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A quotient of the ring of symmetric functions generalizing quantum cohomology

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Abstract. Consider the ring $\mathcal{S}$ of symmetric polynomials in k variables over an arbitrary base ring $\mathbf{k}$. Fix k scalars $a_1, a_2, \ldots, a_k \in \mathbf{k}$. Let I be the ideal of $\mathcal{S}$ generated by $h_{n-k+1} - a_1, h_{n-k+2} - a_2, \ldots, h_n - a_k$, where h_i is the i-th complete homogeneous symmetric polynomial.

The quotient ring $\mathcal{S}/I$ generalizes both the usual and the quantum cohomology of the Grassmannian.

We show that $\mathcal{S}/I$ has a $\mathbf{k}$-module basis consisting of (residue classes of) Schur polynomials fitting into a $k \times (n-k)$-rectangle; and that its multiplicative structure constants satisfy the same S_3-symmetry as those of the Grassmannian cohomology. We conjecture the existence of a Pieri rule (proven in two particular cases) and a positivity property generalizing that of Gromov-Witten invariants.

Keywords: symmetric functions, partitions, Schur functions, Gröbner bases, Grassmannian, cohomology

1 Introduction

Schubert calculus – the quantitative study of the cohomology ring $\mathrm{H}^*(\mathrm{Gr}(k,n))$ of the Grassmannian $\mathrm{Gr}(k,n)$ – is one of the origins of much historical interest in symmetric functions (see, e.g., [4, Part III] and [10, Chp. 3]). From a modern point of view, this cohomology ring can be regarded as a quotient of the ring $\mathcal{S}$ of symmetric polynomials in k variables $x_1, x_2, \ldots, x_k$ modulo the ideal generated by the complete homogeneous symmetric polynomials $h_{n-k+1}, h_{n-k+2}, \ldots, h_n$. The last three decades have seen this cohomology ring generalized and refined in several ways, one of which is the *(small) quantum cohomology ring* $\mathrm{QH}^*(\mathrm{Gr}(k,n))$ originating from 1993 work of Witten. Bertram, Ciocan-Fontanine, Fulton, Postnikov and others have identified the combinatorial structure of this ring (see [11] for a survey); it turns out to be the quotient of the same ring $\mathcal{S}$ modulo the ideal generated by $h_{n-k+1}, h_{n-k+2}, \ldots, h_{n-1}, h_n + (-1)^k q$, where q is a polynomial indeterminate adjoined to the base ring (i.e., we are working over $\mathbb{Z}[q]$ rather

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than over $\mathbb{Z}$). Various properties of $\mathrm{H}^*(\mathrm{Gr}(k,n))$ have been extended to $\mathrm{QH}^*(\mathrm{Gr}(k,n))$, usually with new subtleties and complexities appearing in the process.

Combinatorialists working with $\mathrm{H}^*(\mathrm{Gr}(k,n))$ and $\mathrm{QH}^*(\mathrm{Gr}(k,n))$ commonly rely on geometry (in the case of $\mathrm{QH}^*(\mathrm{Gr}(k,n))$, fairly recent and deep algebraic geometry) to justify the basic properties of these rings (such as the fact that they are free as modules over the base ring, with a basis consisting of projected Schur polynomials $\overline{s_\lambda}$ corresponding to partitions λ that "fit inside a $k \times (n-k)$-rectangle"). While both of these rings are easily defined algebraically as quotients of $\mathcal{S}$, the literature does not seem to contain purely algebraic proofs of their basic structural properties. One purpose of this work is to supply such proofs, establishing two bases of these rings (defined as quotients of $\mathcal{S}$) and the S_3-symmetry of their structure constants in the Schur basis (which are the Littlewood-Richardson numbers in the case of $\mathrm{H}^*(\mathrm{Gr}(k,n))$, and the Gromov-Witten invariants in the case of $\mathrm{QH}^*(\mathrm{Gr}(k,n))$). Our proofs rely on identities for symmetric functions (see, e.g., [7]).

However, we will work in far greater generality, and study the quotient of $\mathcal{S}$ modulo the ideal I generated by $h_{n-k+1} - a_1, h_{n-k+2} - a_2, \ldots, h_n - a_k$, where $a_1, a_2, \ldots, a_k$ are arbitrary elements of our base ring $\mathbf{k}$. (And on occasion, we will work in even more general settings.) The resulting quotient ring $\mathcal{S}/I$ generalizes both $\mathrm{H}^*(\mathrm{Gr}(k,n))$ and $\mathrm{QH}^*(\mathrm{Gr}(k,n))$, and serves as the setting of an "abstract Schubert calculus", which (so far) has not found any geometric meaning. We shall exhibit two $\mathbf{k}$-module bases of this ring $\mathcal{S}/I$ – one consisting of projections $\overline{s_\lambda}$ of Schur polynomials s_λ for partitions λ "fitting inside the $k \times (n-k)$-rectangle" (Theorem 2), and another consisting of analogous projections $\overline{h_\lambda}$ of complete homogeneous symmetric polynomials (Theorem 5). We shall show that the structure constants of $\mathcal{S}/I$ with respect to the first of these two bases exhibit the same S_3-symmetry (Theorem 3) as the Littlewood-Richardson numbers and the Gromov-Witten invariants, as well as a Pieri rule (Theorem 6) subtler than the latter. Finally, we shall conjecture a positivity property (Conjecture 1) for these structure constants, which hints at some hidden geometry or combinatorics behind this "abstract Schubert calculus".

