Heights of spin characters in characteristic 2

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Abstract

Based on our earlier description of the distribution into 2-blocks of the spin characters of the covering groups of symmetric groups we compute the heights of such characters in the blocks containing them. We also give a complete set of labels for the spin characters of minimal height in a 2-block. Another related topic treated here is the determination of the minimal power of 2 dividing a spin character degree and the explicit description of the labels of spin characters with this minimal power of 2 in their degree. Also, an upper bound for the heights of spin characters in 2-blocks is derived, and the labels of spin characters attaining this bound are described.

As an application of our results we show that the 2-blocks of the covering groups of symmetric groups provide further evidence for some important representation theoretical conjectures.

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1 Introduction and preliminaries

It was proved by Schur [11] in 1911 that the finite symmetric groups S_n have covering groups $\widetilde{S}_n$ of order $2|S_n| = 2 \cdot n!$. This means that there is a non-split exact sequence

$$1 \to \langle z \rangle \to \widetilde{S}_n \xrightarrow{\pi} S_n \to 1$$

where $\langle z \rangle$ is a central subgroup of order 2 in $\widetilde{S}_n$.
Those irreducible characters of $\widetilde{S}_n$, which have $\langle z \rangle$ in their kernel, will be referred to as *ordinary characters*. The other irreducible characters of $\widetilde{S}_n$ are referred to as *spin characters*.
It is well-known that the ordinary characters of S_n are labelled canonically by the partitions $\lambda = (\ell_1, \ell_2, \ldots, \ell_m)$ of n; thus $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_m > 0$, $\ell_1 + \cdots + \ell_m = n$. The *length* $\ell(\lambda)$ of λ is defined as m. The set of partitions of n is denoted $\mathcal{P}(n)$ and for $\lambda \in \mathcal{P}(n)$, $[\lambda]$ denotes the corresponding ordinary character of $\widetilde{S}_n$ (resp. of S_n). We also write $\lambda \vdash n$ instead of $\lambda \in \mathcal{P}(n)$. For $\lambda \in \mathcal{P}(n)$ $\quad H_\lambda$ denotes the product of all the hook lengths of λ and then the *hook formula* for the degree of $[\lambda]$ is

$$[\lambda](1) = n!/H_\lambda$$

(see [5], 2.3.21).
Let p be a prime number. The distribution of the ordinary characters into p-blocks is described by a theorem, which is still called the Nakayama Conjecture (see [5], 6.1.21). If $\lambda \in \mathcal{P}(n)$, let $\lambda_{(p)}$ denote its p-core, obtained from λ by removing successively all p-hooks from λ ([5], 2.7.16). Then for $\lambda, \mu \in \mathcal{P}(n)$, $[\lambda]$ and $[\mu]$ are in the same p-block B of S_n if and only if $\lambda_{(p)} = \mu_{(p)}$. In this situation $|\lambda| - |\lambda_{(p)}|$ is a multiple of p, say $|\lambda| - |\lambda_{(p)}| = pw$. The integer w is an invariant of the block B, called the *weight* $w(B)$ of B.
Let sgn $= [1^n]$ denote the sign character of S_n and $\widetilde{S}_n$. An irreducible character χ of $\widetilde{S}_n$ is called *self-associate* if $\chi \cdot \mathrm{sgn} = \chi$. Otherwise χ is called *non self-associate* and χ and $\chi' = \chi \cdot \mathrm{sgn}$ are called a pair of associate characters.
The associate classes of spin characters of $\widetilde{S}_n$ are labelled canonically by the partitions of n into distinct parts, $\lambda = (\ell_1, \ell_2, \ldots, \ell_m)$, $\ell_1 > \ell_2 > \cdots \ell_m > 0$, $\ell_1 + \ell_2 + \cdots + \ell_m = n$. We let $\mathcal{D}(n)$ denote the set of such partitions and divide $\mathcal{D}(n)$ into two subsets as follows:

$$\mathcal{D}^+(n) \quad = \quad \{\lambda = (\ell_1, \ldots, \ell_m) \in \mathcal{D}(n) \mid n - m \ \text{ even}\}$$

$$\mathcal{D}^-(n) \quad = \quad \{\lambda = (\ell_1, \ldots, \ell_m) \in \mathcal{D}(n) \mid n - m \quad \text{odd}\} \, .$$

