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# On random shifted standard Young tableaux and 132-avoiding sorting networks

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**Abstract.** We study shifted standard Young tableaux (SYT). The limiting surface of uniformly random shifted SYT of staircase shape is determined, with the integers in the SYT as heights. This implies via properties of the Edelman–Greene bijection results about random 132-avoiding sorting networks, including limit shapes for trajectories and intermediate permutations. Moreover, the expected number of adjacencies in SYT is considered. It is shown that on average each row and each column of a shifted SYT of staircase shape contains precisely one adjacency.

**Keywords:** Shifted standard Young tableaux, random 132-avoiding sorting networks

A shifted standard Young tableau (SYT) of staircase shape is an increasing filling of the shifted diagram of the partition $(n-1,\ldots,2,1)$ with the integers $1,2,\ldots,\binom{n}{2}$. See Figure 1a for an example and Section 1 for the exact definition.

Shifted diagrams and tableaux are important combinatorial objects that appear in various contexts. In representation theory shifted Young diagrams correspond to projective characters of the symmetric group, and shifted tableaux lend themselves to being studied via RSK-type methods. In the theory of partially ordered sets shifted diagrams alongside non-shifted Young diagrams and rooted trees form the three most interesting families of $d$-complete posets. The most salient property of $d$-complete posets is the fact that their linear extensions (in our case shifted SYT) are enumerated by elegant product formulas. Shifted diagrams also appear as order filters in the root poset of type $B_n$, and shifted SYT play an important role in the enumeration of reduced words of elements of the Coxeter group of type $B_n$. Moreover, as is topical in this paper, shifted SYT are also relevant to the study of certain reduced words in the symmetric group.

The topics of this paper can be divided into three parts.

In Section 2 we study the surface obtained by viewing the integers in random SYT as heights. The study of limit phenomena for partitions and tableaux is an active field of research combining methods from combinatorics, probability theory and analysis.

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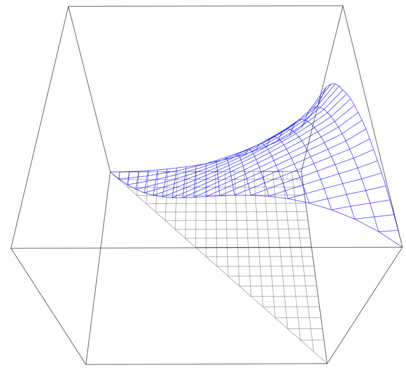
| | | | |
|---|---|---|---|
| 1 | 2 | 4 | 5 |
| | 3 | 6 | 7 |
| | | 8 | 9 |
| | | | 10 |

(a) A shifted SYT of staircase shape.



(b) The limit shape of uniformly random shifted SYT of staircase shape.

**Figure 1**

We refer to [16] for a general survey. Shifted objects have been treated as well, for example Ivanov [8] proves a central limit theorem for the Plancherel measure on shifted diagrams. In the present paper we determine the limiting surface for uniformly random shifted SYT of staircase shape, see Figure 1b and Theorem 2.3. The deduction of our results relies on a paper by Pittel and Romik [14] where the limit shape for random rectangular SYT is determined. In fact, we end up with the same variational problem, and the limit surface for shifted staircase SYT is the surface for square SYT restricted to a triangle. This analogy is in part explained by a combinatorial identity (2.1) relating shifted and non-shifted tableaux. There are very few shapes for which the limit surface has been determined previously. As far as we know the only other case is that of staircase SYT, where again the same limit surface appears, but cut along a different diagonal [1]. Results of this type have applications in other fields of mathematics such as geometric complexity theory [13].

Secondly, we study *132-avoiding sorting networks*, which are by definition reduced words $w_1 \dots w_{\binom{n}{2}}$ of the reverse permutation such that $s_{w_1} \cdots s_{w_k}$ is 132-avoiding for all $1 \leq k \leq \binom{n}{2}$. These objects have received considerable recent interest and also appear in different guises, for example as chains of maximum length in the Tamari lattice [2]. Fishel and Nelson [6] showed that 132-avoiding sorting networks are in bijection with shifted SYT of staircase shape via the Edelman–Greene correspondence. This has been rediscovered several times [17, 4, 10]. They are also in bijection with reduced words of the signed permutation $(-(n-1), -(n-2), \dots, -1)$ via the shift $s_i \mapsto s_{i-1}$ as was remarked in [17, Sec. 1.3]. In Section 3 the Edelman–Greene bijection is used to transfer the limit shape of shifted SYT to determine the limit shapes of intermediate permutations (Theorem 3.2) and trajectories (Theorem 3.3) in random 132-avoiding sorting networks. These results are motivated by a remarkable paper of Angel, Holroyd, Romik and Virág [1] that contains a number of tantalising conjectures about random sorting networks, now proven by Dauvergne [3]. See Section 1 for a description of some of them. Our results are a parallel to their (former) conjectures restricted to a subclass of random sorting networks.

