish the following bounds:

Lemma 4.1. *[6, Lemma 2.6] Let $n \geq 5$. Let do_n denote the number of partitions of n with at least two parts and with all parts odd and distinct. In the conjugacy representation ψ_n, every irreducible except possibly the sign occurs with multiplicity at least $\begin{cases} 4 + do_n, & n \text{ odd;} \\ 3 + do_n, & n \text{ even.} \end{cases}$ This number is at least 5 for odd $n \geq 7$, and at least 4 for even $n \geq 6$.*

The proof of Theorem 4.1 then proceeds by a careful (and tedious) analysis showing that the multiplicity of any irreducible in ψ_μ cannot exceed the above bounds in absolute value. In contrast to the previous proofs of Schur positivity, we are unable to identify a representation-theoretic context for this symmetric function. It would of course be of interest to find such a context.

Finally, we remark (see [9]) that it is easy to find counterexamples showing that the analogous conjecture is false for dominance order.

References

[1] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. 2nd ed. Oxford University Press, 1995.

[2] C. Reutenauer. *Free Lie Algebras*. London Mathematical Society Monographs. Oxford Science Publications, 1993.

[3] L. Solomon. "On the sum of the elements in the character table of a finite group". *Proc. Amer. Math. Soc.* **12**.6 (1961), pp. 962–963. Link.

[4] R. P. Stanley. *Enumerative Combinatorics. Vol. 2*. Cambridge Studies in Advanced Mathematics 62. Cambridge University Press, Cambridge, 1999.

[5] S. Sundaram. "On a variant of Lie_n". *Sém. Lothar. Combin.* **80B** (Proceedings of FPSAC'18) (2018), Art. 19, 12 pp. Link.

[6] S. Sundaram. "On conjugacy classes of S_n containing all irreducibles". *Israel J. Math.* **225**.1 (2018), pp. 321–342. Link.

[7] S. Sundaram. "The conjugacy action of S_n and modules induced from centralisers". *J. Algebraic Combin.* **48**.2 (2018), pp. 179–225. Link.

[8] S. Sundaram. "Variations on the S_n-module Lie_n". 2018. arXiv:1803.09368.

[9] S. Sundaram. "On a positivity conjecture in the character table of S_n". *Electron. J. Combin.* **26**.1 (2019), P1.1, 43 pp. Link.