No proofs are given in this extended abstract; the reader should consult [5] for them.

2 The theorems

2.1 The rings $\mathcal{P}$ and $\mathcal{S}$ in general, and the quotient $\mathcal{P}/J$

Let $\mathbf{k}$ be a commutative ring. Let $\mathbb{N} = \{0,1,2,\ldots\}$. Let $k \in \mathbb{N}$. Let $\mathcal{P}$ denote the polynomial ring $\mathbf{k}[x_1, x_2, \ldots, x_k]$. For each $\alpha \in \mathbb{N}^k$ and each $i \in \{1,2,\ldots,k\}$, we denote the i-th entry of α by α_i (so that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$). For each $\alpha \in \mathbb{N}^k$, we define a monomial x^α by $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$.

Let $\mathcal{S}$ denote the ring of symmetric polynomials in $\mathcal{P}$; in other words, $\mathcal{S}$ is the ring of invariants of the symmetric group S_k acting on $\mathcal{P}$. The following fact goes back to Emil Artin and is proven (e.g.) in [8, (DIFF.1.3)]:

Proposition 1. *The $\mathcal{S}$-module $\mathcal{P}$ is free with basis $(x^\alpha)_{\alpha\in\mathbb{N}^k;\ \alpha_i<i \text{ for each } i}$.*

Now, fix an integer $n \geq k$. For each $i \in \{1, 2, \ldots, k\}$, let a_i be an element of $\mathcal{P}$ with degree $< n - k + i$. (This is clearly satisfied when $a_1, a_2, \ldots, a_k$ are constants in $\mathbf{k}$, but also in some other cases.)

For each $\alpha \in \mathbb{N}^k$, we let $|\alpha|$ denote the sum of the entries of the k-tuple α (that is, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$).

For each $m \in \mathbb{Z}$, we let h_m denote the m-th complete homogeneous symmetric polynomial; this is the element of $\mathcal{S}$ defined by

$$h_m = \sum_{1\leq i_1\leq i_2\leq\cdots\leq i_m\leq k} x_{i_1}x_{i_2}\cdots x_{i_m} = \sum_{\substack{\alpha\in\mathbb{N}^k;\\ |\alpha|=m}} x^\alpha. \tag{2.1}$$

(Thus, $h_0 = 1$, and $h_m = 0$ when $m < 0$.)

Let J be the ideal of $\mathcal{P}$ generated by the k differences $h_{n-k+1} - a_1, h_{n-k+2} - a_2, \ldots, h_n - a_k$.

If M is a $\mathbf{k}$-module and N is a submodule of M, then the projection of any $m \in M$ onto the quotient M/N (that is, the congruence class of m modulo N) will be denoted by $\overline{m}$.

Theorem 1. *The $\mathbf{k}$-module $\mathcal{P}/J$ is free with basis $\left(\overline{x^\alpha}\right)_{\alpha\in\mathbb{N}^k;\ \alpha_i<n-k+i \text{ for each } i}$.*

Example 1. *Let $n = 5$ and $k = 2$. Then, $\mathcal{P} = \mathbf{k}[x_1, x_2]$, and J is the ideal of $\mathcal{P}$ generated by the 2 differences*

$$h_4 - a_1 = \left(x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4\right) - a_1 \qquad \text{and}$$
$$h_5 - a_2 = \left(x_1^5 + x_1^4x_2 + x_1^3x_2^2 + x_1^2x_2^3 + x_1x_2^4 + x_2^5\right) - a_2.$$

Theorem 1 yields that the $\mathbf{k}$-module $\mathcal{P}/J$ is free with basis $\left(\overline{x^\alpha}\right)_{\alpha\in\mathbb{N}^2;\ \alpha_i<3+i \text{ for each } i}$; this basis can also be rewritten as $\left(\overline{x_1^{\alpha_1}x_2^{\alpha_2}}\right)_{\alpha_1\in\{0,1,2,3\};\ \alpha_2\in\{0,1,2,3,4\}}$. As a consequence, any monomial in $\overline{x_1}$ and $\overline{x_2}$ can be written as a linear combination of elements of this basis. For example,

$$\overline{x_1^4} = a_1 - \overline{x_1^3x_2} - \overline{x_1^2x_2^2} - \overline{x_1x_2^3} - \overline{x_2^4} \qquad \text{and} \qquad \overline{x_2^5} = a_2 - a_1\overline{x_1}.$$

These expressions will become more complicated for higher values of n and k.

Theorem 1 is related to the second part of [3, Proposition 2.9]. The $\mathbf{k}$-algebra $\mathcal{P}/J$ somewhat resembles the "splitting algebra" $\operatorname{Split}_A^d(p)$ from [9, §1.3]; the basis in Theorem 1 is similar to the basis in [9, (1.5)]. Whether there is more to this analogy has yet to be understood.