We remark that the limit surface for shifted SYT of staircase shape contains complete information on the limit surface for SYT of square shape. This suggests the perhaps less intuitive idea that the relatively small subset of 132-avoiding sorting networks contains a lot of information on random sorting networks in general.

The third set of results is obtained in Section 4 and concerns patterns in 132-avoiding sorting networks. We first observe that adjacencies in a shifted SYT (that is, integers $i$ and $i+1$ in neighbouring cells) translate directly to adjacencies in a 132-avoiding sorting network (that is, $j$ and $j+1$ next to each other in the reduced word). Corollary 4.3 asserts that the expected number of adjacencies in each column and each row in a shifted SYT of staircase shape is exactly 1. The proof uses promotion and evacuation techniques very similar to the methods used by Schilling, Thiéry, White and Williams [17] to derive results on Yang–Baxter moves (that is, patterns of the form $j(j\pm 1)j$) in 132-avoiding sorting networks. Related results on general sorting networks are due to Reiner [15] and Tenner [18].

This is an extended abstract of [11].

# 1 Background

In this section we fix notation and review some facts about partitions, tableaux and random sorting networks. For $n \in \mathbb{N}$ let $[n] = \{1, \ldots, n\}$. Throughout this paper we denote $N = \binom{n}{2}$.

A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of positive integers. If a partition is strictly decreasing it is called *strict*. The sum $\sum \lambda_i$ is called the *size* of the partition $\lambda$ and is denoted by $|\lambda|$. The number of entries $\lambda_i$ is called the *length* of the partition and is denoted by $\ell(\lambda)$. Define the *staircase* partition as $\Delta_n = (n-1, \ldots, 2, 1)$. The *Young diagram* of a partition $\lambda$ is defined as the set $\lambda^{\mathrm{dg}} = \{(i,j) : i \in [\ell(\lambda)], j \in [\lambda_i]\}$. The elements $(i,j)$ are indexed with matrix notation and typically referred to as *cells* of $\lambda$. Given a strict partition $\lambda$ we also define its *shifted Young diagram* as

$$\lambda^{\mathrm{sh}} = \{(i, j+i-1) : i \in [\ell(\lambda)], j \in [\lambda_i]\}.$$

Thus the shifted Young diagram is obtained from the normal Young diagram by shifting rows to the right, row $i$ by $i-1$ steps.

A *tableau* of *shape* $\lambda^{\mathrm{dg}}$ is a map $T : \lambda^{\mathrm{dg}} \to \mathbb{Z}$. A tableau $T$ is called a *standard Young tableau (SYT)* if $T : \lambda^{\mathrm{dg}} \to [n]$ is a bijection and $T(i,j) < T(i,j+1)$ and $T(i,j) < T(i+1,j)$ whenever the respective cells lie in $\lambda^{\mathrm{dg}}$. Similarly a *shifted standard Young tableau* of *shape* $\lambda^{\mathrm{sh}}$ is a bijection $T : \lambda^{\mathrm{sh}} \to [n]$ such that $T(i,j) < T(i,j+1)$ and $T(i,j) < T(i+1,j)$

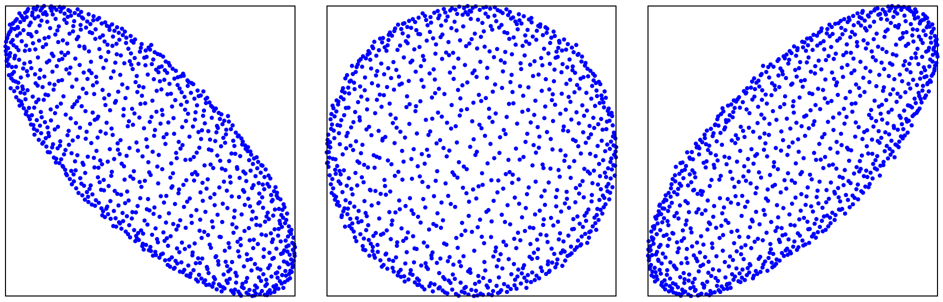


**Figure 2:** The intermediate permutation matrices of $\sigma_{\lfloor \alpha N \rfloor}$ of a 1000-element random sorting network at times $\alpha = \frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$.

whenever the respective cells lie in the shifted Young diagram $\lambda^{\mathrm{sh}}$. Let $\mathcal{T}_n$ denote the set of shifted SYT of shape $\Delta_n^{\mathrm{sh}}$. For example, Figure 1a shows a shifted standard Young diagram of shape $\Delta_5^{\mathrm{sh}}$. The number $f^\lambda$ of SYT of shape $\lambda$ and the number $f_{\mathrm{sh}}^\lambda$ of shifted SYT of shape $\lambda^{\mathrm{sh}}$ are given by nice product formulas involving the hook-lengths of these diagrams [7, 19].

