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The $(s, s+d, \ldots, s+pd)$-core partitions and rational Motzkin paths

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Abstract. In this article, we propose an $(s+d, d)$-abacus for $(s, s+d, \ldots, s+pd)$-core partitions and establish a bijection between $(s, s+d, \ldots, s+pd)$-core partitions and rational Motzkin paths of type $(s+d, -d)$. This result not only gives a lattice path interpretation of the $(s, s+d, \ldots, s+pd)$-core partitions but also counts them with a closed formula. Also we enumerate $(s, s+1, \ldots, s+p)$-core partitions with k corners and self-conjugate $(s, s+1, \ldots, s+p)$-core partitions.

Keywords: simultaneous core partitions, self-conjugate, corners, rational Motzkin paths, generalized Dyck paths

1 Introduction

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of a positive integer n is a finite non-increasing sequence of positive integer parts λ_i such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$. The Young diagram of λ is a finite collection of n boxes arranged in left-justified rows, with the ith row having λ_i boxes. For the Young diagram of λ, the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ is called the conjugate of λ, where λ'_j denotes the number of boxes in the jth column. For each box of the Young diagram in coordinates (i, j), the hook length $h(i, j)$ is the number of boxes weakly below and strictly to the right of the box. For a partition λ, the *beta-set* of λ, denoted $\beta(\lambda)$, is defined to be the set of first column hook lengths of λ. For example, the conjugate of $\lambda = (5, 4, 2, 1)$ is $\lambda' = (4, 3, 2, 2, 1)$ and the *beta-set* of λ is $\beta(\lambda) = \{8, 6, 3, 1\}$.

For a positive integer t, a partition λ is a t-core (partition) if it has no box of hook length t. In the previous example, λ is a t-core for $t = 5, 7$, or $t \geq 9$. For distinct positive integers $t_1, t_2, \ldots, t_p$, we say that a partition λ is a $(t_1, t_2, \ldots, t_p)$-core if it is simultaneously a t_1-core, a t_2-core, …, and a t_p-core. The study of core partitions arose from the representation theory of the symmetric group S_n (see [12] for details). Researches on simultaneous core partitions are motivated by the following result of Anderson [2].

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Theorem 1.1 ([2], Theorem 1)**.** *For relatively prime positive integers s and t, the number of (s,t)-core partitions is*

$$\frac{1}{s+t}\binom{s+t}{s}.$$

In particular, the number of $(s,s+1)$-core partitions is the sth Catalan number $C_s = \frac{1}{s+1}\binom{2s}{s}$.

Since the work of Anderson, results on (s,t)-cores were published by many researchers (see [3, 6, 8, 9, 10, 19, 23]). Also, some researchers concerned with simultaneous core partitions whose cores line up in arithmetic progression (see [1, 4, 7, 20, 21, 22]). Yang-Zhong-Zhou [22] showed that the number of $(s,s+1,s+2)$-core partitions is equal to the sth Motzkin number $M_s = \sum_{k=0}^{\lfloor s/2\rfloor}\binom{s}{2k}C_k$, where C_k is the kth Catalan number. Amdeberhan-Leven [1] and Wang [20] extended this result as follows.

Let an (s,p)*-generalized Dyck path* be a lattice path from $(0,0)$ to (s,s) which stays weakly above the line $y=x$ and consists of vertical steps $(0,p)$, horizontal steps $(p,0)$, and diagonal steps (i,i) for $i=1,2,\ldots,p-1$.

Theorem 1.2 ([1], Theorem 4.2)**.** *For positive integers s and p, the number of $(s,s+1,\ldots,s+p)$-core partitions is equal to the number $C_s^{(p)}$ of (s,p)-generalized Dyck paths, which satisfies the following recurrence relation:*

$$C_s^{(p)} = \sum_{k=1}^{s} C_{k-p}^{(p)} C_{s-k}^{(p)},$$

where $C_s^{(p)} = 1$ for $s<0$.

