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On a fourfold refined enumeration of alternating sign trapezoids

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Abstract. Alternating sign trapezoids have recently been introduced as a generalization of alternating sign matrices. Fischer established a threefold refined enumeration of alternating sign trapezoids and provided three statistics on column strict shifted plane partitions with the same joint distribuition. We extend this result by another statistic that generalizes the number of -1's in alternating sign matrices.

Keywords: Alternating sign trapezoids, plane partitions, constant term formula

1 Introduction

Since their introduction in the early 1980s, alternating sign matrices have aroused great interest among combinatorialists. Mills, Robbins, and Rumsey [10] conjectured them to be equinumerous with descending plane partitions, which had been enumerated by Andrews [1] a few years earlier; it was finally independently proved over a decade later by Zeilberger [11] and Kuperberg [9]. Since then, the spellbinding research of alternating sign matrices has revealed new equinumerous classes of combinatorial objects but finding bijections between them remains one of the most challenging problems. Equally distributed statistics on these objects might finally lead to those eagerly awaited bijections. Embracing this idea, we provide a fourfold refined enumeration of alternating sign trapezoids, a recently defined generalization of alternating sign matrices. Moreover, we establish four statistics on certain column strict shifted plane partitions with the same joint distribution. Thus, we generalize the recent refined enumerations of alternating sign trapezoids and of column strict shifted plane partitions by Fischer [5].

2 Preliminaries

We start by introducing *alternating sign trapezoids* and *column strict shifted plane partitions* together with four statistics on each of these classes of objects.

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Definition 2.1. For given integers $n \geq 1$ and $l \geq 2$, an (n,l)-*alternating sign trapezoid* is an array of -1s, 0s, and $+1$s in a trapezoidal shape with n rows of the following form

$$\begin{array}{ccccccccccc} a_{1,1} & a_{1,2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1,2n+l-2} \\ & a_{2,2} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{2,2n+l-3} & \\ & & \ddots & & & & ⋰ & & \\ & & & a_{n,n} & \cdots & a_{n,n+l-1} & & & \end{array}$$

such that

- the nonzero entries alternate in sign in each row and each column,
- the topmost nonzero entry in each column is 1 (if existent),
- the entries in each row sum up to 1, and
- the entries in the central $l-2$ columns sum up to 0.

An $(n,1)$-*alternating sign trapezoid* is defined as above with the exception that the entry in the bottom row can be 0 or 1.

The entries in each column of an alternating sign trapezoid sum up to 0 or 1. A column whose entries sum up to 1 is called a 1-*column*. If, in addition, the bottom entry of a 1-column is 0, we call that column a 10-*column*. Note that the number of 1-columns in any (n,l)-alternating sign trapezoid is exactly n if $l \neq 1$; otherwise, it is n or $n-1$.

$$\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & \\ & & 0 & 0 & 0 & 0 & 1 & 0 & & \\ & & & 1 & 0 & 0 & 0 & & & \end{array}$$

Figure 1: $(4,4)$-alternating sign trapezoid with weight QR^2T^2

We introduce four different statistics on alternating sign trapezoids by associating the following weight to (n,l)-alternating sign trapezoids if $l \geq 2$:

$$Q^{\#\,-1\text{s}}\, R^{\#\text{ 1-columns within the } n \text{ leftmost columns}} \\ \times S^{\#\text{ 10-columns within the } n \text{ leftmost columns}}\, T^{\#\text{ 10-columns within the } n \text{ rightmost columns}}.$$

An example of a $(4,4)$-alternating sign trapezoid is given in Figure 1. For $(n,1)$-alternating sign trapezoids, however, we have to adapt the weight in the following way:

$$Q^{\#\,-1\text{s}}\, R^{\#\text{ 1-columns within the } n \text{ leftmost columns}} \\ \times S^{\#\text{ 10-columns within the } n-1 \text{ leftmost columns}}\, T^{\#\text{ 10-columns within the } n-1 \text{ rightmost columns}} \\ \times (S+T-Q)^{[\text{the central column is a 10-column}]},$$

where we make use of the *Iverson bracket*: For any logical proposition P, $[P] = 1$ if P is satisfied and $[P] = 0$ otherwise.