For $i \in [n-1]$ let $s_i = (i, i+1)$ denote the $i$-th *adjacent transposition*. The *reverse permutation* $w_0 \in \mathfrak{S}_n$ is defined by $w_0(i) = n - i + 1$ for $i \in [n]$. A *reduced word* of $w_0$ is a word $w = w_1 \cdots w_N$ in the alphabet $[n-1]$ such that $w_0 = s_{w_1} \cdots s_{w_N}$.

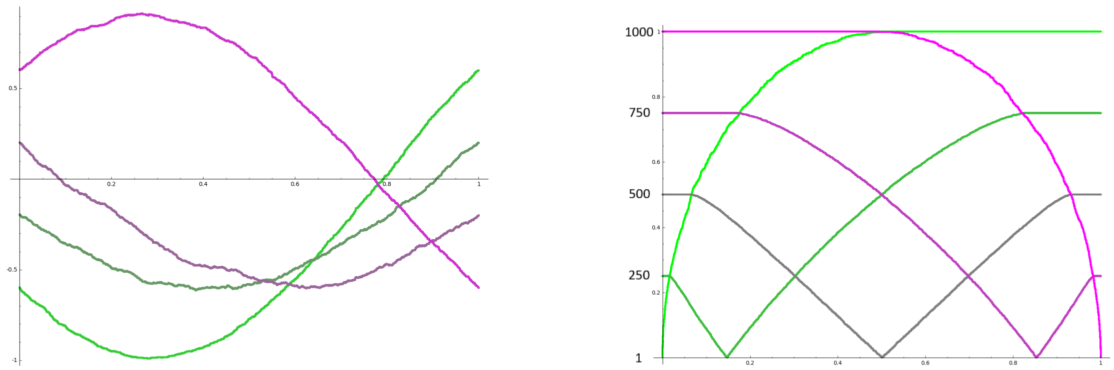
Angel, Holroyd, Romik and Virág introduced *n-element random sorting networks* as the set of reduced words of the reverse permutation $w_0 \in \mathfrak{S}_n$ equipped with the uniform probability measure in [1]. In the same paper, Angel et al. pose several striking conjectures about random sorting networks, which have now been proven by Dauvergne [3].

Suppose $w = w_1 \ldots w_N$ is a sorting network. Then $w_1 \ldots w_k$ defines the *intermediate permutation* $\sigma_k = s_{w_1} \cdots s_{w_k} \in \mathfrak{S}_n$ for all $k \in [N]$. One of the consequences of [3, Thm. 2] (previously [1, Conj. 2]) is that asymptotically the 1s in the permutation matrices of intermediate configurations of random sorting networks lie inside ellipses. In particular, at half-time the permutation matrix is supported on a disc. Figure 2 provides an illustration.

For $0 \leq \alpha \leq 1$, [3, Thm. 1] (previously [1, Conj. 1]) states that the scaled trajectories defined by $f_{w,i}(\alpha) = 2\sigma_{\alpha N}^{-1}(i)/n - 1$ for $\alpha N \in \mathbb{Z}$, and by linear interpolation otherwise, converge to random sine curves. See Figure 3a.

The *permutahedron* is an embedding of $\mathfrak{S}_n$ into a sphere in $\mathbb{R}^n$ defined by $\sigma \mapsto (\sigma^{-1}(1), \ldots, \sigma^{-1}(n))$. Random sorting networks correspond to paths on the permutahedron. The strongest theorem, [3, Thm. 4] (previously [1, Conj. 3]), which implies both of the previous ones, states that these paths are close to great circles.

This paper considers similar questions restricted to *132-avoiding sorting networks*, that is, those reduced words $w_1 \ldots w_N$ of the reverse permutation in $\mathfrak{S}_n$ such that $s_{w_1} \cdots s_{w_k}$




(a) (b)

**Figure 3:** Scaled trajectories in a random (a) ordinary and (b) 132-avoiding sorting network with 1000 elements.

is 132-avoiding for all $k \in [N]$. For background on pattern avoidance in permutations the reader is referred to [9]. With a *random 132-avoiding sorting network*, we will refer to uniform distribution among all such networks of the same length.