2.2 The case of symmetric $a_1, a_2, \ldots, a_k$ and the quotient $\mathcal{S}/I$

To state our next result, we need some more notations. We define the notion of a *partition* as in [7, Chapter 2]. A *part* of a partition λ means a nonzero entry of λ. If λ is a partition and p is a positive integer, then λ_p will denote the p-th entry of λ (so that $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$). If λ and μ are two partitions, then we say that $\mu \subseteq \lambda$ if and only if each positive integer p satisfies $\mu_p \leq \lambda_p$. A *skew partition* means a pair (λ, μ) of two partitions satisfying $\mu \subseteq \lambda$; such a pair is denoted by λ/μ. We refer to [7, §2.7] for the definition of a *vertical i-strip* (where $i \in \mathbb{N}$). The *size* of a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ is defined as $\lambda_1 + \lambda_2 + \lambda_3 + \cdots$, and is denoted by $|\lambda|$.

We let ω be the partition $(n-k, n-k, \ldots, n-k)$ with k entries equal to $n-k$. This partition ω is called the *$k \times (n-k)$-rectangle*, and often denoted by $\left((n-k)^k\right)$.

We let $P_{k,n}$ denote the set of all partitions that have at most k parts and have the property that each of their parts is $\leq n-k$. Equivalently, $P_{k,n}$ is the set of all partitions λ satisfying $\lambda \subseteq \omega$.

For any partition λ, we let s_λ denote the Schur polynomial (in $x_1, x_2, \ldots, x_k$) corresponding to the partition λ. (This is what is called $s_\lambda(x_1, x_2, \ldots, x_k)$ in [7, Chapter 2].) Note that $s_\lambda = 0$ if λ has more than k parts.

Assume from now on that $a_1, a_2, \ldots, a_k$ belong to $\mathcal{S}$.

Let I be the ideal of $\mathcal{S}$ generated by the k differences $h_{n-k+1} - a_1, h_{n-k+2} - a_2, \ldots, h_n - a_k$. Thus, I has the same generators as J, but is an ideal of $\mathcal{S}$ rather than of $\mathcal{P}$.

Theorem 2. *The $\mathbf{k}$-module $\mathcal{S}/I$ is free with basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$.*

The $\mathbf{k}$-algebra $\mathcal{S}/I$ generalizes several constructions in the literature:

- If $\mathbf{k} = \mathbb{Z}$ and $a_1 = a_2 = \cdots = a_k = 0$, then $\mathcal{S}/I$ becomes the cohomology ring $\mathrm{H}^*(\mathrm{Gr}(k,n))$ of the Grassmannian $\mathrm{Gr}(k,n)$ of k-dimensional subspaces in an n-dimensional space (see, e.g., [4, §9.4]); the elements of the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ correspond to the Schubert classes.

- If $\mathbf{k} = \mathbb{Z}[q]$ and $a_1 = a_2 = \cdots = a_{k-1} = 0$ and $a_k = -(-1)^k q$, then $\mathcal{S}/I$ becomes isomorphic to the quantum cohomology ring $\mathrm{QH}^*(\mathrm{Gr}(k,n))$ of the same Grassmannian (see [11]). Indeed, our ideal I becomes the J^q_{kn} of [11, (6)] in this case, and Theorem 2 generalizes the fact that the quotient $(\Lambda_k \otimes \mathbb{Z}[q]) / J^q_{kn}$ in [11, (6)] has basis $(s_\lambda)_{\lambda \in P_{kn}}$.

Assume from now on that $a_1, a_2, \ldots, a_k$ belong to $\mathbf{k}$.

Let us now study the structure constants of the $\mathbf{k}$-algebra $\mathcal{S}/I$ in the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$. For each $\mu \in P_{k,n}$, let $\mathrm{coeff}_\mu : \mathcal{S}/I \to \mathbf{k}$ be the $\mathbf{k}$-linear map that sends $\overline{s_\mu}$ to 1 while sending all other $\overline{s_\lambda}$ (with $\lambda \in P_{k,n}$) to 0. Thus, $\left(\mathrm{coeff}_\mu\right)_{\mu \in P_{k,n}}$ is the dual basis to the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ of $\mathcal{S}/I$.

For every partition $\nu = (\nu_1, \nu_2, \ldots, \nu_k) \in P_{k,n}$, we let $\nu^\vee$ denote the partition $(n-k-\nu_k, n-k-\nu_{k-1}, \ldots, n-k-\nu_1) \in P_{k,n}$. This partition $\nu^\vee$ is called the *complement* of ν.

For any three partitions $\alpha, \beta, \gamma \in P_{k,n}$, let $g_{\alpha,\beta,\gamma} = \operatorname{coeff}_{\gamma^\vee}\left(\overline{s_\alpha s_\beta}\right) \in \mathbf{k}$. These scalars $g_{\alpha,\beta,\gamma}$ are thus the structure constants of the $\mathbf{k}$-algebra $\mathcal{S}/I$ in the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ (although slightly reindexed). They satisfy the following S_3-symmetry:

Theorem 3. *For any $\alpha, \beta, \gamma \in P_{k,n}$, we have*

$$g_{\alpha,\beta,\gamma} = g_{\alpha,\gamma,\beta} = g_{\beta,\alpha,\gamma} = g_{\beta,\gamma,\alpha} = g_{\gamma,\alpha,\beta} = g_{\gamma,\beta,\alpha} = \operatorname{coeff}_\omega\left(\overline{s_\alpha s_\beta s_\gamma}\right).$$

Theorem 4. *Each $\nu \in P_{k,n}$ and $f \in \mathcal{S}/I$ satisfy* $\operatorname{coeff}_\omega\left(\overline{s_\nu} f\right) = \operatorname{coeff}_{\nu^\vee}(f)$.