Theorem 1.3 ([20], Theorem 1.6)**.** *] For relatively prime positive integers s and d, the number of $(s,s+d,s+2d)$-core partitions is*

$$\frac{1}{s+d}\sum_{k=0}^{\lfloor s/2\rfloor}\binom{s+d}{k,k+d,s-2k}.$$

Recently, Baek-Nam-Yu [4] obtained an alternative proof for Theorem 1.3 and found a formula for the number of $(s,s+d,s+2d,s+3d)$-core partitions.

Theorem 1.4 ([4], Theorem 5.7)**.** *For relatively prime positive integers s and d, the number of $(s,s+d,s+2d,s+3d)$-core partitions is*

$$\frac{1}{s+d}\sum_{k=0}^{\lfloor s/2\rfloor}\left\{\binom{s+d-k}{k}+\binom{s+d-k-1}{k-1}\right\}\binom{s+d-k}{s-2k}.$$

An idea of counting paths was used in [1, 2, 3, 6, 10, 11] and counting the lattice points method was used in [4, 8, 9, 13, 20]. A poset structure is the main tool to get the formulae for counting simultaneous core partitions in [19, 22, 23].

In this article, we define the "rational Motzkin path", which generalizes the idea of the Motzkin path, and give a generalization of Theorems 1.2 to 1.4 by using rational Motzkin paths of type $(s+d, -d)$ with a specific restriction (see Definition 2.5). The following is the main result of this article.

Theorem 1.5. *Let s and d be relatively prime positive integers. For a given integer $p \geq 2$, the number of $(s, s+d, \dots, s+pd)$-core partitions is equal to the number of rational Motzkin paths of type $(s+d, -d)$ without UF^iU steps for $i = 0, 1, \dots, p-3$ if $p \geq 3$, that is*

$$\frac{1}{s+d}\binom{s+d}{d} + \sum_{k=1}^{\lfloor s/2 \rfloor} \sum_{\ell=0}^{r} \frac{1}{k+d}\binom{k+d}{k-\ell}\binom{k-1}{\ell}\binom{s+d-\ell(p-2)-1}{2k+d-1},$$

where $r = \min(k-1, \lfloor (s-2k)/(p-2) \rfloor)$.

As a corollary, by setting $d = 1$, we obtain a closed formula for the number of $(s, s+1, \dots, s+p)$-core partitions. Also, we give a bijection between the set of (s, p)-generalized Dyck paths and that of Motzkin paths of length s with a restriction. Furthermore, we count the number of $(s, s+1, \dots, s+p)$-core partitions with k corners. At the end of this article, we enumerate self-conjugate $(s, s+1, \dots, s+p)$-core partitions.

2 Counting $(s, s+d, \dots, s+pd)$-core partitions

2.1 The $(s+d, d)$-abacus diagram

James-Kerber [12] introduced the abacus diagram which has played important roles in the theory of core partitions (see [3, 8, 14, 16, 17]). The *s-abacus diagram* is a diagram with infinitely many rows labeled by nonnegative integers such that the smallest index is at the bottom, and s columns labeled by $0, 1, \dots, s-1$, whose position in (i, j) is labeled by $si + j$, where $i \geq 0$ and $j = 0, 1, \dots, s-1$. The *s-abacus* of a partition λ is obtained from the s-abacus diagram by placing a bead on each position which the number at this position belongs to $\beta(\lambda)$. Positions without beads are called spacers. It is well-known that λ is an s-core if and only if the s-abacus of λ has no spacer below a bead in any column. Equivalently, one can have the following.

Lemma 2.1 ([12], Lemma 2.7.13). *For a partition λ, λ is an s-core if and only if $x \in \beta(\lambda)$ implies $x - s \in \beta(\lambda)$ whenever $x - s > 0$.*

We now introduce the $(s+d, d)$-abacus diagram, which generalizes the definition of the s-abacus diagram and the two-way abacus diagram suggested by Anderson [2].