The connection between 132-avoiding sorting networks and shifted SYT is the following. Let $w = w_1 \ldots w_N$ be a 132-avoiding sorting network. Then, for $k \in [N]$, define a SYT $Q_{w_1 \ldots w_k}$ by letting its $j$-th column consist of the indices $m \in [k]$ such that $w_m = j$. Furthermore, define a shifted SYT $Q^{\rightarrow}_{w_1 \ldots w_k}$ by shifting the rows of $Q_{w_1 \ldots w_k}$. For example, the reduced word $w = 1213423121$ corresponds to the shifted SYT in Figure 1a.

**Theorem 1.1** ([6, Thm. 3.3 and Thm. 4.6])**.** *For all $n \in \mathbb{N}$, the map $w \mapsto Q^{\rightarrow}_w$ is a bijection from $n$-element 132-avoiding sorting networks to shifted SYT of shape $\Delta_n^{\mathrm{sh}}$. The map $w \mapsto Q_w$ agrees with the restriction of the Edelman–Greene correspondence to 132-avoiding sorting networks.*

The same bijection was also described in [10], [17, Fig. 4] in terms of heaps, and [4, Prop. 5.2] in terms of descent sets.

Reversing a sorting network preserves the property of being 132-avoiding. See [10] for a proof.

**Proposition 1.2.** *A reduced word $w_1 \ldots w_N$ is a 132-avoiding sorting network if and only if $w_N \ldots w_1$ is.*

## 2 The limit shape

In this section we present a limit shape for random shifted SYT of staircase shape. We may interpret a shifted SYT $T \in \mathcal{T}_n$ as the graph of a function $L_T$ by viewing the entries

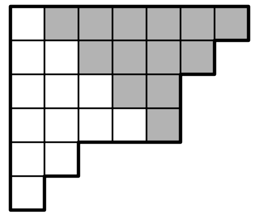


**Figure 4:** The shift-symmetric partition $\Lambda = (7,6,5,5,2,1)$ of the strict partition $\lambda = (6,4,2,1)$.

as heights. Our main result, Theorem 2.3, states that with a suitable choice of scaling, the functions $L_{T^{(n)}}$ converge with probability 1 to the surface depicted in Figure 1b, for a sequence of tableaux $T^{(n)} \in \mathcal{T}_n$ chosen uniformly at random. The proof of Theorem 2.3 relies heavily on the work of Pittel and Romik [14]. In this extended abstract we only outline the ideas that lead to Theorem 2.3.

An important concept for connecting the shifted case to the setting of Pittel and Romik is the following definition. Given a strict partition $\lambda$ define a partition $\Lambda$ by letting its Young diagram equal

$$\Lambda^{\mathrm{dg}} = \left\{(i, j+1) : (i,j) \in \lambda^{\mathrm{sh}}\right\} \cup \left\{(j,i) : (i,j) \in \lambda^{\mathrm{sh}}\right\}.$$

It is easy to see that this really is the Young diagram of a partition. See Figure 4. We call $\Lambda$ the *shift-symmetric* partition corresponding to $\lambda$. The motivation for this definition is the fact that shifted hook-lengths of the cells in $\lambda^{\mathrm{sh}}$ correspond to hook-lengths of cells in $\Lambda^{\mathrm{dg}}$.

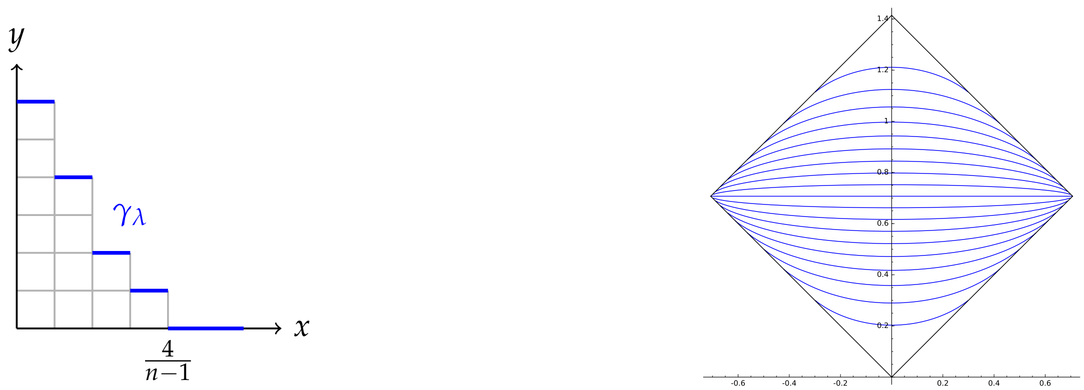
The following proposition is not new and can be obtained by extracting coefficients from an identity of symmetric functions found in [12, Chap. III, Sec. 8, Example 9.(b)]. Alternatively, one can give a simple proof using induction and the hook-length formula.

