Séminaire Lotharingien de Combinatoire **84B** (2020)
Article #96, 12 pp.

Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics (Online)

Billiards, Channels, and Perfect Matching 2-Divisibility

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Abstract. Let m_G denote the number of perfect matchings of the graph G. We introduce a number of combinatorial tools for determining the parity of m_G and giving a lower bound on the power of 2 dividing m_G. In particular, we introduce certain vertex sets called channels, which correspond to elements in the kernel of the adjacency matrix of G modulo 2. A result of Lovász states that the existence of a nontrivial channel is equivalent to m_G being even. We strengthen this result by showing that the number of channels gives a lower bound on the power of 2 dividing m_G when G is planar. We describe a number of local graph operations which preserve the number of channels. We also establish a surprising connection between 2-divisibility of m_G and dynamical systems by showing an equivalency between channels and billiard paths. We exploit this relationship to show that $2^{(\gcd(m+1,n+1)-1)/2}$ divides the number of domino tilings of the $m \times n$ rectangle.

Keywords: perfect matchings, domino tilings, divisibility, billiard paths

1 Introduction

Given a graph G, a *perfect matching* of G is a subset of edges μ such that each vertex of G is contained in a unique edge in μ. We let m_G denote the number of distinct perfect matchings of G. The problem of determining m_G arises in various mathematical contexts, particularly in tiling problems. Exact formulas for m_G over an infinite family of graphs are quite rare. One notable exact formula is for $G = \mathcal{R}_{m\times n}$, the rectangular subgraph of the square lattice with m rows of n vertices. In this case, the famous result of Kasteleyn [3] gives

$$m_G^4 = \prod_{j=1}^{n}\prod_{k=1}^{m}\left(4\cos^2\frac{j\pi}{n+1} + 4\cos^2\frac{k\pi}{m+1}\right).$$

From this product we may extract certain number theoretic information. In particular, m_G is always divisible by $2^{(\gcd(n+1,m+1)-1)/2}$ [6]. Studying similar 2-divisibility patterns

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is a common theme in the literature on domino tilings, which are equivalent to perfect matchings of subgraphs of the square lattice. It is often the case that the 2-component of the prime factorization of m_G follows a predictable pattern, even when an exact formula for m_G is elusive or unwieldy. In Propp's perfect matching problem anthology [7], he gives a number of conjectured and known power of 2 patterns for various graphs. Most results along these lines rely on a theorem of Ciucu [2] that requires the graph have reflective symmetry, which excludes most graphs from consideration. The results we introduce here require only planarity and sometimes a degree condition, thus providing a uniform (partial) explanation of power of 2 patterns in terms of the geometry of the graph. Our foundational construction is based on the following result known to Lovász.

Proposition 1.1 ([5, Problem 5.18])**.** *Let G be any graph. Then m_G is even if and only if there is a nonempty vertex set $C \subseteq V$ such that every vertex in G is adjacent to an even number of vertices in C.*

Vertex sets C satisfying the hypothesis of Proposition 1.1 are called *channels*. (We also count the empty set as a trivial channel.) Lovász's result already shows the importance of channels for determining the parity of m_G. The main theorem of this extended abstract shows that channels have even more to say for planar graphs.

Theorem 1.2 (Channeling twos)**.** *Let G be a planar graph. Then the number of distinct channels in G divides m_G^2.*

Since the number of channels will always be a power of 2, Theorem 1.2 gives a lower bound on the power of 2 dividing m_G for any planar graph. We show the strength of this theorem in a number of examples throughout the abstract. In particular, we will show that $2^{(\gcd(n+1,m+1)-1)/2}$ divides $m_{\mathcal{R}_{m\times n}}$.

Many of our results are tailored for subgraphs of the square lattice, where perfect matchings are equivalent to domino tilings of a region. When possible, however, we will state results in greater generality. Our most fascinating result is a characterization of channels in terms of dynamical systems. We state the result here for subgraphs of the square lattice, and describe the general case in Section 3.

Let G be a full subgraph of the square lattice such that each internal (bounded) face of G is a unit square. In the dual language of domino tilings, such graphs correspond to simply connected regions of the plane. Since G is bipartite, we 2-color the vertices of G black and white. An example of such a graph is shown in Figure 1.