Definition 2.2. Let s and d be relatively prime positive integers. The $(s+d,d)$-*abacus diagram* is a diagram with infinitely many rows labeled by integers and $s+d+1$ columns labeled by $0,1,\dots,s+d$, whose position in (i,j) is labeled by $(s+d)i+dj$, where $i \in \mathbb{Z}$ and $j = 0,1,\dots,s+d$. For a partition λ, the $(s+d,d)$-*abacus* of λ is obtained from the $(s+d,d)$-abacus diagram by placing a *bead* on each position which the number at this position belongs to $\beta(\lambda)$. Again, a position without a bead is called a *spacer*.

Example 2.3. If $\lambda = (6,4,3,1,1,1,1)$, then λ is a $(5,8,11,\dots)$-core partition and its beta-set is $\beta(\lambda) = \{12,9,7,4,3,2,1\}$. Figure 1 shows the Young diagram with the hook lengths and the $(8,3)$-abacus of λ.

12	7	6	4	2	1
9	4	3	1		
7	2	1			
4					
3					
2					
1					

				$\vdots$					
24	27	30	33	36	39	42	45	48	$i=3$
16	19	22	25	28	31	34	37	40	$i=2$
8	11	14	17	20	23	26	29	32	$i=1$
0	③	6	⑨	⑫	15	18	21	24	$i=0$
−8	−5	−2	①	④	⑦	10	13	16	$i=-1$
−16	−13	−10	−7	−4	−1	②	5	8	$i=-2$
−24	−21	−18	−15	−12	−9	−6	−3	0	$i=-3$
				$\vdots$					

Figure 1: The Young diagram of the partition $(6,4,3,1,1,1,1)$ with the hook lengths and its $(8,3)$-abacus

This modified abacus diagram is useful when we consider $(s,s+d,\dots,s+pd)$-core partitions with $p \geq 2$. For a given $(s,s+d,\dots,s+pd)$-core partition λ, if we consider the $(s+d,d)$-abacus of λ, then a bead on the position in (i,j) implies that positions in $(i-1,j-p+1),(i-1,j-p+2),\dots,(i-1,j+1)$ are also beads whenever these positions are labeled by positive integers as in Figure 1. We now have the following lemma.

Lemma 2.4. *For a given $p \geq 2$ and the $(s+d,d)$-abacus of an $(s,s+d,\dots,s+pd)$-core λ, we define a function $f : \{0,1,\dots,s+d\} \to \mathbb{Z}$ as follows: For a column number j, $f(j)$ is defined to be the smallest row number i satisfying that the position in (i,j) is a spacer being labeled by a nonnegative integer. Then f satisfies that*

(a) $f(0) = 0$ *and* $f(s+d) = -d$,

(b) $f(j-1)$ *is exactly one of the values* $f(j)-1$, $f(j)$, *and* $f(j)+1$, *for* $1 \leq j \leq s+d$,

(c) $f(j-1) = f(j)-1$ *implies that* $f(j-p+1), f(j-p+2), \dots, f(j-2) \geq f(j-1)$, *for* $p-1 \leq j \leq s+d$.

As depicted in Figure 1, for the $(8,3)$-abacus of $\lambda = (6,4,3,1,1,1,1)$, we have $f(0) = 0$, $f(1) = 1$, $f(2) = 0$, $f(3) = 1$, $f(4) = 1$, $f(5) = 0$, $f(6) = -1$, $f(7) = -2$, and $f(8) = -3$. We see that f agrees with Lemma 2.4.

2.2 Rational Motzkin paths of type (s,t)

A *Motzkin path of length* s is a lattice path from $(0,0)$ to $(s,0)$ which stays weakly above the x-axis and consists of up steps $(1,1)$, down steps $(1,-1)$, and flat steps $(1,0)$. We introduce a path which generalizes the idea of the Motzkin path.