**Proposition 2.1.** *Let $\lambda$ be a strict partition and $\Lambda$ its shift-symmetric partition. Then*

$$f^{\Lambda} = (f^{\lambda}_{\mathrm{sh}})^2 \cdot \binom{2\,|\lambda|}{|\lambda|} \cdot 2^{-\ell(\lambda)}. \tag{2.1}$$

Proposition 2.1 allows us to transfer many of the results of Pittel and Romik to the shifted staircase. The first important step is to obtain an analogue of [14, Lem. 1], which introduces a *hook-integral* in order to estimate the probability that a fixed subdiagram $\lambda^{\mathrm{sh}} \subseteq \Delta_n^{\mathrm{sh}}$ contains precisely the integers $1, \ldots, k$ in a random SYT of shape $\Delta_n^{\mathrm{sh}}$ chosen uniformly at random.

Fix $n$ and let $\lambda$ be a partition with $|\lambda| = k$. Define a function $\gamma_\lambda : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ by $\gamma_\lambda(x) = \lambda_{\lceil (n-1)x \rceil}/(n-1)$, where by convention $\lambda_i = 0$ for $i > \ell(\lambda)$. See Figure 5a.




(a) The function $\gamma_\lambda$ for $\lambda = (6, 4, 2, 1)$.

(b) The curves $v = \tilde{g}_\alpha(u)$ for $\alpha = 0.05, 0.1, \ldots, 0.95$.

**Figure 5**

**Lemma 2.2.** *Let $\alpha \in (0, 1)$, $k = k(n)$ be a sequence such that $k/N \to \alpha$ as $n \to \infty$, and let $\mathbb{P}_n$ denote the uniform probability measure on $\mathcal{T}_n$. Then, as $n \to \infty$,*

$$\mathbb{P}_n\big(T \in \mathcal{T}_n : T(\lambda^{\mathrm{sh}}) = [k]\big) = \exp\Big( - \big(1 + o(1)\big)\frac{n^2}{2}\big(I(\gamma_\Lambda) + H(\alpha) + C\big)\Big)$$

*uniformly over all strict partitions $\lambda$ of $k$ with $\lambda_1 < n$, where*

$$C = \frac{3}{2} - 2\ln 2, \qquad I(\gamma) = \int_0^1 \int_0^1 \ln\Big|\gamma(x) + \gamma^{-1}(y) - x - y\Big| \,\mathrm{d}y\mathrm{d}x,$$

$$H(\alpha) = -\alpha \ln(\alpha) - (1-\alpha)\ln(1-\alpha), \quad \gamma^{-1}(y) = \inf\{x \in [0,1] : \gamma(x) \le y\}\,,$$

*and $\Lambda$ denotes the shift-symmetric partition of $\lambda$.*

Results of the type of Lemma 2.2 lead to a so-called large deviation principle, which heuristically can be understood as follows: Suppose that $n$ is large, and let $\lambda$ be the strict partition of size $k = \lfloor \alpha N \rfloor$ with shift-symmetric partition $\Lambda$ minimising $I(\gamma_\Lambda)$. Let $\mu$ be another strict partition of size $k$ with shift-symmetric partition $M$, which deviates from $\lambda$, meaning that $I(\gamma_M) > I(\gamma_\Lambda) + \varepsilon$. Then the probability that $\mu$ contains the numbers $1, \ldots, k$ in a random shifted SYT $T \in \mathcal{T}_n$ is exponentially smaller (by a factor $\exp(-\varepsilon n^2/2)$) than the probability that $\lambda$ contains these numbers. Hence the shape formed by the entries $1, \ldots, k$ in a random shifted SYT will be close to the minimising partition $\lambda$ with high probability. This leads to the variational problem of identifying the function $\gamma$ in a certain search space depending on $\alpha$ that minimises the integral $I(\gamma)$.