Now we define a *billiard path* on G to be any collection of paths traced out by billiard balls placed on black vertices of G and launched at 45 degree angles. When a billiard ball reaches a wall, it reflects at a 90 degree angle and proceeds in its new direction, continuing until it is caught by a corner or returns to its start position. (If the billiard brushes past a corner or hits one head-on, the situation is more complicated—the path

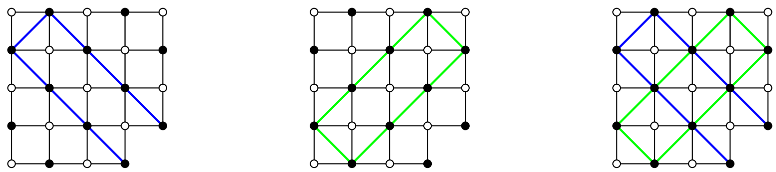


Figure 1: The three nonempty billiard paths in a full subgraph G of the square lattice that is composed of unit squares.

splits into two paths continuing in different directions. See Section 3 for a more precise definition.)

Remarkably, channels and billiard paths are intrinsically connected. Let G' be the *inner subgraph* of G, the subgraph formed by removing all vertices of G which are incident to the unbounded face and all edges incident to those vertices.

Theorem 1.3. *Let G satisfy the assumptions described above, and let G' be the inner subgraph of G. Further assume that the dual graph of G is 2-connected. Then the number of billiard paths in G is twice the number of channels which use only the black vertices of G'.*

In particular, a bipartite version of channeling twos implies that the number of billiard paths for G divides $2m_{G'}$. For the graph G in Figure 1, the inner subgraph G' is shown in Figure 2. Since G has 4 billiard paths and satisfies the hypothesis of Theorem 1.3, it follows that $m_{G'}$ is divisible by 2. Indeed, there are 4 perfect matchings of G'.

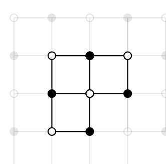


Figure 2: The inner subgraph G' of the graph G defined in Figure 1.

The connection between 2-divisibility, channels, and dynamical systems explains both the sensitivity and the regularity of perfect matching 2-divisibility. Small changes to G can result in entirely different billiard dynamics, with the effects visible in $m_{G'}$. The dynamics can also induce a regularity in the 2-divisibility of m_G. The well-known theory of arithmetic billiards describes billiard paths for rectangles in terms of divisibility properties of the rectangle side lengths. In Section 3.2, we exploit these results to explain the factor of $2^{(\gcd(m+1,n+1)-1)/2}$ dividing m_G for the $m \times n$ grid graph.

Billiard paths give a global explanation of channel structure for many graphs. Sometimes we are instead interested in local behavior. For instance, we may have a family of graphs which are globally similar, but differ locally. To relate these graphs, we introduce in Section 2.1 a set of channel-preserving graph operations and show that they may be applied repeatedly to reduce many graphs to a set of independent vertices.

The paper is organized as follows. Section 2 gives definitions and context for Theorem 1.2. In Section 2.1, we discuss certain graph moves that always preserve the number of channels in a graph. Section 2.2 introduces a useful graph move called *diagonal contraction*. We describe an application of diagonal contraction and channels to the Aztec diamond family of graphs. Section 3 describes billiard paths for a large class of graphs called *inner semi-Eulerian graphs*. The results described in the introduction are applied to the rectangle grid graph, connecting its 2-divisibility to the theory of arithmetic billiards. In the full version [1] of this extended abstract, we also give a combinatorial proof of Proposition 1.1, a fast algorithm using billiard paths for determining the 2-divisibility of certain graphs, and a number of additional examples and theorems.

2 Channels

All graphs in this paper are undirected, finite, and contain no self-loops. If a graph is bipartite, we will consider its vertices to be colored black and white. Additionally, all matchings discussed will be perfect matchings, and thus the word "perfect" will be omitted in the future for brevity. For a graph $G = (V, E)$, V denotes the vertex set, E denotes the edge set, and A denotes the adjacency matrix.