2.3 Complete homogeneous symmetric polynomials

Another basis of $\mathcal{S}/I$ can be obtained from the complete homogeneous symmetric polynomials. For each $\ell \in \mathbb{N}$ and each ℓ-tuple $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathbb{Z}^\ell$ of integers, we define the symmetric polynomial $h_\nu \in \mathcal{S}$ by $h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_\ell}$. Thus, in particular, h_ν is defined for any partition ν.

Theorem 5. *The family $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ is a basis of the $\mathbf{k}$-module $\mathcal{S}/I$.*

The transfer matrix between the two bases $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ and $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ does not appear easy to describe. One remarkable result that helps relating these bases is the following formula:

Proposition 2. *Let m be a positive integer. Then,*

$$h_{n+m} \equiv \sum_{j=0}^{k-1} (-1)^j a_{k-j} s_{\left(m,1^j\right)} \bmod I,$$

where $\left(m, 1^j\right)$ stands for the partition $\left(m, \underbrace{1, 1, \ldots, 1}_{j \text{ ones}}\right)$.

Let us return to the basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ of "abstract Schubert classes". A natural task to attempt when faced with such a basis is finding a Pieri rule – i.e., a formula for expanding a product $\overline{s_\lambda h_j}$ in this basis, where $\lambda \in P_{k,n}$ and $j \in \{0, 1, \ldots, n-k\}$. Such a rule indeed exists. To state it, we shall use the notation $c^\gamma_{\alpha,\beta}$ for the Littlewood–Richardson coefficients (see, e.g., [7, Definition 2.5.8] for their definition). Our Pieri rule now says the following:

Theorem 6. *Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Then,*

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\nu \in P_{k,n};\\ \nu \subseteq \lambda}} c^{\lambda}_{(n-k-j+1,1^{i-1}),\nu} \overline{s_\nu}.$$

Note that the Littlewood–Richardson coefficients $c^{\lambda}_{(n-k-j+1,1^{i-1}),\nu}$ are nonnegative, but may be > 1. For example, if $n = 7$ and $k = 3$, then

$$\overline{s_{(4,3,2)} h_2} = \overline{s_{(4,4,3)}} + a_1 \left(\overline{s_{(4,2)}} + \overline{s_{(3,2,1)}} + \overline{s_{(3,3)}} \right) - a_2 \left(\overline{s_{(4,1)}} + \overline{s_{(2,2,1)}} + \overline{s_{(3,1,1)}} + 2\overline{s_{(3,2)}} \right) + a_3 \left(\overline{s_{(2,2)}} + \overline{s_{(2,1,1)}} + \overline{s_{(3,1)}} \right).$$

Note that Theorem 6 generalizes [1, (22)]. Also, in the cases $j = 1$ and $j = k - 1$, Theorem 6 takes the following simpler forms:

Proposition 3. *Let $\lambda \in P_{k,n}$. Assume that $k > 0$.*

(a) *If $\lambda_1 < n - k$, then*

$$\overline{s_\lambda h_1} = \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a single box}}} \overline{s_\mu}.$$

(b) *Let $\overline{\lambda}$ be the partition $(\lambda_2, \lambda_3, \lambda_4, \ldots)$. If $\lambda_1 = n - k$, then*

$$\overline{s_\lambda h_1} = \sum_{\substack{\mu \in P_{k,n};\\ \mu/\lambda \text{ is a single box}}} \overline{s_\mu} + \sum_{i=0}^{k-1} (-1)^i a_{1+i} \sum_{\substack{\mu \in P_{k,n};\\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.$$

Proposition 4. *Let $\lambda \in P_{k,n}$. Assume that $k > 0$.*

(a) *We have*

$$\overline{s_\lambda h_{n-k}} = \overline{s_{(n-k,\lambda_1,\lambda_2,\lambda_3,\ldots)}} - \sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu \in P_{k,n};\\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.$$

(b) *If $\lambda_k > 0$, then*

$$\overline{s_\lambda h_{n-k}} = - \sum_{i=1}^{k} (-1)^i a_i \sum_{\substack{\mu \in P_{k,n};\\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \overline{s_\mu}.$$

Pieri rules for multiplication by $\overline{e_j}$ (instead of $\overline{h_j}$) appear to be more complicated: For example, for $n = 5$ and $k = 3$, we have

$$\overline{s_{(2,2,1)} e_2} = a_1 \overline{s_{(2,2)}} - 2a_2 \overline{s_{(2,1)}} + a_3 \left(\overline{s_{(2)}} + \overline{s_{(1,1)}} \right) + a_1^2 \overline{s_{(1)}} - 2a_1 a_2 \overline{s_{()}}.$$

We also have shown a "straightening law" that writes an $\overline{s_\mu}$ with $\mu \notin P_{k,n}$ as a linear combination of "smaller" Schur polynomials (i.e., of $\overline{s_\nu}$ with $|\nu| < |\mu|$); this generalizes the "rim hook algorithm" of [1]. We refer to [5, §10] for the details.