A function $g : [-\sqrt{2}/2, \sqrt{2}/2] \to [0, \sqrt{2}]$ is called *$\alpha$-admissible* if it is 1-Lipschitz and satisfies

$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \big(g(u) - |u|\big)\,\mathrm{d}u = \alpha\,.$$

As is explained in [14, Sec. 2.2] our problem is equivalent the following formulation: For each $\alpha \in (0,1)$ find the unique $\alpha$-admissible function $g$ which is symmetric, that is, $g(-u) = g(u)$, and minimises the integral

$$K(g) = -\frac{1}{2}\int_{-\sqrt{2}/2}^{\sqrt{2}/2}\int_{-\sqrt{2}/2}^{\sqrt{2}/2} g'(s)g'(t)\ln|s-t|\,\mathrm{d}s\mathrm{d}t\,. \tag{2.2}$$

The only difference between our situation and the situation in [14] is the fact that our search space is a subset of theirs since we require that $\Lambda$ is the shift-symmetric partition of a strict partition. In [14, Sec. 2 and 3] Pittel and Romik show that the variational problem (2.2) without the assumption $g(-u) = g(u)$ has the unique solution $\tilde{g}_\alpha$. The family of functions $\tilde{g}_\alpha$ is illustrated in Figure 5b. Since this solution already exhibits the additional symmetry $\tilde{g}_\alpha(-u) = \tilde{g}_\alpha(u)$, we may apply it to the shifted case as well.

Our limit shape result is an analogue of [14, Thm. 1]. The obtained limit shape is the same as the limit shape for random SYT of square shape except that the domain is restricted from a square to a triangle. Let

$$\overline{L} : \big\{(u,v) \in \mathbb{R}^2 : -\sqrt{2}/2 \leq u \leq 0, |u| \leq v \leq \sqrt{2} - |u|\,\big\} \to \mathbb{R}_{\geq 0}$$

be the surface defined by the level curves given by $\tilde{g}_\alpha(u)$ for $\alpha \in (0,1)$. Let

$$L : \big\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq 1\big\} \to \mathbb{R}_{\geq 0}$$

be the rotated version of $\overline{L}$.

**Theorem 2.3.** *For $n \in \mathbb{N}$ let $\Delta_n$ denote the staircase partition of size $N = \binom{n}{2}$, $\mathcal{T}_n$ the set of shifted SYT of shape $\Delta_n^{\mathrm{sh}}$, and $\mathbb{P}_n$ the uniform probability measure on $\mathcal{T}_n$. Then for all $\epsilon > 0$*

$$\lim_{n\to\infty} \mathbb{P}_n\left(T \in \mathcal{T}_n : \max_{(i,j)\in\Delta_n^{\mathrm{sh}}} \left|\frac{T(i,j)}{N} - L\Big(\frac{i}{n},\frac{j}{n}\Big)\right| > \epsilon\right) = 0\,. \tag{2.3}$$

*Moreover for all $p \in (0,1/2)$ and all $q \in (0,p/2)$ such that $2p+q<1$*

$$\lim_{n\to\infty} \mathbb{P}_n\left(T \in \mathcal{T}_n : \max_{\substack{(i,j)\in\Delta_n^{\mathrm{sh}}\\ \sigma(i/n,j/n)>n^{-q}}} \left|\frac{T(i,j)}{N} - L\Big(\frac{i}{n},\frac{j}{n}\Big)\right| > n^{-p}\right) = 0\,, \tag{2.4}$$

*where $\sigma(x,y) = \min\{xy,(1-x)(1-y)\}$.*

In particular (2.3) provides point-wise convergence to the limit surface, while (2.4) specifies the rate of convergence if we assume a sufficient distance to the sides.

# 3 Intermediate permutations and trajectories

This section considers the limit of intermediate permutation matrices in random 132-avoiding sorting networks, a parallel to [3, Thm. 2] (previously [1, Conj. 2]). The (Rothe) *diagram* $D(\sigma)$ of a permutation $\sigma$ is the set of cells left unshaded when we shade all the cells weakly to the east and south of 1-entries in the permutation matrix $M(\sigma)$.

**Theorem 3.1** ([10, Thm. 3.1, Cor. 3.4]). *Let $w = w_1 \cdots w_N$ be a 132-avoiding sorting network. Then the Young diagram of the shape of $Q_{w_1 \dots w_k}$ is $D(\sigma_k)$.*

Hence, the diagrams of intermediate permutation matrices are obtained by rotating and shifting the level curves $v = \tilde{g}_\alpha(u)$ of the limit surface $\bar{L}$.