Ayyer, Behrend, and Fischer [2] showed that $n \times n$-alternating sign matrices are equinumerous with $(n-1, 3)$-alternating sign trapezoids. As a corollary of [2, Theorem 1.2], the statistic Q generalizes the number of -1s in alternating sign matrices.

Definition 2.2. A *shifted Young diagram* is a finite collection of cells arranged in rows of strictly decreasing lengths such that each row is indented by one cell compared to the row above. The *shape* of a shifted Young diagram is the sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of its row lengths. Note that λ is a *strict partition*, that is, a sequence of strictly decreasing positive integers.

Figure 2: A shifted Young diagram of shape $(5, 3, 2)$ and a column strict shifted plane partition of class 4 of the same shape with weight Q^2R^3ST for $d = 3$

A *column strict shifted plane partition* is a filling of a shifted Young diagram with positive integers such that the entries weakly decrease along each row and strictly decrease down each column. It is of *class* k if the first entry of each row i is exactly $k + \lambda_i$, that is, exactly k plus its corresponding row length.

Note that we cannot always associate a class to a given column strict shifted plane partition. Column strict shifted plane partitions of class 2 correspond to *descending plane partitions*.

We introduce four different statistics on column strict shifted plane partitions of class k of which two depend on a fixed parameter $d \in \{1, \ldots, k\}$:

- Q counts the number of parts equal to $\{2, 3, \ldots, j - i + k\} \setminus \{j - i + d\}$,
- R counts the number of rows,
- S counts the number of parts equal to $j - i + d$, and
- T counts the number of 1s,

where i is the row and j is the column of the respective part.

An example of a shifted Young diagram and a column strict shifted plane partition is presented in Figure 2. Note that the parts counted by the statistic Q generalize the parts in descending plane partitions that are referred to as *special parts* by Mills, Robbins, and Rumsey [10] and enumerated by Behrend, Di Francesco, and Zinn-Justin [3].

Fischer [5] established refined enumerations of alternating sign trapezoids and column strict shifted plane partition taking account of the statistics S, T, and P. We extend her proof by adding the fourth statistic in order to prove the following main theorem of this paper:

Theorem 2.3. *Let $n, l \geq 1$ and $1 \leq d \leq l-1$. Then the joint distribution of the corresponding statistics Q, R, S, and T on (n,l)-alternating sign trapezoids and on column strict shifted plane partitions of class $l-1$ with at most n entries in the first row coincide.*

Note that we can generalize Theorem 2.3 by providing a combinatorial interpretation for the case $d = 0$.

3 Weighted Enumeration of Alternating Sign Trapezoids

First, we provide the generating function of (n,l)-alternating sign trapezoids. For this purpose, we heavily exploit the correspondence between alternating sign trapezoids and truncated *monotone triangles*, both as defined below. For the sake of simplicity, we assume that $l \geq 2$ throughout the extended abstract. However, note that Theorem 2.3 includes the case $l = 1$.

Definition 3.1. For a given integer $n \geq 1$, a *monotone triangle of order n* is an array of integers in a triangular shape with n rows of the following form

$$\begin{array}{ccccccccc} & & & & a_{1,1} & & & & \\ & & & a_{2,1} & & a_{2,2} & & & \\ & & \dots & & \dots & & \dots & & \\ & a_{n-1,1} & & a_{n-1,2} & & \dots & & a_{n-1,n-1} & \\ a_{n,1} & & a_{n,2} & & a_{n,3} & & \dots & & a_{n,n} \end{array}$$

such that the entries strictly increase along rows and weakly increase both along $\nearrow$-diagonals and $\searrow$-diagonals.

Definition 3.2. For given integers $p, q \geq 0$ and $n \geq 1$ such that $p + q \leq n$ as well as a weakly decreasing sequence $\mathbf{s} = (s_1, s_2, \dots, s_p)$ and a weakly increasing sequence $\mathbf{t} = (t_{n-q+1}, t_{n-q+2}, \dots, t_n)$ of nonnegative integers, we define an $(\mathbf{s},\mathbf{t})$*-tree* as an array of integers which arises from a monotone triangle of order n by truncating the diagonals as follows: for each $1 \leq i \leq p$, we delete the s_i bottom entries of the i^{th} $\nearrow$-diagonal; for each $n-q+1 \leq i \leq n$, we delete the t_i bottom entries of the i^{th} $\searrow$-diagonal. All diagonals are counted from left to right.