Definition 2.5. Let *free rational Motzkin path of type* (s,t) be a lattice path from $(0,0)$ to (s,t) which consists of up steps $U = (1,1)$, down steps $D = (1,-1)$, and flat steps $F = (1,0)$. A *rational Motzkin path of type* (s,t) is a free rational Motzkin path which stays weakly above the line $y = tx/s$.

Figure 2 shows all rational Motzkin paths of type $(5,-2)$. We note that if $P = P_1P_2\cdots P_{s+1}$ is a rational Motzkin path of type $(s+1,-1)$, then P_{s+1} must be a down step and the subpath $\bar{P} = P_1P_2\cdots P_s$ is a Motzkin path of length s.

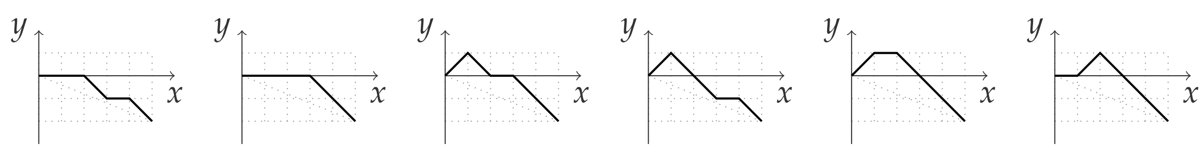

Figure 2: All rational Motzkin paths of type $(5,-2)$

Proposition 2.6. *Let* s *and* d *be relatively prime positive integers. For* $p \geq 2$*, there is a bijection between the set of* $(s, s+d, \ldots, s+pd)$*-core partitions and that of rational Motzkin paths of type* $(s+d, -d)$ *without* UF^iU *steps for* $i = 0, 1, \ldots, p-3$ *if* $p \geq 3$.

Example 2.7. If $\lambda = (9,5,3,2,2,1,1,1,1)$, then λ is a $(5,8,11,14)$-core partition and $\beta(\lambda) = \{17,12,9,7,6,4,3,2,1\}$. Figure 3 shows the corresponding path

$$P = UFUDDDDD$$

of λ, a rational Motzkin path of type $(8,-3)$ without UU steps.

To count the number of rational Motzkin paths of type $(s+d,-d)$, we use the cyclic shifting of paths (see [5, 15]). For a path $P = P_1P_2\cdots P_s$, the cyclic shift $\sigma(P)$ of P is $\sigma(P) = P_2P_3\cdots P_sP_1$. Iteratively, $\sigma^i(P) = P_{i+1}\cdots P_sP_1\cdots P_i$, for $i = 1,\ldots,s-1$ and $\sigma^0(P) = P$.

Lemma 2.8. *For relatively prime positive integers* s *and* d*, let* $P = P_1P_2\cdots P_{s+d}$ *be a free rational Motzkin path of type* $(s+d,-d)$*. Then there exists a unique cyclic shift* $\sigma^j(P)$ *of* P *such that* $\sigma^j(P)$ *is a rational Motzkin path of type* $(s+d,-d)$ *for* $j = 0,1,2,\ldots,s+d-1$.

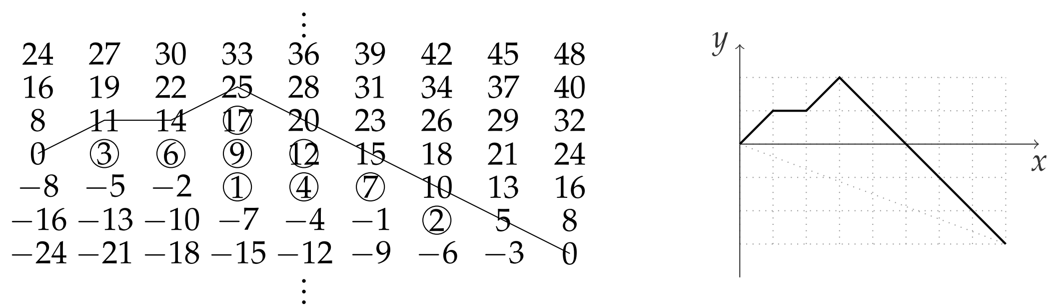

Figure 3: The corresponding rational Motzkin path of the partition $(9, 5, 3, 2, 2, 1, 1, 1, 1)$

Now we can enumerate the rational Motzkin paths of type $(s + d, -d)$.