To each $\lambda \in \mathcal{D}^+(n)$ corresponds a self-associate spin character $\langle\lambda\rangle$ and to each $\lambda \in \mathcal{D}^-(n)$ corresponds a pair $\langle\lambda\rangle$ and $\langle\lambda\rangle'$ of associate spin characters. For $\lambda \in \mathcal{D}(n)$ we let $\overline{H}_\lambda$ denote the product of the bar lengths of λ, (see [7], [3]). Then the *bar formula* for the degree of $\langle\lambda\rangle$ is

$$\langle\lambda\rangle(1) = 2^{\left[\frac{n-\ell(\lambda)}{2}\right]} n!/\overline{H}_\lambda$$

(see [3], Theorem (10.7)).

For odd primes p a result analogous to the Nakayama Conjecture holds for spin characters ([4], [2]); instead of removing p-hooks you have to remove p-bars. In this case a p-block cannot contain ordinary and spin characters at the same time. The weight of a block of spin characters is defined analogously to the weight of blocks of ordinary characters.

In this paper we consider the case $p = 2$, where the characters of a 2-block B of S_n may be considered as the ordinary characters in a unique 2-block $\widetilde{B}$ of $\widetilde{S}_n$. Then $\widetilde{B}$ also contains some spin characters. The distribution of spin characters into 2-blocks was described in [1] (see Theorem 1.1 for the exact statement). The weight $w(\widetilde{B})$ of a 2-block of $\widetilde{S}_n$ is defined as $w(B)$, where B is the 2-block of S_n contained in $\widetilde{B}$. We consider the following questions:

(I) What is the minimal power of 2 dividing the degree of spin characters of $\widetilde{S}_n$ and what are the labels of these characters ?

(II) What are the possible heights of the spin characters in a given 2-block $\widetilde{B}$ of $\widetilde{S}_n$ and what are the labels of characters of minimal height ?

To these questions we remark the following:

The first part of (I) was answered by Wagner [12] and in [8] the power of 2 dividing a spin character degree was computed. In Section 2 we give a complete answer to question (I) including an essentially different proof of Wagner's result: The minimal power of 2 in a spin character degree is 2^t where $t = \left[\frac{n-s(n)}{2}\right]$. Here $s(n)$ is the number of summands in the 2-adic decomposition of n. The number of spin characters of $\widetilde{S}_n$ with a minimal 2-power in their degree depend in a complicated way on n. Thus there is no result analogous to Macdonald's beautiful result for the ordinary characters (Theorem 2.1 below).

The second author has proved that the number of characters of a given height in a p-block of S_n depends only on the weight of the block for any prime p. A similar statement can be made for p-blocks of spin characters of $\widetilde{S}_n$ for odd primes p. Here there is a modification in that also the sign of the p-bar-core of the block plays a rôle, i.e. whether the p-bar-core is in $\mathcal{D}^+$ or in $\mathcal{D}^-$. But in any of the cases the heights of characters lie between 0 and $(w - \sum_i a_i)/(p-1)$ where $w = \sum_i a_i p^i$ is the p-adic decomposition of w. The maximal height occurs in some but not all blocks for $p = 2$. For instance a 2-block of S_n of weight 8 does not contain a character of height 7. In Section 3 we prove that also the number of spin characters of a given height in a 2-block $\widetilde{B}$ of $\widetilde{S}_n$ depends only on the weight $w = w(\widetilde{B})$ of $\widetilde{B}$. Moreover the heights of spin characters in a 2-block of weight w range between

$$\left[\frac{2w - s(w)}{2}\right] \quad \text{and} \quad \left[\frac{3w - 2\,s(w)}{2}\right] .$$

It is also possible to describe explicitly the labels of spin characters of the minimal possible height. Such characters always exist whereas there may be no spin characters of the maximal possible height.
In the final section we check some conjectures concerning block invariants for the 2-blocks of $\widetilde{S}_n$. A recent conjecture of Robinson [10] is verified for all p-blocks of $\widetilde{S}_n$.