We say that an $(\mathbf{s},\mathbf{t})$-tree has *bottom row* $\mathbf{k} = (k_1, \dots, k_n)$ if the following holds true: for all i such that $1 \leq i \leq p$ or $n-q+1 \leq i \leq n$, the integer k_i is the bottom entry of the i^{th} $\nearrow$-diagonal or the i^{th} $\searrow$-diagonal, respectively; and, for all $p < i < n-q+1$, the integer k_i is equal to the entry $a_{n,i}$ in the bottom row of the original monotone triangle.

$$\begin{array}{cccc} & 1 & & \\ -3 & & 3 & \\ & 2 & & 3 \\ -1 & & 2 & \end{array}$$

Figure 3: $((2),(1))$-tree with bottom row $(-3,-1,2,3)$

We can transform an alternating sign trapezoid into a tree; see [6] for the detailed construction. To illustrate its main features, we number the n leftmost columns of an (n,l)-alternating sign trapezoid from $-n$ to -1 and the n rightmost columns from 1 to n. The 1-*column vector* $\mathbf{c} = (c_1, \ldots, c_n)$ records the positions of the 1-columns of the alternating sign trapezoid; hence, $-n \le c_1 < \cdots < c_m < 0 < c_{m+1} < \cdots < c_n \le n$ for some $0 \le m \le n$. The construction above yields an $(\mathbf{s},\mathbf{t})$-tree with bottom row $(c_1, \ldots, c_m, c_{m+1} + l - 3, \ldots, c_n + l - 3)$ such that $\mathbf{s} = (-c_1 - 1, \ldots, -c_m - 1)$ and $\mathbf{t} = (c_{m+1} - 1, \ldots, c_n - 1)$. Regarding the statistics of alternating sign trapezoids, we make the following observations: a -1 in the alternating sign trapezoid corresponds to an entry $a_{i,j}$ in the tree which has two neighbouring entries $a_{i+1,j}$ and $a_{i+1,j+1}$ in the row below such that $a_{i+1,j} < a_{i,j} < a_{i+1,j+1}$. The positions of the 1-columns are reflected in the bottom row of the tree, and 10-columns cause the corresponding diagonals in the tree to have twice the same bottom entries. In Figure 3, we present the tree corresponding to the $(4,4)$-alternating sign trapezoid with 1-column vector $(-3,-1,1,2)$ in Figure 1.

To enumerate monotone triangles and trees, we use operator formulae and constant term expressions. To this end, we need to introduce several operators and notations. First, we define the *symmetriser* **Sym** and the *antisymmetriser* **ASym** of a function $f(x_1, \ldots, x_n)$. Let $\mathfrak{S}_n$ be the symmetric group of degree n. Then

$$\begin{aligned} \mathbf{Sym}_{x_1,\ldots,x_n} f(x_1, \ldots, x_n) &:= \sum_{\sigma \in \mathfrak{S}_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \text{ and} \\ \mathbf{ASym}_{x_1,\ldots,x_n} f(x_1, \ldots, x_n) &:= \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \end{aligned}$$

We use $\mathbf{Sym}_{\mathbf{x}}$ and $\mathbf{ASym}_{\mathbf{x}}$ as an abbreviation if $\mathbf{x} = (x_1, \ldots, x_n)$ is clear from the context. Furthermore, $\mathrm{CT}_{\mathbf{x}} f(\mathbf{x}) = \mathrm{CT}_{x_1,\ldots,x_n} f(x_1, \ldots, x_n)$ denotes the constant term of the function f with respect to the variables $x_1, \ldots, x_n$. Finally, we define the *shift operator* $\mathrm{E}_x f(x) := f(x+1)$, the *forward difference operator* $\Delta_x := \mathrm{E}_x - \mathrm{id}$, and the *backward difference operator* $\delta_x := \mathrm{id} - \mathrm{E}_x^{-1}$, where id denotes the standard identity operator. We use the notation $\mathrm{E}_a f(a) := \mathrm{E}_x f(x)|_{x=a}$ for a given a variable x and an integer a. This abbreviatory notation is correspondingly used for other operator expressions.