Proposition 2.9. *Let s and d be relatively prime positive integers. For a given integer $0 \leq k \leq \lfloor s/2 \rfloor$, the number of rational Motzkin paths of type $(s + d, -d)$ having k up steps is*

$$\frac{1}{s+d}\binom{s+d}{k,\, k+d,\, s-2k}.$$

Consequently, the number of rational Motzkin paths of type $(s + d, -d)$ is

$$\frac{1}{s+d}\sum_{k=0}^{\lfloor s/2 \rfloor}\binom{s+d}{k,\, k+d,\, s-2k}.$$

By Propositions 2.6 and 2.9, we give an alternating proof of Theorem 1.3 using path enumeration. Also, we can use the cyclic shifting for rational Motzkin paths of type $(s + d, -d)$ without UF^iU steps for $i = 0, 1, \ldots, p - 3$. For a free rational Motzkin path P of type $(s + d, -d)$, we say that P is *without cyclic UF^iU steps* if there is no UF^iU steps for any cyclic shift of P.

Proposition 2.10. *Let s and d be relatively prime positive integers. For integers $p \geq 3$ and $1 \leq k \leq \lfloor s/2 \rfloor$, the number of rational Motzkin paths of type $(s + d, -d)$ having k up steps and no UF^iU steps for all $i = 0, 1, \ldots, p - 3$ is*

$$\frac{1}{k+d}\sum_{\ell=0}^{r}\binom{k+d}{k-\ell}\binom{k-1}{\ell}\binom{s+d-\ell(p-2)-1}{2k+d-1},$$

where $r = \min(k - 1, \lfloor (s - 2k)/(p - 2) \rfloor)$.

Proof. Let $\mathcal{M}_D(s + d, -d; k, p)$ be the set of free rational Motzkin paths of type $(s + d, -d)$ consisting of k U's, $k + d$ D's, and $s - 2k$ F's which starts with a down step and has no cyclic UF^iU steps for all $i = 0, 1, \dots, p-3$. From Lemma 2.8, there are $k+d$ cyclic shifts of a rational Motzkin path, which starts with a down step so that the number of rational Motzkin paths of type $(s+d, -d)$ with k up steps and without UF^iU steps for all $i = 0, 1, \dots, p-3$ is

$$\frac{1}{k+d} \left| \mathcal{M}_D(s+d, -d; k, p) \right|.$$

For a path $P \in \mathcal{M}_D(s+d, -d; k, p)$, let $\tilde{P}$ denote the subpath obtained from P by deleting all flat steps. Then, $\tilde{P} = Q_1 Q_2 \cdots Q_{2k+d}$ is a path consisting of k U's and $k+d$ D's which starts with a down step. Now, we partition $\mathcal{M}_D(s+d, -d; k, p)$ into k sets according to the number of UU steps of $\tilde{P}$. For $0 \le \ell \le k-1$, let $\mathcal{M}_D^{\ell}(s+d, -d; k, p)$ be the set of $P \in \mathcal{M}_D(s+d, -d; k, p)$ for which $\tilde{P}$ has ℓ UU steps so that

$$\left| \mathcal{M}_D(s+d, -d; k, p) \right| = \sum_{\ell=0}^{k-1} \left| \mathcal{M}_D^{\ell}(s+d, -d; k, p) \right|.$$

Hence, it is enough to show that

$$\left| \mathcal{M}_D^{\ell}(s+d, -d; k, p) \right| = \binom{k+d}{k-\ell} \binom{k-1}{\ell} \binom{s+d-\ell(p-2)-1}{2k+d-1}.$$