2.4 Positivity?

The structure constants of the $\mathbb{Z}[q]$-algebra $\operatorname{QH}^*(\operatorname{Gr}(k,n))$ are polynomials in the indeterminate q, whose coefficients are the famous Gromov-Witten invariants $C^d_{\lambda\mu\nu}$. These Gromov-Witten invariants $C^d_{\lambda\mu\nu}$ are nonnegative integers (as follows from their geometric interpretation, but also from the "Quantum Littlewood-Richardson Rule" [2, Theorem 2]). This appears to generalize to the general case of $\mathcal{S}/I$:

Conjecture 1. *Let $b_i = (-1)^{n-k-1} a_i$ for each $i \in \{1,2,\ldots,k\}$. Let $\lambda,\mu,\nu \in P_{k,n}$. Then, $(-1)^{|\lambda|+|\mu|-|\nu|} \operatorname{coeff}_\nu\left(\overline{s_\lambda s_\mu}\right)$ is a polynomial in $b_1,b_2,\ldots,b_k$ with coefficients in $\mathbb{N}$.*

We have verified this conjecture for all $n \leq 8$ using SageMath.

2.5 Deforming symmetric functions

We have so far studied a quotient $\mathcal{S}/I$ of the ring $\mathcal{S}$ of symmetric polynomials in k variables $x_1,x_2,\ldots,x_k$. But $\mathcal{S}$ itself is a quotient of a larger ring – the ring Λ of symmetric functions in infinitely many variables. Let us briefly introduce Λ and then state a version of Theorem 2 for this larger ring (requiring, however, that $a_1,a_2,\ldots,a_k$ lie in $\mathbf{k}$ rather than in $\mathcal{S}$).

Let Λ be the ring of symmetric functions in infinitely many indeterminates $x_1,x_2,x_3,\ldots$ over $\mathbf{k}$, defined as in [7, Chapter 2]. We shall use boldfaced notations for symmetric functions in Λ in order to distinguish them from symmetric polynomials in $\mathcal{S}$. In particular:

- For any $i \in \mathbb{Z}$, we let $\mathbf{h}_i$ be the i-th complete homogeneous symmetric function in Λ. (This is called h_i in [7, Definition 2.2.1].)
- For any $i \in \mathbb{Z}$, we let $\mathbf{e}_i$ be the i-th elementary symmetric function in Λ. (This is called e_i in [7, Definition 2.2.1].)
- For any partition λ, we let $\mathbf{s}_\lambda$ be the corresponding Schur function in Λ. (This is called s_λ in [7, Definition 2.2.1].)

If $\mathbf{f} \in \Lambda$ is a symmetric function, then $\mathbf{f}(x_1,x_2,\ldots,x_k)$ is a symmetric polynomial in $\mathcal{S}$; the map

$$\Lambda \to \mathcal{S}, \qquad \mathbf{f} \mapsto \mathbf{f}(x_1,x_2,\ldots,x_k)$$

is a surjective $\mathbf{k}$-algebra homomorphism. The kernel of this homomorphism is the ideal $\langle \mathbf{e}_{k+1},\mathbf{e}_{k+2},\mathbf{e}_{k+3},\ldots\rangle$ of Λ (where the notation $\langle\cdot\rangle$ means the ideal generated by whatever is inside the brackets). Hence, $\mathcal{S} \cong \Lambda / \langle \mathbf{e}_{k+1},\mathbf{e}_{k+2},\mathbf{e}_{k+3},\ldots\rangle$, so that

$$\mathcal{S}/I \cong \Lambda / \left(\langle \mathbf{h}_{n-k+1}-a_1, \mathbf{h}_{n-k+2}-a_2,\ldots,\mathbf{h}_n - a_k\rangle + \langle \mathbf{e}_{k+1},\mathbf{e}_{k+2},\mathbf{e}_{k+3},\ldots\rangle\right).$$

This suggests a further generalization: What if we replace $\mathbf{e}_{k+1}, \mathbf{e}_{k+2}, \mathbf{e}_{k+3}, \ldots$ by $\mathbf{e}_{k+1} - b_1, \mathbf{e}_{k+2} - b_2, \mathbf{e}_{k+3} - b_3, \ldots$ for some constants $b_1, b_2, b_3, \ldots$? At least in the case when $a_1, a_2, \ldots, a_k \in \mathbf{k}$, this generalization still satisfies an analogue of Theorem 2:

Theorem 7. *Assume that $a_1, a_2, \ldots, a_k$ as well as $b_1, b_2, b_3, \ldots$ are elements of $\mathbf{k}$. Let K be the ideal $\langle \mathbf{h}_{n-k+1} - a_1, \mathbf{h}_{n-k+2} - a_2, \ldots, \mathbf{h}_n - a_k \rangle + \langle \mathbf{e}_{k+1} - b_1, \mathbf{e}_{k+2} - b_2, \mathbf{e}_{k+3} - b_3, \ldots \rangle$ of Λ. Then, Λ / K is a free $\mathbf{k}$-module with basis $\left(\overline{\mathbf{s}_\lambda}\right)_{\lambda \in P_{k,n}}$.*

3 Proof methods

We shall now give a brief overview of the techniques that enter into the proofs of the above results. See [5] for the missing details.