**Theorem 3.2.** *Let $\sigma_0 = id$ and $\sigma_k = s_{w_1} \cdots s_{w_k}$ for $k \in [N]$, where $w = w_1 \dots w_N$ is a sorting network. Let $\mathbb{P}_n$ be the uniform probability measure on $\mathcal{R}_n^{132}$, the set of $n$-element 132-avoiding sorting networks. Finally, let*

$$J_w(\alpha) = \{ j \in [n] : \sigma_{\lfloor \alpha N \rfloor}(j) \le \sigma_{\lfloor \alpha N \rfloor}(1) \}$$

*and $J_w^c(\alpha) = [n] \setminus J_n(\alpha)$. For all $0 \le \alpha \le 1$, $\epsilon > 0$,*

$$\mathbb{P}_n \left( w \in \mathcal{R}_n^{132} : \max_{j \in J_w(\alpha)} \left| \frac{\sigma_{\lfloor \alpha N \rfloor}(j)}{n} - \frac{1}{\sqrt{2}} \left( \tilde{g}_\alpha \left( \frac{-j}{n\sqrt{2}} \right) - \frac{j}{n\sqrt{2}} \right) \right| > \epsilon \right) \to 0,$$

*as $n \to \infty$. By symmetry, for all $0 \le \alpha \le 1$, $\epsilon > 0$,*

$$\mathbb{P}_n \left( w \in \mathcal{R}_n^{132} : \max_{j \in J_w^c(\alpha)} \left| \frac{\sigma_{\lfloor \alpha N \rfloor}(j)}{n} + \frac{1}{\sqrt{2}} \left( \tilde{g}_{1-\alpha} \left( \frac{-j}{n\sqrt{2}} \right) - \frac{j}{n\sqrt{2}} \right) - 1 \right| > \epsilon \right) \to 0,$$

*as $n \to \infty$.*

Next, inspired by the former sine trajectories conjecture [1, Conj. 1] of Angel et al. (now [3, Thm. 1]), we study trajectories in random 132-avoiding sorting networks. The *trajectory* of the element $i \in [n]$ in $w = w_1 \dots w_N$ is the function $k \mapsto \sigma_k^{-1}(i)$. The *scaled trajectory* $f_i(\alpha) = f_{w,i}(\alpha)$ of $i$ in an $n$-element 132-avoiding sorting network $w$ is defined by $f_i(\alpha) = \sigma_{\alpha N}^{-1}(i)/n$ for $\alpha N \in \mathbb{Z}$, and by linear interpolation for other $\alpha \in [0, 1]$. Figure 3b contains some examples. Using Theorem 3.1 and Proposition 1.2, we get that the trajectories are given by the limit shape as follows.

**Theorem 3.3.** *Fix $m/n = \beta$. Let $D_\beta = \{ (x, y) \in \mathbb{R}^2 : 0 \le x \le y \le 1, y = \beta \text{ or } x = \beta \}$. Define $\mathfrak{f}_\beta(\alpha) = y - x$, where $L^{-1}(\alpha) = (x, y) \in D_\beta$, and*

$$\mathfrak{t}_\beta(\alpha) = \begin{cases} \beta & \text{if } 0 \le \alpha \le \frac{1 - \sqrt{1-\beta^2}}{2}, \\ \mathfrak{f}_\beta(\alpha) & \text{if } \frac{1-\sqrt{1-\beta^2}}{2} < \alpha < \frac{1+\sqrt{2\beta - \beta^2}}{2}, \\ 1 - \beta & \text{if } \frac{1+\sqrt{2\beta-\beta^2}}{2} \le \alpha \le 1. \end{cases}$$

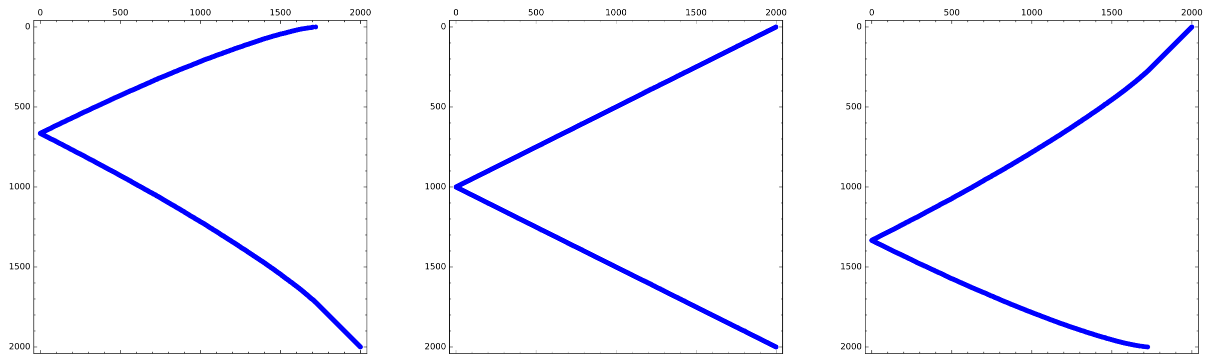



**Figure 6:** The intermediate permutation matrices of a random 132-avoiding sorting network with 1000 elements at times $\alpha = \frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$.