Fischer and Riegler [7] provided a weighted enumeration of monotone triangles:

Theorem 3.3. *The generating function of monotone triangles of order n with bottom row $\mathbf{k} =$*

$(k_1, \dots, k_n)$ *with respect to the statistic Q is given by* $M_n(\mathbf{k})$ *with* $M_n(\mathbf{x})$ *defined as*

$$\mathrm{CT}_{\mathbf{y}}\left(\mathbf{ASym}_{\mathbf{y}}\left(\prod_{i=1}^{n}(1+y_i)^{x_i}\prod_{1\le i<j\le n}(Q-(1-Q)y_i+y_j+y_iy_j)\right)\prod_{1\le i<j\le n}(y_j-y_i)^{-1}\right). \tag{3.1}$$

The crucial observation is that if we repeatedly apply $-\Delta_{x_i}$ and δ_{x_i} to $M_n(\mathbf{x})|_{Q=1}$, we enumerate monotone triangles with truncated diagonals. By generalising the difference operators, we obtain the following enumeration formula for trees with respect to the statistic Q as a corollary of [4, Theorem 5].

Theorem 3.4. *The generating function of* $(\mathbf{s},\mathbf{t})$*-trees with* $\mathbf{s} = (s_1, s_2, \dots, s_p)$*,* $\mathbf{t} = (t_{n-q+1}, t_{n-q+2}, \dots, t_n)$ *and bottom row* $\mathbf{k} = (k_1, \dots, k_n)$ *with respect to the statistic Q is given by*

$$\prod_{i=1}^{p}\left(-{}^{Q}\Delta_{k_i}\right)^{s_i}\prod_{i=n-q+1}^{n}{}^{Q}\delta_{k_i}^{t_i}M_n(\mathbf{k}),$$

where ${}^{Q}\Delta_x := (Q-(1-Q)\Delta_x)^{-1}\Delta_x$ *and* ${}^{Q}\delta_x := (Q-(Q-1)\delta_x)^{-1}\delta_x$.

We use the correspondence between alternating sign trapezoids and trees to obtain enumeration formulae. First, we consider alternating sign trapezoids with prescribed 1-column vectors. The following theorem can be proved by similar means as [8, Theorem 4.4]:

Lemma 3.5. *The generating function of* (n,l)*-alternating sign trapezoids with* 1*-column vector* $\mathbf{c}$ *with respect to the statistics Q, S, and T is given by*

$$\prod_{i=1}^{m}\left(\mathrm{id}-\frac{S}{Q}\delta_{c_i}\right)\left(\mathrm{id}+{}^{Q}\Delta_{c_i}\right)\left(-{}^{Q}\Delta_{c_i}\right)^{-c_i-1}\prod_{i=m+1}^{n}\left(\mathrm{id}+\frac{T}{Q}\Delta_{c_i}\right)\left(\mathrm{id}-{}^{Q}\delta_{c_i}\right){}^{Q}\delta_{c_i}^{c_i-1}M_n(\tilde{\mathbf{c}}), \tag{3.2}$$

where $\tilde{\mathbf{c}} = (c_1, \dots, c_m, c_{m+1}+l-3, \dots, c_n+l-3)$.

Instead of evaluating the previous polynomial at $\tilde{\mathbf{c}}$, we can shift the argument by suitable operators and take the constant term. In particular, (3.2) is equal to

$$\begin{aligned}
&\mathrm{CT}_{\mathbf{x}}\left(\prod_{i=1}^{m}\mathrm{E}_{x_i}^{c_i}\left(\mathrm{id}-\frac{S}{Q}\delta_{c_i}\right)\left(\mathrm{id}+{}^{Q}\Delta_{c_i}\right)\left(-{}^{Q}\Delta_{c_i}\right)^{-c_i-1}\right.\\
&\qquad\left.\times\prod_{i=m+1}^{n}\mathrm{E}_{x_i}^{c_i+l-3}\left(\mathrm{id}+\frac{T}{Q}\Delta_{c_i}\right)\left(\mathrm{id}-{}^{Q}\delta_{c_i}\right){}^{Q}\delta_{c_i}^{c_i-1}M_n(\mathbf{x})\right)\\
&=\mathrm{CT}_{\mathbf{x}}\left(\prod_{i=1}^{m}\mathrm{E}_{x_i}^{-1}\frac{Q-(S-Q)\Delta_{x_i}}{Q-(1-Q)\Delta_{x_i}}\left(\frac{-\delta_{x_i}}{Q-(1-Q)\Delta_{x_i}}\right)^{-c_i-1}\right.\\
&\qquad\left.\times\prod_{i=m+1}^{n}\mathrm{E}_{x_i}^{l-2}\frac{Q+(T-Q)\Delta_{x_i}}{Q+(1-Q)\Delta_{x_i}}\left(\frac{-\Delta_{x_i}}{Q+(1-Q)\delta_{x_i}}\right)^{-c_i-1}M_n(\mathbf{x})\right).
\end{aligned}\tag{3.3}$$