Sometimes we will be interested in planar graphs G. Such graphs admit a dual graph, with vertices given by the faces of G and edges between faces separated by an edge in G. If the same face is on both sides of an edge of G, then that edge corresponds to a self-loop in the dual graph. The *external face* of G is the face which is unbounded, and all other faces are *internal faces* of G. The *reduced dual graph* of G is the dual graph of G with the vertex corresponding to the external face of G removed. We say a vertex of G is *external* if it is incident to the external face, and we say it is *internal* otherwise.

Let G be a graph with adjacency matrix A. The 2-kernel $\ker_2 A$ is the kernel of A considered as a matrix over $\mathbf{Z}/2\mathbf{Z}$. Then a vector x in $\ker_2 A$ has entries in $\mathbf{Z}/2\mathbf{Z}$ and can be lifted to a vector $\tilde{x}$ with entries $0, 1 \in \mathbf{Z}$. The condition $Ax = 0$ then becomes $A\tilde{x} = 2y$ for some integral vector y. Because each row of x corresponds to a vertex in G, we may interpret x as the indicator function for a vertex set C, where a row with a 1 indicates the vertex is in C and a row with a 0 indicates the vertex is not in C. This leads to the following interpretation of 2-kernel elements.

Definition 2.1. Let $G = (V, E)$ be any graph. A *channel* is a set C of vertices such that every vertex in G is adjacent to an even number of vertices in C. In other words, a channel satisfies

$$|\{(v, v') \in E \mid v' \in C\}| \text{ is even for all } v \in V.$$

Let the set of channels in G be denoted $\mathcal{C}(G)$. If G is bipartite, let $\mathcal{C}_B(G)$ (resp. $\mathcal{C}_W(G)$) be the subspace of $\mathcal{C}(G)$ consisting of channels that use only black (resp. white) vertices.

The 2-kernel also has an additive structure as a $\mathbf{Z}/2\mathbf{Z}$ vector space. This transfers to $\mathcal{C}(G)$ by defining the sum of $C_1, C_2 \in \mathcal{C}(G)$ to be the symmetric difference of C_1 and C_2.

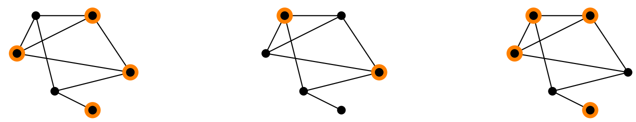


Figure 3: A graph G with its three nonempty channels indicated by shading. Any two of these form a basis for the space $\mathcal{C}(G)$.

For planar graphs G, there exists a matrix K (called a *Kasteleyn matrix* [4] of G) obtained from A by adding signs to some entries such that $\det K = m_G^2$. We note that K and A are equivalent modulo 2 and thus have the same 2-kernel. The next theorem follows by analyzing the Smith normal form of K and identifying 2-kernels with channels (see [1] for more details).

Theorem 2.2 (Channeling twos)**.** *If $\{C_1, ..., C_n\}$ is a linearly independent set of channels in a graph G with a Kasteleyn matrix (in particular, for planar G), then*

$$2^n \text{ divides } m_G^2.$$

If additionally G is bipartite, and $\{C_1, ..., C_n\} \subseteq \mathcal{C}_B(G)$, then

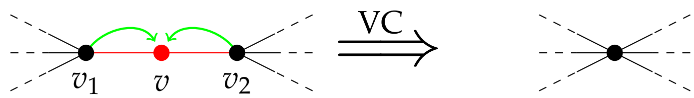
$$2^n \text{ divides } m_G.$$

Remark 2.3. Despite the fact that Proposition 1.1 holds for an arbitrary graph, Theorem 2.2 does not. For example, the complete bipartite graph $K_{3,3}$ has $|\mathcal{C}_B(K_{3,3})| = 2^2$, but $m_{K_{3,3}} = 6$ is not divisible by 2^2. Thus the assumption of a Kasteleyn matrix for G cannot be weakened much further.