We give a brief survey of the results in characteristic 2 which are needed here. It follows from the Nakayama Conjecture that the 2-blocks of S_n (and thus of $\widetilde{S}_n$) are labelled canonically by the 2-cores of integers t satisfying $t \equiv n \pmod{n}$. It is easy to see that the only 2-cores at all are the partitions κ_k, $k \geq 0$, where $\kappa_k = (k, k-1, \ldots, 1)$ is a "triangular" partition of $k(k+1)/2$. The main result of [1] is as follows. For a partition $\lambda = (\ell_1, \ldots, \ell_m) \in \mathcal{D}(n)$ we set

$$\text{dbl}\,(\lambda) = \left(\left[\frac{\ell_1+1}{2}\right], \left[\frac{\ell_1}{2}\right], \left[\frac{\ell_2+1}{2}\right], \left[\frac{\ell_2}{2}\right], \ldots, \left[\frac{\ell_m+1}{2}\right], \left[\frac{\ell_m}{2}\right]\right),$$

the doubling of λ. Then

Theorem 1.1 *([1]) Let $\lambda \in \mathcal{D}(n)$. Then $\langle\lambda\rangle$ and $[dbl(\lambda)]$ belong to the same 2-block of $\widetilde{S}_n$.*

For the study of the powers of 2 dividing spin character degrees a theory of $\bar{4}$-cores and $\bar{4}$-quotients for partitions $\lambda \in \mathcal{D}(n)$ plays a rôle, which is similar to that of p-cores and p-quotients in the study of ordinary character degrees.
Given $\lambda \in \mathcal{D}(n)$ it is possible to define its $\bar{4}$-core $\lambda_{(\bar{4})}$ (which is a partition of the form $(4t+1, 4t-3, \ldots, 5, 1)$ or $(4t+3, 4t-1, \ldots, 7, 3)$) and its $\bar{4}$-quotient $\lambda^{(\bar{4})}$ (see [1]). The $\bar{4}$-quotient is a partition and the following relation is satisfied:

$$|\lambda| = |\lambda_{(\bar{4})}| + 2|\lambda^{(\bar{4})}|$$

Moreover, $\mathrm{dbl}(\lambda_{(\bar{4})}) = (\mathrm{dbl}(\lambda))_{(2)}$, which implies that $|\lambda^{(\bar{4})}|$ is the weight of the 2-block containing $\langle\lambda\rangle$. From $\lambda_{(\bar{4})}$ and $\lambda^{(\bar{4})}$ one may easily recover the partition λ. Suppose $\rho = \lambda^{(\bar{4})}$ is the $\bar{4}$-quotient of λ, say $\rho = (i^{2m_i+\varepsilon_i})$ (written exponentially) with $\varepsilon_i \in \{0,1\}$. Then we set $\rho_o = (i^{m_i})$ and $\rho_e = (i^{\varepsilon_i})$.
Let λ_o (resp. λ_e) denote the partition consisting of all odd (resp. even) parts of λ. Then $\lambda_e = 2\rho_e$. There is a combinatorial process associating to each partition α with odd distinct parts a new partition $\mu(\alpha)$; this is described in [1, §3], [8, §4] and [9, §7]. With this notation, $\rho_o = \mu(\lambda_o)$. There is for instance an explicit formula (Theorem 3.1 below) for the height of $\langle\lambda\rangle$ in the 2-block containing it based on the partitions ρ_o and ρ_e, involving the 2-powers in the character degrees $[\rho_o](1)$ and $\langle\rho_e\rangle(1)$.

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2 On the 2-part of spin character degrees

First we fix some further notation.
Let $n \in \mathbb{N}$, then

$$\begin{aligned} \nu_2(n) &= \text{2-adic valuation of } n \\ \mathcal{O}(n) &= \{\lambda = (\ell_1, \ldots, \ell_m) \vdash n \mid \ell_i \text{ odd for } i = 1, \ldots, l\} \\ \mathcal{M}_0(n) &= \{\rho \vdash n \mid \nu_2([\rho](1)) = 0\} \\ m_0(n) &= |\mathcal{M}_0(n)| \end{aligned}$$

For a partition α of n, we set

$$\begin{array}{rcl} s(\alpha) & = & s(|\alpha|), \\ \alpha_{(2)} & = & \text{2-core of } \alpha, \end{array}$$

and we write $\alpha \in \mathcal{M}_0$ as an abbreviation of $\alpha \in \mathcal{M}_0(|\alpha|)$.