We note that if a path P belongs to $\mathcal{M}_D^{\ell}(s+d, -d; k, p)$, then $\tilde{P} = Q_1 Q_2 \cdots Q_{2k+d}$ is a path of the form

$$D^{a_1} U^{b_1} D^{a_2} U^{b_2} \cdots D^{a_{k-\ell}} U^{b_{k-\ell}} D^{a_{k-\ell+1}},$$

where a_i and b_i are integers satisfying $a_i, b_i \ge 1$ for $i = 1, 2, \dots, k-\ell$, $a_{k-\ell+1} \ge 0$,

$$a_1 + a_2 + \cdots + a_{k-\ell+1} = k+d \qquad \text{and} \qquad b_1 + b_2 + \cdots + b_{k-\ell} = k.$$

Since P can be written as

$$Q_1 F^{c_1} Q_2 F^{c_2} \cdots Q_{2k+d} F^{c_{2k+d}},$$

where c_i's are nonnegative integers satisfying $c_1 + c_2 + \cdots + c_{2k+d} = s - 2k$ and $c_i \ge p-2$ if $Q_i = Q_{i+1} = U$, one can see that $\left| \mathcal{M}_D^{\ell}(s+d, -d; k, p) \right|$ is equal to the number of solution tuples $((a_i), (b_i), (c_i))$. It is easy to see that the number of solutions (a_i) and (b_i) are $\binom{k+d}{k-\ell}$ and $\binom{k-1}{k-\ell-1} = \binom{k-1}{\ell}$, respectively. If (a_i) and (b_i) are given, then they determine ℓ indices i such that $c_i \ge p-2$. Hence, the number of solutions (c_i) is equal to the number of nonnegative integer solutions to $y_1 + y_2 + \cdots + y_{2k+d} = s - 2k - \ell(p-2)$, that is $\binom{s+d-\ell(p-2)-1}{2k+d-1}$, for ℓ satisfying that $s+d-\ell(p-2)-1 \ge 2k+d-1$. This completes the proof. □

Remark 2.11. The number of rational Motzkin paths of type $(s+d, -d)$ without up step and UF^iU steps for all $i = 0, 1, \dots, p-3$ is equal to the number of rational Motzkin paths of type $(s+d, -d)$ without up step, that is $\binom{s+d}{d}/(s+d)$ by Proposition 2.9. It follows from Propositions 2.6 and 2.10 that we have proven Theorem 1.5.

3 The $(s, s+1, \ldots, s+p)$-core partitions revisited

From Theorem 1.5, we obtain a closed formula for the number of $(s, s+1, \ldots, s+p)$-core partitions.

Corollary 3.1. *For positive integers s and $p \geq 2$, the number of $(s, s+1, \ldots, s+p)$-core partitions is equal to the number of Motzkin paths of length s without UF^iU steps for $i = 0, 1, \ldots, p-3$ if $p \geq 3$, that is*

$$1 + \sum_{k=1}^{\lfloor s/2 \rfloor} \sum_{\ell=0}^{r} N(k, \ell+1) \binom{s - \ell(p-2)}{2k},$$

where $N(k, \ell+1) = \frac{1}{k}\binom{k}{\ell+1}\binom{k}{\ell} = \frac{1}{k+1}\binom{k+1}{\ell+1}\binom{k-1}{\ell}$ is the Narayana number which counts the number of Dyck paths of order k having $\ell+1$ peaks and $r = \min(k-1, \lfloor (s-2k)/(p-2) \rfloor)$.

From Theorem 1.2 and Corollary 3.1, we see that the (s, p)-generalized Dyck paths and the Motzkin paths of length s without UF^iU steps for $i = 0, 1, \ldots, p-3$ if $p \geq 3$ are equinumerous. We now provide a bijection between sets of these paths.