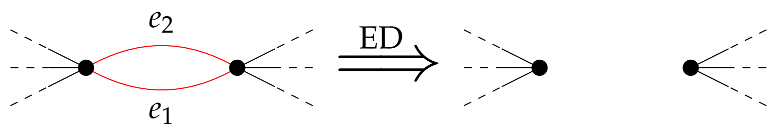
2.1 Channel-preserving moves

In this section, we describe a set of local graph moves which unconditionally preserve channels and (for two of the moves) perfect matchings of our graph. In many cases we can reduce the graph to a collection of vertices via these moves. We begin by introducing our operations of interest.

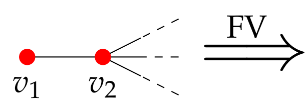
A *2-valent vertex contraction* may be applied to any vertex v of degree two that is adjacent to distinct vertices v_1, v_2. The resulting graph is formed by contracting the edges incident to v and deleting self-loops if they occur.



A *doubled edge deletion* may be applied to any pair of edges e_1, e_2 that share the same endpoints. This operation removes e_1 and e_2 from the graph.



A *forced vertex pair removal* may be applied to distinct adjacent vertices v_1, v_2 such that v_1 has degree one. The resulting graph is formed by removing v_1, v_2 and all edges incident to these vertices.



We call these three moves *channel-preserving moves*. As the name suggests, applying a channel-preserving move to a graph preserves the number of channels in that graph. A graph is called *reducible* if it can be reduced to a set of degree 0 vertices using only channel-preserving moves.

Theorem 2.4. *Let G be a reducible graph. Then the number of degree 0 vertices remaining after G has been fully reduced is the dimension of $\mathcal{C}(G)$. In particular, this number is independent of the choice of channel-preserving moves used to reduce the graph.*

Example 2.5. The graph G shown in Figure 3 is a reducible graph. Figure 4 shows a possible sequence of channel-preserving moves. Because G reduces to two vertices of degree 0, G must have 2^2 channels. This is indeed the case; Figure 3 shows the three nonempty channels.

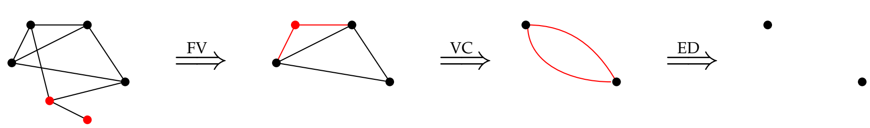


Figure 4: Reducing a graph to independent vertices.

One structural property that implies reducibility is the following. We call a planar graph called *inner Eulerian* if all internal vertices have even degree. (In particular, this property holds for many subgraphs of the square lattice.)

Theorem 2.6. *Let G be an inner Eulerian bipartite graph. Then G is reducible.*

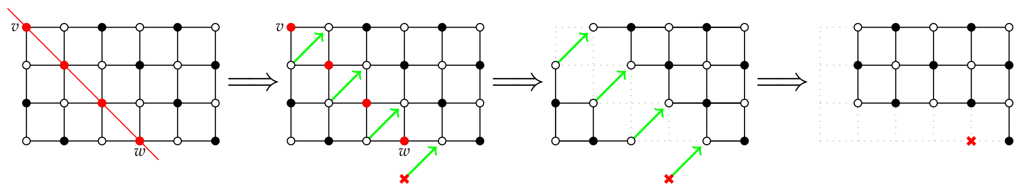


Figure 5: Contracting the highlighted diagonal by deleting the diagonal vertices and merging the vertices immediately opposite.

2.2 Contracting diagonals

When our region is a subgraph of the square lattice, there is a useful sequence of channel-preserving moves available called a *diagonal contraction*. Pick a degree 2 vertex v that is a corner of the graph. Then v defines a unique diagonal passing through it, as in the first step of Figure 5.

We say that the diagonal is *contractible* if each internal vertex and each internal face it intersects have degree 4. In this case, diagonal contraction proceeds by selecting all vertices on the diagonal between v and w, the last vertex on the diagonal before it reaches the external face. For each selected vertex v_1, delete v_1 and combine each neighbor of v_1 with its mirror image across the diagonal, as shown in Figure 5. If a vertex combines with a missing vertex (denoted by a red "x" in the figure), then that vertex is deleted.