3.1 A fundamental identity

The first step towards understanding I and J is an identity between certain polynomials (Lemma 1 below). To state it, we shall use the notations h_m and e_m for complete homogeneous symmetric polynomials and elementary symmetric polynomials in general. Thus, for any $m \in \mathbb{Z}$ and any p elements $y_1, y_2, \ldots, y_p$ of a commutative ring, we set

$$h_m \left(y_1, y_2, \ldots, y_p\right) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq p} y_{i_1} y_{i_2} \cdots y_{i_m} \qquad \text{and} \tag{3.1}$$

$$e_m \left(y_1, y_2, \ldots, y_p\right) = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq p} y_{i_1} y_{i_2} \cdots y_{i_m}. \tag{3.2}$$

Thus, $h_0 \left(y_1, y_2, \ldots, y_p\right) = 1$ and $e_0 \left(y_1, y_2, \ldots, y_p\right) = 1$. Also, $e_m \left(y_1, y_2, \ldots, y_p\right) = 0$ for all $m > p$. Also, for any $m < 0$, we have $h_m \left(y_1, y_2, \ldots, y_p\right) = 0$ and $e_m \left(y_1, y_2, \ldots, y_p\right) = 0$. Finally, what we have previously called h_m without any arguments can now be rewritten as $h_m \left(x_1, x_2, \ldots, x_k\right)$.

Lemma 1. *Let $i \in \{1, 2, \ldots, k+1\}$ and $p \in \mathbb{N}$. Then,*

$$h_p \left(x_i, x_{i+1}, \ldots, x_k\right) = \sum_{t=0}^{i-1} (-1)^t e_t \left(x_1, x_2, \ldots, x_{i-1}\right) h_{p-t} \left(x_1, x_2, \ldots, x_k\right).$$

(If $i = k+1$, then the term $h_p \left(x_i, x_{i+1}, \ldots, x_k\right)$ on the left hand side is understood to be h_p of an empty alphabet; this is 1 when $p = 0$ and 0 otherwise.)

This is easily proven (e.g., using generating functions) and likely folklore.

3.2 Proving Theorems 1 and 2

We refer to [6, detailed version, §3] for the definition of a Gröbner basis over $\mathbf{k}$. (When $\mathbf{k}$ is a field, this definition is classical and appears throughout the literature.) We endow the set of monomials in $\mathcal{P}$ with a degree-lexicographic term order, where the variables are ordered by $x_1 > x_2 > \cdots > x_n$.

Proposition 5. *The family*

$$\left(h_{n-k+i}\left(x_i, x_{i+1}, \ldots, x_k\right) - \sum_{t=0}^{i-1} (-1)^t e_t\left(x_1, x_2, \ldots, x_{i-1}\right) a_{i-t} \right)_{i \in \{1,2,\ldots,k\}}$$

is a Gröbner basis of the ideal J.

This is not hard to show using Lemma 1 and Buchberger's first criterion (see, e.g., [6, detailed version, Proposition 3.9]). See [12, Theorem 1.2.7] for a similar result. Now, Theorem 1 easily follows from Proposition 5 using the Macaulay-Buchberger basis theorem (e.g., [6, detailed version, Proposition 3.10]).

Let us now assume that $a_1, a_2, \ldots, a_k$ belong to $\mathcal{S}$. A simple induction argument shows the following:

Lemma 2. *Let i be an integer such that $i > n - k$. Then,*

$$h_i \equiv (\textit{some symmetric polynomial of degree } < i) \bmod I.$$

Using this lemma and the Jacobi-Trudi identity, we can easily obtain:

Lemma 3. *Let P_k denote the set of all partitions with at most k parts. Let $\lambda \in P_k$ be a partition such that $\lambda \notin P_{k,n}$. Then,*

$$s_\lambda \equiv (\textit{some symmetric polynomial of degree } < |\lambda|) \bmod I.$$

On the other hand, a well-known fact from commutative algebra states the following:

Lemma 4. *Let M be a free $\mathbf{k}$-module with a finite basis $(b_s)_{s \in S}$. Let $(a_u)_{u \in U} \in M^U$ be a family that spans M. Assume that $|U| = |S|$. Then, $(a_u)_{u \in U}$ is a basis of the $\mathbf{k}$-module M. (In other words: A spanning family of M whose size equals the size of a basis must itself be a basis, as long as the sizes are finite.)*

Now, Theorem 2 can be proven by the following strategy: The family $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ spans the $\mathbf{k}$-module $\mathcal{S}/I$ (by Lemma 3), while the family $\left(x^\alpha\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans the $\mathcal{S}$-module $\mathcal{P}$ (by Proposition 1). Hence, the family $\left(\overline{s_\lambda x^\alpha}\right)_{\lambda \in P_{k,n};\ \alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ spans the $\mathbf{k}$-module $\mathcal{P}/J$. But its size is easily seen to be $\dbinom{n}{k} \cdot k! = n(n-1)\cdots(n-k+1)$, which is exactly the size of the basis $\left(\overline{x^\alpha}\right)_{\alpha\in\mathbb{N}^k;\ \alpha_i<n-k+i \text{ for each } i}$ of the $\mathbf{k}$-module $\mathcal{P}/J$ (see Theorem 1). Hence, Lemma 4 shows that the former family is a basis of $\mathcal{P}/J$ as well. This readily yields Theorem 2.