3.1 A bijection between generalized Dyck paths and restricted Motzkin paths

For a given $p \geq 2$, let P be a Motzkin path of length s without UF^iU steps for $i = 0, 1, \ldots, p-3$ if $p \geq 3$. Then each U step of P is followed by either F^jD for some $j \geq 0$ or F^kU for some $k \geq p-2$. Hence, we can decompose P into the following $p+1$ units:

$$\begin{aligned}
&\bar{U}_p := UF^{p-2} \\
&\bar{D}_p := D \qquad \text{(which is not following } UF^i \text{ for all } i = 0, 1, \ldots, p-3) \\
&\bar{F}_1 := F \qquad \text{(which is not following } UF^i \text{ for all } i = 0, 1, \ldots, p-3) \\
&\bar{F}_i := UF^{i-2}D \quad \text{for } i = 2, 3, \ldots, p-1\,.
\end{aligned}$$

We now construct a simple bijection ϕ between (s, p)-generalized Dyck paths and Motzkin paths of length s without UF^iU steps for $i = 0, 1, \ldots, p-3$ if $p \geq 3$, for fixed $p \geq 2$ as follows.

For a given Motzkin path P of length s without UF^iU steps for $i = 0, 1, \ldots, p-3$ if $p \geq 3$, we define $\phi(P)$ to be the path obtained from P by replacing each unit $\bar{A}$ with A for $A \in \{U_p, D_p, F_i \,|\, i = 1, 2, \ldots, p-1\}$, where $U_p = (0, p)$, $D_p = (p, 0)$, $F_i = (i, i)$ for $i = 1, 2, \ldots, p-1$.

We note that if P is decomposed into k $\bar{U}_p$'s, k $\bar{D}_p$'s, and c_i $\bar{F}_i$'s for $i = 1, 2, \ldots, p-1$, then $\phi(P)$ is a path from $(0, 0)$ to (s, s) since

$$k(p-1) + k + \sum_{i=1}^{p-1} ic_i = s = kp + \sum_{i=1}^{p-1} ic_i\,.$$

Moreover, P never goes below the x-axis if and only if $\phi(P)$ never goes below the line $y = x$. Hence, $\phi(P)$ is an (s, p)-generalized Dyck path, and therefore ϕ is a bijection between (s, p)-generalized Dyck paths and Motzkin paths of length s without UF^iU steps for $i = 0, 1, \dots, p-3$ if $p \geq 3$.

Example 3.2. Let $p = 4$ and $P = UFFUFFFUDDUFDUFFDD$ so that P is a Motzkin path of length 18 without UU and UFU steps. Hence, P can be written as

$$P = \bar{U}_4\bar{U}_4\bar{F}_1\bar{F}_2\bar{D}_4\bar{F}_3\bar{U}_4\bar{D}_4\bar{D}_4,$$

and therefore $Q = \phi(P) = U_4U_4F_1F_2D_4F_3U_4D_4D_4$ which is an $(18, 4)$-generalized Dyck path. See Figure 4.

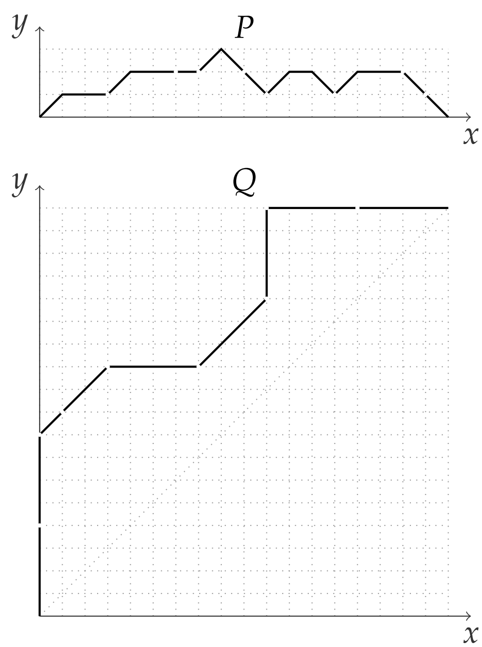

Figure 4: A Motzkin path and the corresponding generalized Dyck path

3.2 The $(s, s+1, \dots, s+p)$-core partitions with k corners

For a partition λ, the number of distinct parts in λ is equal to the number of corners in the Young diagram of λ. Many researchers were interested in corners of a partition, and Huang-Wang [11] found formulae for the number of some simultaneous core partitions with specified number of corners.