Theorem 2.7. *Let G be a subgraph of the square lattice. Let G' be the result of applying a diagonal contraction to a contractible diagonal from a black corner vertex v to a vertex w. If w has degree 2, then*

$$|\mathcal{C}_B(G)| = 2|\mathcal{C}_B(G')| \text{ and } |\mathcal{C}_W(G)| = |\mathcal{C}_W(G')|.$$

Otherwise,

$$|\mathcal{C}_B(G)| = |\mathcal{C}_B(G')| \text{ and } |\mathcal{C}_W(G)| = |\mathcal{C}_W(G')|.$$

Example 2.8. Let us apply diagonal contraction to the a well-known class of graphs. The *Aztec diamond* of rank n is a diamond of side length n in the square lattice. The Aztec diamonds of rank 1, 2, and 3 are shown in Figure 6.

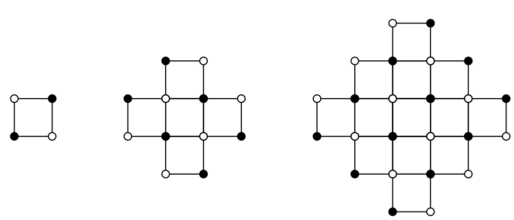
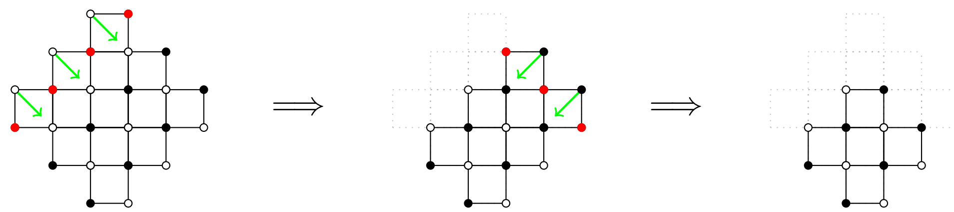


Figure 6: The Aztec diamonds of rank 1, 2, and 3.

Let G_n be the rank n Aztec diamond. Performing two diagonal contractions as shown below reduces G_n to G_{n-1}.



Since both diagonal contractions end on a vertex of degree 2, $|\mathcal{C}(G_n)| = 2^2|\mathcal{C}(G_{n-1})|$. An easy induction argument then shows that

$$|\mathcal{C}(G_n)| = 2^{2n}.$$

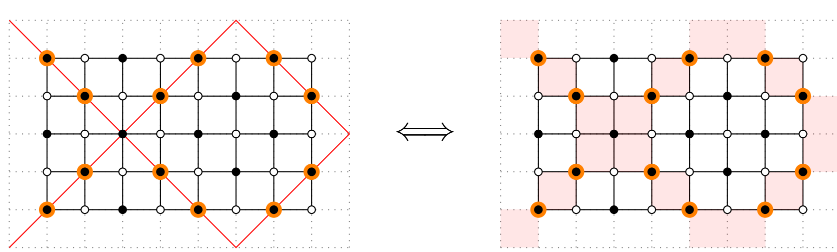
Since there are 2^{2n} channels in G_n, by channeling twos it follows that 2^n divides the number of matchings of G_n. Indeed, it is well-known that G_n has $2^{\binom{n+1}{2}}$ matchings.

3 Billiards and Channels

For an arbitrary graph, it is not clear how to identify all channels contained within it. In this section we give a geometric approach to channel construction based on a phenomenon that can be observed in the channels of a rectangle grid graph.

3.1 Billiard paths

In the rectangle, we note that channels tend to form along diagonal lines as in the following figure. This pattern was studied by Tomei and Vieira in [8], where they described it in terms of polygonal tilings of the rectangle. We propose an alternative description. By extending these diagonals, we find that these lines form a path which reflects off the edges of a larger rectangle, as shown below. The channel vertices are vertices in the interior of this larger rectangle which intersect exactly one line from this path.



Notice that we may recover the path, given that it was a path through black vertices, by remembering just the faces that it passes through. This viewpoint of the path will allow us to define a similar structure on a wide class of graphs called inner semi-Eulerian graphs.