*Finally, let $\mathcal{R}_n^{132}$ denote the set of n-element 132-avoiding sorting networks and let $\mathbb{P}_n$ be the uniform probability measure on $\mathcal{R}_n^{132}$. Then for all $\epsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}_n\Big(w \in \mathcal{R}_n^{132} : \sup_{0\le\alpha\le1} \big|f_{w,\lfloor\beta n\rfloor}(\alpha) - \mathfrak{t}_\beta(\alpha)\big| > \epsilon\Big) = 0.$$

Informally, we can trace the trajectory of $\lfloor\beta n\rfloor$ by following the limit shape along $y = \beta$ until $x = y$, and then along $x = \beta$. If the height $\alpha = L(x', y')$ is given by some point $(x', y')$ along this curve, then the trajectory of $\lfloor\beta n\rfloor$ is at height $y' - x'$ at time $\alpha$. This combined with the implicit definition of $L(x, y)$ means that it is difficult to compute the trajectories of arbitrary elements explicitly. However, an accessible special case, the scaled trajectory of 1 (the curve starting at the origin in Figure 3b) converges in probability to

$$\mathfrak{t}_0(\alpha) = \begin{cases} 2\sqrt{\alpha - \alpha^2} & \text{if } 0 \le \alpha \le \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < \alpha \le 1. \end{cases}$$

By symmetry, the trajectory of $n$ is given by the transformation $\alpha \mapsto 1 - \alpha$.

# 4 Adjacencies

Motivated by the former great circle conjecture [1, Conj. 3] (now [3, Thm. 4]) and trying to understand the geometry of random 132-avoiding sorting networks on the permutahedron, we next study adjacencies.

Let $w$ be a reduced word of the longest element in $\mathfrak{S}_n$. An index $k \in [N-1]$ is called an *adjacency* of $w$ if $|w_{k+1} - w_k| = 1$. In the permutahedron, an adjacency corresponds to a pair of adjacent edges at an angle of $\frac{\pi}{3}$. In the case of $|w_{k+1} - w_k| > 1$ the edges corresponding to $w_k$ and $w_{k+1}$ are orthogonal. Adjacencies in a 132-avoiding sorting network $w$ correspond directly to adjacencies in the SYT $Q_w^{\rightarrow}$ as follows.

Let $\lambda^{\mathrm{dg}}$ be a Young diagram. A pair $(T, u)$ of a cell $u = (i, j) \in \lambda^{\mathrm{dg}}$ and a standard tableaux $T$ of shape $\lambda$ is called a *horizontal adjacency* if $T(i, j+1) = T(u) + 1$. The pair $(T, u)$ is called a *vertical adjacency* if $T(i+1, j) = T(u) + 1$. The same definitions apply to shifted diagrams $\lambda^{\mathrm{sh}}$. For example, consider the tableau of Figure 1a. Then the horizontal adjacencies are $(T, (1,1))$, $(T, (1,3))$, $(T, (2,3))$ and $(T, (3,3))$, whereas $(T, (1,2))$ and $(T, (3,4))$ are the vertical adjacencies.

**Proposition 4.1.** *Let $w$ be a 132-avoiding sorting network. Then $(w_k, w_{k+1}) = (j, j+1)$ if and only if $(Q_{\vec{w}}, (Q_{\vec{w}})^{-1}(k))$ is a horizontal adjacency. Similarly $(w_k, w_{k+1}) = (j+1, j)$ if and only if $(Q_{\vec{w}}, (Q_{\vec{w}})^{-1}(k))$ is a vertical adjacency.*

We next enumerate adjacencies in Young tableaux.

**Theorem 4.2.** *Let $\lambda$ be a (possibly strict) partition and $A_c(\lambda)$ be the set of horizontal adjacencies $(T, (i, c))$ in column $c$ of any (possibly shifted) standard Young tableau $T$ of shape $\lambda$. Then $|A_c(\lambda)|$ is equal to the number of (possibly shifted) SYT of shape $\lambda$ with largest entry in or after column $c+1$.*

Techniques similar to those used in the proof of Theorem 4.2 also appear in [17]. Theorem 4.2 implies the following two results on the expected number of adjacencies.

**Corollary 4.3.** *The expected number of horizontal (resp. vertical) adjacencies in column $c < n-1$ (resp. row $r < n-1$) of a uniformly random shifted staircase SYT is equal to 1.*

**Corollary 4.4.** *The expected number of adjacencies in a random 132-avoiding sorting network of length $N$ is $2(n-2)$.*

Compare this with the result of Schilling et al. below.

**Theorem 4.5** ([17, Thm. 1.3])**.** *The expected number of $i\ (i+1)\ i$, $1 \le i \le n-1$, in a random 132-avoiding sorting network of length $N$ is 1.*

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