We analyse how the operators in (3.3) interact with the argument of the antisymmetriser in (3.1): The effect of the shift operator E_{x_i} is the multiplication by $1+y_i$. Therefore, the application of the forward difference operator Δ_{x_i} or of the backward difference operator δ_{x_i} is equivalent to the multiplication by y_i or by $y_i(1+y_i)^{-1}$, respectively. This observation implies that (3.3) equals

$$\begin{aligned}\mathrm{CT}_{\mathbf{y}}\Bigg(\mathbf{ASym}_{\mathbf{y}}\Bigg(&\prod_{i=1}^{m}(-y_i)^{-c_i-1}(1+y_i)^{c_i}\left(Q-(1-Q)y_i\right)^{c_i+1}\frac{Q-(S-Q)y_i}{Q-(1-Q)y_i}\\&\times\prod_{i=m+1}^{n}y_i^{c_i-1}(1+y_i)^{c_i+l-3}(Q+y_i)^{-c_i+1}\frac{Q+Ty_i}{Q+y_i}\\&\times\prod_{1\le i<j\le n}\left(Q-(1-Q)y_i+y_j+y_iy_j\right)\Bigg)\prod_{1\le i<j\le n}(y_j-y_i)^{-1}\Bigg).\end{aligned}\tag{3.4}$$

Thus far, we have considered (n,l)-alternating sign trapezoids with prescribed 1-column vector $\mathbf{c}$. To sum over all c_i such that $-n\le c_1<\cdots<c_m<0<c_{m+1}<\cdots<c_n\le n$, we ignore the upper and lower bound in the summation since the polynomial in (3.4) has no constant term if $c_1<n$ or $c_n>n$. Hence, by using some geometric series evaluation, we obtain that the argument of the antisymmetriser in (3.4) is equal to

$$\begin{aligned}&\prod_{i=1}^{m}\frac{1}{1+y_i}\left(\frac{-y_i}{(1+y_i)\left(Q-(1-Q)y_i\right)}\right)^{m-i}\\&\qquad\times\left(1-\prod_{j=1}^{i}\left(\frac{-y_j}{(1+y_j)\left(Q-(1-Q)y_j\right)}\right)\right)^{-1}\frac{Q-(S-Q)y_i}{Q-(1-Q)y_i}\\&\times\prod_{i=m+1}^{n}(1+y_i)^{l-2}\left(\frac{y_i(1+y_i)}{Q+y_i}\right)^{i-m-1}\left(1-\prod_{j=i}^{n}\left(\frac{y_j(1+y_j)}{Q+y_j}\right)\right)^{-1}\frac{Q+Ty_i}{Q+y_i}\\&\qquad\times\prod_{1\le i<j\le n}\left(Q-(1-Q)y_i+y_j+y_iy_j\right)(y_j-y_i)^{-1}.\end{aligned}\tag{3.5}$$

Before summing over all m such that $0\le m\le n$, we apply the symmetriser to the expression (3.5). To this end, we use the following trick by Fischer [5]: We set $\mathfrak{S}_n^m:=\{\sigma\in\mathfrak{S}_n\mid\sigma(i)<\sigma(j)\,\forall\,1\le i<j\le m\vee m+1\le i<j\le n\}$ and define

$$\mathbf{Subsets}_{x_1,\ldots,x_m}^{x_{m+1},\ldots,x_n}f(x_1,\ldots,x_n):=\sum_{\sigma\in\mathfrak{S}_n^m}f\left(x_{\sigma(1)},\ldots,x_{\sigma(n)}\right).$$