Definition 3.1. We say that a bipartite planar graph G is *inner semi-Eulerian* if every internal black vertex of G has even degree. Let G be inner semi-Eulerian and let F denote the set of internal faces in G. We say that a subset of faces $B \subseteq F$ of G is a *billiard path* if the following hold:

- if b is an internal black vertex, then either all faces incident to b are in B, no faces incident to b are in B, or every second face incident to b is in B.
- if b is an external black vertex, then either all internal faces incident to b are in B or no internal faces incident to b are in B.

Denote the set of billiard paths in G by $\mathcal{B}(G)$.

When G is a full subgraph of the square lattice with all internal faces being unit squares, this agrees with the intuitive notion of billiard paths as the paths traced out by a collection of billiard balls. Note that $\varnothing$ and F are trivially billiard paths for every graph. As with channels, we may define the sum of two billiard paths to be their symmetric difference, making $\mathcal{B}(G)$ a vector space over $\mathbf{Z}/2\mathbf{Z}$.

We have a canonical basis for $\mathcal{B}(G)$ such that the basis billiard paths are mutually disjoint. Indeed, define a graph $G_{\mathcal{B}}$ with vertex set F and edges between f and f' if they satisfy one of the following:

- f and f' are incident to the same internal black vertex b and are separated by an odd number of faces incident to b.
- f and f' are incident to the same external black vertex b.

Then the connected components of $G_{\mathcal{B}}$ are independent billiard paths that span $\mathcal{B}(G)$. This is called the *path basis* for G. This is particularly useful since as we shall soon see, billiard paths in G are equivalent to channels in a subgraph of G.

Definition 3.2. Let G be inner semi-Eulerian. Then the *inner subgraph* of G, denoted G', is the induced subgraph on the internal vertices of G. Given any inner semi-Eulerian graph H, an *outer completion* of H is an inner semi-Eulerian graph G such that $G' = H$.

Given an inner semi-Eulerian graph H, we may always construct an outer completion G such that $G' = H$ by taking a copy of the boundary of H, expanding it so that H lies within it, and adding edges between the two copies of the boundary as needed to make the graph inner semi-Eulerian. Given a billiard path in G, we may construct an associated channel in G' as follows. Let $B \in \mathcal{B}(G)$ be a billiard path. Define

$$\mathrm{ch}\colon \mathcal{B}(G) \longrightarrow \mathcal{C}_B(G')$$

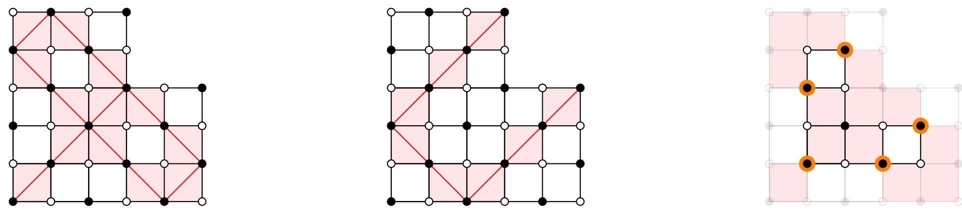


Figure 7: A graph with its two path basis elements (shown formally as the set of shaded faces and informally as the path drawn above it). To the right we show the corresponding channel on the inner subgraph.

by setting the vertices in $\mathrm{ch}(B)$ to be the internal black vertices b of G for which exactly half of the faces incident to b are in B.

Lemma 3.3. *The map* ch *is a group homomorphism from* $\mathcal{B}(G)$ *to* $\mathcal{C}_B(G')$.

We can now state our main result on billiard paths. Recall that the reduced dual graph of G is the dual graph with the vertex corresponding to the external face of G removed.

Theorem 3.4. *Let G be inner semi-Eulerian and let G' be the inner subgraph of G. Let c be the number of connected components of the reduced dual graph of G. Then*

$$|\mathcal{B}(G)| = 2^c|\mathcal{C}_B(G')|.$$

In particular, $2^{-c}|\mathcal{B}(G)|$ divides the number of matchings of G'.