Theorem 7 is not hard to obtain from Theorem 2 by defining a $\mathbf{k}$-algebra endomorphism φ of Λ that sends each $\mathbf{e}_i$ to $\mathbf{e}_i + b_{i-k}$ (where we set $b_{i-k} = 0$ for all $i \leq k$).

3.3 Proving the S_3-symmetry

We have two proofs of Theorem 4; both require significant amounts of work. Referring again to [5] for the details, we shall just telegraph the main ingredients of the first proof here. One of them are the *skewing operators* as defined (e.g.) in [7, §2.8]. They have the property that $\left(\mathbf{s}_\mu\right)^\perp\left(\mathbf{s}_\lambda\right) = \mathbf{s}_{\lambda/\mu}$ for any two partitions λ and μ, where $\mathbf{s}_{\lambda/\mu} \in \Lambda$ is the skew Schur function (understood to be 0 unless $\mu \subseteq \lambda$). We also recall the two Pieri rules for $\mathbf{e}_i$ in Λ ([7, (2.7.2)] and [7, version with solutions (ancillary file), Lemma 12.83.3(b)]):

Proposition 6. *Let λ be a partition, and let $i \in \mathbb{N}$. Then,* $\mathbf{s}_\lambda \mathbf{e}_i = \sum\limits_{\substack{\mu \text{ is a partition;} \\ \mu/\lambda \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu.$

Corollary 1. *Let λ be a partition, and let $i \in \mathbb{N}$. Then,* $\mathbf{e}_i^\perp \mathbf{s}_\lambda = \sum\limits_{\substack{\mu \text{ is a partition;} \\ \lambda/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu.$

An identity of Bernstein ([7, Exercise 2.9.1(b)]) serves a crucial role as well:

Proposition 7. *Let λ be a partition. Let $m \in \mathbb{Z}$ be such that $m \geq \lambda_1$. Then,*

$$\sum_{i\in\mathbb{N}} (-1)^i \mathbf{h}_{m+i} \mathbf{e}_i^\perp \mathbf{s}_\lambda = \mathbf{s}_{(m,\lambda_1,\lambda_2,\lambda_3,\ldots)}.$$

Proposition 7 leads quickly to Proposition 4, but also (via Corollary 1) to the following corollary:

Corollary 2. *Let λ be a partition with at most k parts. Let $\overline{\lambda}$ be the partition $(\lambda_2, \lambda_3, \lambda_4, \ldots)$. Then,*

$$\mathbf{s}_\lambda = \sum_{i=0}^{k-1} (-1)^i \mathbf{h}_{\lambda_1+i} \sum_{\substack{\mu \text{ is a partition;} \\ \overline{\lambda}/\mu \text{ is a vertical } i\text{-strip}}} \mathbf{s}_\mu.$$

Now, for each $p \in \mathbb{Z}$, we let Q_p be the $\mathbf{k}$-submodule of $\mathcal{S}/I$ spanned by the $\overline{s_\lambda}$ with $\lambda \in P_{k,n}$ satisfying $\lambda_k \leq p$. Clearly, $Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots$ and $\operatorname{coeff}_\omega\left(Q_{n-k-1}\right) = 0$.

From Corollary 2, it is not hard to obtain Proposition 3, but also the following lemma:

Lemma 5. *Let λ be a partition with at most k parts. If $\lambda_1 = n - k + 1$, then $\overline{s_\lambda} \in Q_0$.*

Using Proposition 6 and Lemma 5, we can find:

Lemma 6. *Let $i \in \mathbb{N}$ and $\lambda \in P_{k,n}$. Then,* $\overline{e_i s_\lambda} \equiv \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a vertical } i\text{-strip}}} \overline{s_\mu} \operatorname{mod} Q_0$.

Hence:

Lemma 7. *Let $i \in \mathbb{N}$ and $p \in \mathbb{Z}$. Then,* $\overline{e_i} Q_p \subseteq Q_{p+1}$.

We furthermore recall a property of the Littlewood–Richardson coefficients $c^\lambda_{\mu,\nu}$ (part of [7, Remark 2.5.9]):

Proposition 8. *Let λ and μ be two partitions.*
(a) *We have* $\mathbf{s}_{\lambda/\mu} = \sum_{\nu \text{ is a partition}} c^\lambda_{\mu,\nu} \mathbf{s}_\nu$.
(b) *If ν is a partition, then $c^\lambda_{\mu,\nu} = 0$ unless $(\nu \subseteq \lambda$ and $|\mu| + |\nu| = |\lambda|)$.*

Next, let $\mathcal{Z}$ be the $\mathbf{k}$-submodule of Λ with basis $(\mathbf{s}_\lambda)_{\lambda \in P_{k,n}}$. We thus can define a $\mathbf{k}$-linear map $\delta : \mathcal{Z} \to \mathcal{S}/I$ by setting

$$\delta(\mathbf{s}_\lambda) = \overline{s_{\lambda^\vee}} \qquad \text{for every } \lambda \in P_{k,n}.$$

The following is easy to see from Proposition 8:

Lemma 8. *We have $\mathbf{f}^\perp(\mathcal{Z}) \subseteq \mathcal{Z}$ for each $\mathbf{f} \in \Lambda$.*

Comparing Corollary 1 with Lemma 6, we easily find:

Lemma 9. *Let $i \in \mathbb{Z}$ and $\mathbf{f} \in \mathcal{Z}$. Then,* $\delta\left(\mathbf{e}_i^\perp \mathbf{f}\right) \equiv \overline{e_i} \delta(\mathbf{f}) \operatorname{mod} Q_0$.