It follows that

$$\mathbf{Sym}_{x_1,\ldots,x_n}f(x_1,\ldots,x_n)=\mathbf{Subsets}_{x_1,\ldots,x_m}^{x_{m+1},\ldots,x_n}\mathbf{Sym}_{x_1,\ldots,x_m}\mathbf{Sym}_{x_{m+1},\ldots,x_n}f(x_1,\ldots,x_n).$$

That is, we first apply $\mathbf{Sym}_{y_1,\ldots,y_m}$ and $\mathbf{Sym}_{y_{m+1},\ldots,y_n}$ to (3.5) by means of the following antisymmetriser lemma [8]:

Lemma 3.6. *Let $n \geq 1$. Then*

$$\mathbf{ASym}_{\mathbf{x}}\left(\prod_{i=1}^{n} \frac{\left(\frac{x_i(1+x_i)}{Q+x_i}\right)^{i-1}}{1-\prod_{j=i}^{n}\frac{x_j(1+x_j)}{Q+x_j}} \prod_{1\leq i<j\leq n}(Q-(1-Q)x_i+x_j+x_ix_j)\right)$$
$$= \prod_{i=1}^{n}\frac{Q+x_i}{Q-x_i^2}\prod_{1\leq i<j\leq n}\frac{(Q(1+x_i)(1+x_j)-x_ix_j)(x_j-x_i)}{Q-x_ix_j}.$$

Eventually, we obtain

$$\prod_{i=1}^{m}\frac{Q-(S-Q)y_i}{Q(1+y_i)^2-y_i^2}\prod_{1\leq i<j\leq m}\frac{Q-y_iy_j}{Q(1+y_i)(1+y_j)-y_iy_j}\prod_{i=m+1}^{n}(1+y_i)^{l-2}\frac{Q+Ty_i}{Q-y_i^2}$$
$$\times\prod_{m+1\leq i<j\leq n}\frac{Q(1+y_i)(1+y_j)-y_iy_j}{Q-y_iy_j}\prod_{i=1}^{m}\prod_{j=m+1}^{n}\frac{Q-(1-Q)y_i+y_j+y_iy_j}{y_j-y_i}. \quad (3.6)$$

Next, we need to apply the operator $\mathbf{Subsets}_{y_1,\dots,y_m}^{y_{m+1},\dots,y_n}$ to (3.6) and take the constant term. To simplify the computation, we divide (3.6) by the polynomial $\prod_{1\leq i<j\leq n}(Q(1+y_i)(1+y_j)-y_iy_j)(Q-y_iy_j)$, which is symmetric and, thus, invariant under the application of $\mathbf{Subsets}_{y_1,\dots,y_m}^{y_{m+1},\dots,y_n}$. However, we need to incorporate its constant term $Q^{2\binom{n}{2}}$. We get

$$Q^{2\binom{n}{2}}\prod_{i=1}^{m}(Q-(S-Q)y_i)\prod_{i,j=1}^{m}\frac{1}{Q(1+y_i)(1+y_j)-y_iy_j}\prod_{i=m+1}^{n}(1+y_i)^{l-2}(Q+Ty_i)$$
$$\times\prod_{i,j=m+1}^{n}\frac{1}{Q-y_iy_j}\prod_{i=1}^{m}\prod_{j=m+1}^{n}\frac{Q-(1-Q)y_i+y_j+y_iy_j}{(y_j-y_i)\left(Q(1+y_i)(1+y_j)-y_iy_j\right)\left(Q-y_iy_j\right)}. \quad (3.7)$$

This expression can be written in determinantal form. For this purpose, we consider the *Cauchy determinant*

$$\det_{1\leq i,j\leq n}\left(\frac{1}{x_i+y_j}\right)=\frac{\prod_{1\leq i<j\leq n}(x_j-x_i)(y_j-y_i)}{\prod_{i,j=1}^{n}(x_i+y_j)}$$

and set $x_i=\frac{Q(1+y_i)}{Q-(1-Q)y_i}$ for all $1\leq i\leq m$ and $x_i=-\frac{Q}{y_i}$ for all $m+1\leq i\leq n$. This yields that