Thus our study of channels in appropriate graphs H (in particular, all inner semi-Eulerian graphs) reduces to the study of billiard paths in an outer completion G. Billiard paths are considerably easier to work with since every face of G is contained in a unique element of the path basis of G. In general there is no such basis for the channels of H; vertices of H may be contained in no channel and there may be no channel basis for H with pairwise disjoint elements. However, any path basis element can be found by starting with a face of G and adding additional faces as required by the definition of billiard paths. In [1], we use this to give an algorithm for constructing the path basis for certain grid graphs which is almost linear in the perimeter of the graph.

3.2 Arithmetic billiards

Let us find the billiard paths for the rectangle grid graph $\mathcal{R}_{m+1\times n+1}$, an outer completion of $\mathcal{R}_{m-1\times n-1}$. For this graph, we may use our interpretation of billiard paths as the paths traced out by billiard balls travelling at 45 degree angles. We begin by straightening out the billiard paths; to do so, we tile the plane with copies of our rectangle.

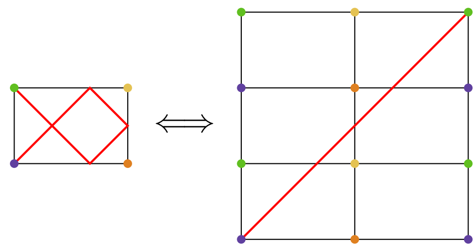


Figure 8: A rectangle used to tile the plane. The coloring is provided as a visual indicator of the rectangle's orientation in the tessellation. Any billiard path on the left corresponds to a line of slope 1 on the right and vice versa.

We can then lift the billiard path to a straight line of slope 1 in the tessellation. A billiard path between two corners of $\mathcal{R}_{m+1\times n+1}$ will lift to the diagonal of a square in the tessellation. Since $\mathcal{R}_{m+1\times n+1}$ has side lengths m and n, this square will have side length $\operatorname{lcm}(m,n)$, so the path will pass through $\operatorname{lcm}(m,n)$ faces.

Suppose at least one of $m+1$ and $n+1$ is even so that exactly two corners of $\mathcal{R}_{m+1\times n+1}$ are black. Of the billiard paths on black vertices, one passes through both of the black corners of the rectangle and passes through $\operatorname{lcm}(m,n)$ faces. The others lift to a diagonal of a square of side length $2\operatorname{lcm}(m,n)$ (twice as long since they must return to their starting point) and hence pass through $2\operatorname{lcm}(m,n)$ squares. Since every internal face is part of a unique path basis element, we may now count the billiard paths for $\mathcal{R}_{m+1\times n+1}$.

Theorem 3.5. *The rectangle grid graph $\mathcal{R}_{m+1\times n+1}$ with $(m+1)(n+1)$ even has a path basis of size*

$$\frac{\gcd(m,n)+1}{2}.$$

Proof. There are mn total internal faces in $\mathcal{R}_{m+1\times n+1}$. From the above, $mn - \operatorname{lcm}(m,n)$ of these are part of a path basis element not passing through a corner. Each such basis element uses $2\operatorname{lcm}(m,n)$ internal faces. Thus there are

$$\frac{mn - \operatorname{lcm}(m,n)}{2\operatorname{lcm}(m,n)} = \frac{\gcd(m,n)-1}{2}$$

non-corner path basis elements. Adding back the last path basis element gives the claim. □

Corollary 3.6. *The rectangle grid graph $\mathcal{R}_{m-1\times n-1}$ with $(m-1)(n-1)$ even has*

$$|\mathcal{C}_B(\mathcal{R}_{m-1\times n-1})| = 2^{\frac{\gcd(m,n)-1}{2}}.$$

Proof. The inner subgraph of $\mathcal{R}_{m+1\times n+1}$ is $\mathcal{R}_{m-1\times n-1}$, and the reduced dual graph of $\mathcal{R}_{m\times n}$ is connected. Thus the result follows by Theorem 3.4. □

Corollary 3.7. *The number of matchings of $\mathcal{R}_{m-1\times n-1}$ is divisible by*

$$2^{\frac{\gcd(m,n)-1}{2}}.$$

Proof. If $m-1$ and $n-1$ are both odd, then $m_G = 0$ and the claim follows. Otherwise, the hypothesis of the previous corollary holds, and we may channel twos to arrive at the result. □

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