From this lemma, we obtain by induction (using Lemmas 7 and 8 as well as the identity $(\mathbf{fg})^\perp = \mathbf{g}^\perp \circ \mathbf{f}^\perp$ for skewing operators) the following fact:

Lemma 10. *Let $p \in \mathbb{N}$. Let $i_1, i_2, \ldots, i_p \in \mathbb{Z}$ and $\mathbf{f} \in \mathcal{Z}$. Then,* $\delta\left(\left(\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_p}\right)^\perp \mathbf{f}\right) \equiv \overline{e_{i_1}e_{i_2}\cdots e_{i_p}} \delta(\mathbf{f}) \operatorname{mod} Q_{p-1}$.

But if $\lambda \in P_{k,n}$, then the dual Jacobi-Trudi formula can be used to express $\mathbf{s}_\lambda$ as a sum of products of the form $\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_{n-k}}$ with $i_1, i_2, \ldots, i_{n-k} \in \mathbb{Z}$. Hence, Lemma 10 yields:

Lemma 11. *Let $\lambda \in P_{k,n}$ and $\mathbf{f} \in \mathcal{Z}$. Then,* $\delta\left((\mathbf{s}_\lambda)^\perp \mathbf{f}\right) \equiv \overline{s_\lambda} \delta(\mathbf{f}) \operatorname{mod} Q_{n-k-1}$ *and thus* $\operatorname{coeff}_\omega\left(\delta\left((\mathbf{s}_\lambda)^\perp \mathbf{f}\right)\right) = \operatorname{coeff}_\omega\left(\overline{s_\lambda}\delta(\mathbf{f})\right)$ *(since* $\operatorname{coeff}_\omega(Q_{n-k-1}) = 0$*).*

Fixing $\mu \in P_{k,n}$ and applying Lemma 11 to $\mathbf{f} = \mathbf{s}_{\mu^\vee}$, we obtain (after some work):

Lemma 12. *Let $\lambda \in P_{k,n}$ and $\mu \in P_{k,n}$. Then,* $\operatorname{coeff}_\omega\left(\overline{s_\lambda s_\mu}\right) = \operatorname{coeff}_\omega\left(\delta\left((\mathbf{s}_\lambda)^\perp \mathbf{s}_{\mu^\vee}\right)\right) = \operatorname{coeff}_\omega\left(\delta\left(\mathbf{s}_{\mu^\vee/\lambda}\right)\right) = \begin{cases} 1, & \text{if } \lambda = \mu^\vee; \\ 0, & \text{if } \lambda \neq \mu^\vee \end{cases}$.

This quickly yields Theorem 4. Theorem 3, in turn, easily follows from Theorem 4.

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References

[1] A. Bertram, I. Ciocan-Fontanine, and W. Fulton. "Quantum Multiplication of Schur Polynomials". *J. Algebra* **219**.2 (1999), pp. 728–746. Link.

[2] A. Buch, A. Kresch, K. Purbhoo, and H. Tamvakis. "The puzzle conjecture for the cohomology of two-step flag manifolds". *J. Algebraic Comb.* **44**.4 (2016), pp. 973–1007. Link.

[3] A. Conca, C. Krattenthaler, and J. Watanabe. "Regular Sequences of Symmetric Polynomials". *Rend. Sem. Mat. Univ. Padova* **121** (2009), pp. 179–199. Link.

[4] W. Fulton. *Young Tableaux*. Cambridge University Press, 1996. Link.

[5] D. Grinberg. "A basis for a quotient of symmetric polynomials (draft)". Link.

[6] D. Grinberg. "t-unique reductions for Mészáros's subdivision algebra". 2018. arXiv:1704.00839v6.

[7] D. Grinberg and V. Reiner. "Hopf algebras in Combinatorics". 2018. arXiv:1409.8356v5.

[8] D. Laksov, A. Lascoux, P. Pragacz, and A. Thorup. "The LLPT Notes". 2018. Link.

[9] D. Laksov and A. Thorup. "Splitting algebras and Schubert calculus". *Indiana Univ. Math. J.* **61**.3 (2012), pp. 1253–1312.

[10] L. Manivel. *Symmetric Functions, Schubert Polynomials, and Degeneracy Loci*. AMS, 2001.

[11] A. Postnikov. "Affine approach to quantum Schubert calculus". *Duke Math. J.* **128** (3), pp. 473–509. Link.

[12] B. Sturmfels. *Algorithms in Invariant Theory*. 2nd ed. Springer, 2008.

[13] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 7.3)*. 2017. Link.