$$\det_{1\leq i,j\leq n}\left(\begin{cases}\frac{Q-(1-Q)y_i}{Q(1+y_i)(1+y_j)-y_iy_j}, & 1\leq i\leq m\\ \frac{-y_i}{Q-y_iy_j}, & m+1\leq i\leq n\end{cases}\right)$$

is equal to

$$(-1)^{n-m} Q^{\binom{n}{2}} \prod_{i=1}^{m} (Q-(1-Q)y_i) \prod_{i,j=1}^{m} \frac{1}{Q(1+y_i)(1+y_j)-y_iy_j} \prod_{1 \le i<j \le m} (y_j-y_i)^2$$
$$\times \prod_{i=m+1}^{n} y_i \prod_{i,j=m+1}^{n} \frac{1}{Q-y_iy_j} \prod_{m+1 \le i<j \le n} (y_j-y_i)^2$$
$$\times \prod_{i=1}^{m} \prod_{j=m+1}^{n} \frac{(y_j-y_i)\,(Q-(1-Q)y_i+y_j+y_iy_j)}{(Q(1+y_i)(1+y_j)-y_iy_j)\,(Q-y_iy_j)}.$$

Simple row and column transformations of the determinant's underlying matrix show that (3.7) equals

$$\frac{Q^{\binom{n}{2}}}{\prod_{1 \le i<j \le n} (y_j-y_i)^2} \det_{1 \le i,j \le n} \left(\begin{cases} \frac{Q-(S-Q)y_i}{Q(1+y_i)(1+y_j)-yiy_j}, & 1 \le i \le m \\ (1+y_i)^{l-2} \frac{Q+Ty_i}{Q-y_iy_j}, & m+1 \le i \le n \end{cases} \right).$$

It can be shown that the application of $\mathbf{Subsets}_{y_1,\ldots,y_m}^{y_{m+1},\ldots,y_n}$ and the summation over all $1 \le m \le n$ finally yield

$$\frac{Q^{\binom{n}{2}}}{\prod_{1 \le i<j \le n} (y_j-y_i)^2} \det_{1 \le i,j \le n} \left(R \frac{Q-(S-Q)y_i}{Q(1+y_i)(1+y_j)-y_iy_j} + (1+y_i)^{l-2} \frac{Q+Ty_i}{Q-y_iy_j} \right), \quad (3.8)$$

where the exponent of R takes account of m.

The determinantal formula (3.8) is our first expression for the fourfold refined enumeration of (n,l)-alternating sign trapezoids. We transform it into a determinant involving binomial coefficients. Our key tool is the following formula by Behrend, Di Francesco, and Zinn-Justin [3, (43)-(47)]:

Lemma 3.7. *For a given power series f in variables x and y, it holds that*

$$\left. \frac{\det_{1 \le i,j \le n} (f(x_i,y_j))}{\prod_{1 \le i< \le n} (x_j-x_i)\,(x_j-x_i)} \right|_{\mathbf{x}=\mathbf{y}=\mathbf{0}} = \det_{0 \le i,j \le n-1} \left(\left[x^iy^j\right] f(x,y) \right);$$

here, $[x^iy^j]\, f(x,y)$ denotes the coefficient of x^iy^j in the series expansion of f.

We set

$$f(x,y) = R\frac{Q-(S-Q)x}{Q(1+x)(1+y)-xy} + (1+x)^{l-2} \frac{Q+Tx}{Q-xy}$$

and extract the coefficient $[x^iy^j]\, f(x,y)$:

$$R(-1)^{i+j} \sum_{k \ge 0} \binom{j}{k} Q^{-k} \left(\binom{i-1}{k-1} + \binom{i-1}{k} SQ^{-1} \right) + \binom{l-2}{i-j} Q^{-j} + \binom{l-2}{i-j-1} TQ^{-j-1};$$

note that we set the binomial coefficient $\binom{n}{k} := 0$ for $k < 0$. Some manipulation and Lemma 3.7 finally yield that (3.8) and hence the generating function of (n, l)-alternating sign trapezoids with respect to the statistics Q, S, and T is equal to

$$\det_{0 \le i,j \le n-1} \left(R \sum_{k=0}^{i} T^{i-k} \sum_{m=0}^{j} \binom{j}{m} Q^{k-m} \left(\binom{k+l-3}{k-m} + \binom{k+l-3}{k-m-1} S Q^{-1} \right) + \delta_{i,j} \right); \quad (3.9)$$

it can be shown that this is even true if $l = 1$.