Theorem 3.3 ([11], Theorems 3.1 and 3.8)**.** *For positive integers s and k, the number of $(s, s+1)$-core partitions with k corners is the Narayana number $N(s, k+1) = \frac{1}{s}\binom{s}{k+1}\binom{s}{k}$, and* *the number of $(s, s+1, s+2)$-core partitions with k corners is $\binom{s}{2k}C_k$, where C_k is the kth Catalan number.*

Huang-Wang also suggested an open problem for enumerating $(s, s+1, \ldots, s+p)$-cores with k corners, and we give an answer to this problem.

Theorem 3.4. *For positive integers s, $p \geq 2$, and $1 \leq k \leq \lfloor s/2 \rfloor$, the number of $(s, s+1, \ldots, s+p)$-core partitions with k corners is*

$$\sum_{\ell=0}^{r} N(k, \ell+1)\binom{s-\ell(p-2)}{2k},$$

where $r = \min(k-1, \lfloor (s-2k)/(p-2) \rfloor)$.

3.3 Self-conjugate $(s, s+1, \ldots, s+p)$-core partitions

A partition whose conjugate is equal to itself is called *self-conjugate*. From now on, we focus on self-conjugate partitions. Ford–Mai–Sze [10] found the number of self-conjugate (s, t)-core partitions.

Theorem 3.5 ([10], Theorem 1)**.** *For relatively prime integers s and t, the number of self-conjugate (s, t)-core partitions is*

$$\binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor}.$$

In particular, the number of self-conjugate $(s, s+1)$-core partitions is equal to the number of symmetric Dyck paths of order s, that is $\binom{s}{\lfloor s/2 \rfloor}$.

Motivated by Theorem 3.5, in a previous work [7], the authors showed that the number of self-conjugate $(s, s+1, s+2)$-core partitions is equal to the number of symmetric Motzkin paths of length s, and then gave a conjecture for the number of self-conjugate $(s, s+1, \ldots, s+p)$-cores. Recently, this was proved by Yan-Yu-Zhou.

Theorem 3.6 ([21], Theorems 2.14, 2.19, and 2.22)**.** *For positive integers s and p, the number of self-conjugate $(s, s+1, \ldots, s+p)$-core partitions is equal to the number of symmetric (s, p)-generalized Dyck paths.*

Now, we give a closed formula for the number of self-conjugate $(s, s+1, \ldots, s+p)$-core partitions. Here, we give a useful lemma from the OEIS.

Lemma 3.7 ([18], Sequence A088855)**.** *For nonnegative integers k and ℓ such that $\ell < k$, the number of symmetric Dyck paths of order k having ℓ UU steps is*

$$\binom{\lfloor \frac{k-1}{2} \rfloor}{\lfloor \frac{\ell}{2} \rfloor}\binom{\lfloor \frac{k}{2} \rfloor}{\lfloor \frac{\ell+1}{2} \rfloor}.$$

Theorem 3.8. *For positive integers $s, p \geq 2$, the number of self-conjugate $(s, s+1, \ldots, s+p)$-core partitions is*

$$1 + \sum_{k=1}^{\lfloor s/2 \rfloor} \sum_{\ell=0}^{r} \binom{\lfloor \frac{k-1}{2} \rfloor}{\lfloor \frac{\ell}{2} \rfloor} \binom{\lfloor \frac{k}{2} \rfloor}{\lfloor \frac{\ell+1}{2} \rfloor} \binom{\lfloor \frac{s-\ell(p-2)}{2} \rfloor}{k},$$

where $r = \min(k-1, \lfloor (s-2k)/(p-2) \rfloor)$.

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