4 Weighted Enumeration of Column Strict Shifted Plane Partitions

In order to enumerate column strict shifted plane partitions, we transform them into a family of nonintersecting lattice paths: Each row corresponds to a path that only consists of vertical and horizontal unit steps. If p is the first entry of the corresponding row, then the path starts at $(-1, p-1)$, and every row ends on the x-axis; the heights of the vertical steps are the entries of the row diminished by 1. Figure 4 displays the family of nonintersecting lattice paths corresponding to the column strict shifted plane partition in Figure 2.

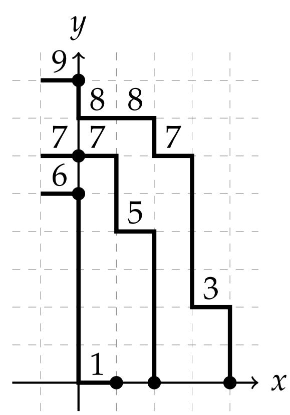

Figure 4: Family of nonintersecting lattice paths corresponding to the column strict shifted plane partition in Figure 2.

This construction yields a bijective correspondence between column strict shifted plane partitions of class $l-1$ with at most n entries in the first row and the family of nonintersecting lattice paths using only horizontal $\leftarrow$ and vertical $\uparrow$ unit steps with start points $S \subseteq \{S_i := (i, 0) \mid 0 \le i \le n-1\}$ and end points $E \subseteq \{E_i := (0, i+l-1) \mid 0 \le i \le n-1\}$ such that $S_i \in S$ if and only if $E_i \in E$.

By this interpretation of column strict shifted plane partitions as a family of nonintersecting lattice paths and by the Lindström-Gessel-Viennot lemma, it can be shown that column strict shifted plane partitions of class $l-1$ with at most n entries in the first row are enumerated by

$$\det_{0\le i,j\le n-1}\left(\binom{i+j+l-1}{i}+\delta_{i,j}\right). \tag{4.1}$$

This was first proved by Andrews [1]. In fact, this determinant (4.1) can be obtained from (3.9) by setting $Q=R=S=T=1$.

Andrews' result can be generalized: By setting $Q=1$ in (3.9), we obtain

$$\det_{0\le i,j\le n-1}\left(R\sum_{k=0}^{i}T^{i-k}\left(\binom{k+j+l-3}{k}+\binom{k+j+l-3}{k-1}S\right)+\delta_{i,j}\right).$$

This is the generating function of column strict shifted plane partitions of class $l-1$ with at most n entries in the first row with respect to the statistics R, S, and T, which was proved by Fischer [5]. In particular, we see that

$$\sum_{k=0}^{i}T^{i-k}\binom{k+j+l-3}{k}$$

is the generating function of lattice paths from $(i,0)$ to $(0,j+l-1)$ where the line $y=x+d$ is reached by a vertical step, and T counts the number of horizontal steps at height 0. As a straightforward consequence,

$$\sum_{k=0}^{i}T^{i-k}\sum_{m=0}^{j}\binom{j}{m}\binom{k+l-3}{k-m}Q^{k-m}$$

is the generating function of lattice paths from $(i,0)$ to $(0,j+l-1)$ where the line $y=x+d$ is reached by a vertical step, T takes the number of horizontal steps at height 0 into account, and, in addition, Q counts the number of horizontal steps which are under the line $y=x+l-1$ and have at least height 1, that is, which are not already counted by T. Similarly,

$$\sum_{k=0}^{i}T^{i-k}\sum_{m=0}^{j}\binom{j}{m}\binom{k+l-3}{k-m-1}SQ^{k-m}$$

is the generating function of lattice paths from $(i,0)$ to $(0,j+l-1)$ where the line $y=x+d$ is reached by a horizontal step which S keeps track of, T counts the number of horizontal steps at height 0, and Q counts the number of horizontal steps which are beneath the line $y=x+l-1$ that are not already taken into account by the statistics T and S. As a result, (3.9) is also the generating function of column strict shifted plane partitions of class $l-1$ with at most n entries in the first row with respect to the statistics Q, R, S, and T. This completes the proof of Theorem 2.3.

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