For a partition $\lambda = (\ell_1, \ldots, \ell_m) \in \mathcal{D}(n)$, we set

$$\begin{array}{rcl} \lambda^{(\bar{4})} & = & \bar{4}\text{-quotient of } \lambda \text{ (see [1, § 3])}, \\ \alpha(\lambda) & = & \left[\dfrac{n-m}{2}\right]. \end{array}$$

For later use we recall

Theorem 2.1 *(Macdonald [6])*
If $n = \sum_{i=1}^{s} 2^{k_i}$, $k_1 > k_2 > \ldots > k_s$, *then* $m_0(n) = 2^{k_1 + \ldots + k_s}$.

In fact, the set $\mathcal{M}_0(n)$ can be described explicitly using the 2-core tower (see [9, §6]). Using the notation of the Theorem, a partition α belongs to $\mathcal{M}_0(n)$ if and only if there is exactly one 2-core (1) in the k_i-th layer of the 2-core tower of α, for $i = 1, \ldots, s$, and all other 2-cores in the 2-core tower of α are $\emptyset$.

It is the aim of this section to give a description of the set of partitions labelling spin characters of minimal 2-part in their degree.

First we consider only partitions into distinct odd parts.

Proposition 2.2 *Let* $n > 1$, $s = s(n)$ *and let* $\lambda \in \mathcal{D}(n) \cap \mathcal{O}(n)$. *Let* $\kappa = dbl(\lambda)_{(2)}$, *so* $|\kappa| = \frac{k(k+1)}{2}$ *for some* $k \in \mathbb{N}_0$.
Then we have

$$\nu_2(\langle\lambda\rangle(1)) \geq \begin{cases} \frac{n}{2} & \text{if } k = 0 \\ \frac{n-1}{2} & \text{if } k = 1 \text{ or } 2 \\ \frac{n+2}{2} & \text{if } k = 3 \\ \frac{n+7}{2} & \text{if } k = 5 \\ \left[\frac{n+(|\kappa|+3)/2}{2}\right] & \text{otherwise} \end{cases}$$

Proof. Let $\rho = \lambda^{(\bar{4})}$, $r = |\rho|$, so $n = |\kappa| + 4r$. By [9, 7.12] or [8, 4.8] we have

$$\nu_2(\langle\lambda\rangle(1)) = n - s - r - d_2(\rho)$$

where

$$d_2(\rho) = \nu_2(H_\rho).$$

Using

$$s \leq s(\kappa) + s(r),$$

we obtain

$$r + d_2(\rho) \leq r + \nu_2(r!) = 2r - s(r) = \frac{n - |\kappa|}{2} - s(r) \leq \frac{n - |\kappa|}{2} - (s - s(\kappa))$$

Hence

$$\nu_2(\langle\lambda\rangle(1)) \geq \frac{n + |\kappa| - 2s(\kappa)}{2}$$

It is easy to check that for $k \in \{0, 1, 2, 3, 5\}$ the stated expressions follow. For $k = 4$ or $k > 5$ one has $s(\kappa) < \frac{|\kappa|}{4}$, and thus

$$\nu_2(\langle\lambda\rangle(1)) > \frac{n + |\kappa|/2}{2}$$

Since

$$\left\lceil \frac{n + |\kappa|/2}{2} \right\rceil = \left[\frac{n + (|\kappa| + 3)/2}{2} \right],$$

the assertion follows.

Corollary 2.3 *Let $n \in \mathbb{N}$, $n > 1$, $s = s(n)$ and $\lambda \in \mathcal{D}(n) \cap \mathcal{O}(n)$. Then*

$$\nu_2(\langle\lambda\rangle(1)) \geq \left[\frac{n - s}{2} \right] + 1\,.$$

Definition 2.4 *For $n \in \mathbb{N}$, set*

$$\bar{\mathcal{M}}_1(n) = \{\lambda \in \mathcal{D}(n) \mid \nu_2(\langle\lambda\rangle(1)) = \left[\frac{n - s(n)}{2} \right] + 1\}$$

Proposition 2.5 *For $n \in \mathbb{N}$ we have*

$$\bar{\mathcal{M}}_1(n) \cap \mathcal{O}(n) = \begin{cases} \{(3)\} & \text{for } n = 3 \\ \{\lambda \in \mathcal{D}(n) \cap \mathcal{O}(n) \mid |\lambda| - 4|\lambda^{(\bar{4})}| \leq 3,\ s(\lambda^{(\bar{4})}) = 1, \lambda^{(\bar{4})} \in \mathcal{M}_0\} \\ & \text{for } n > 3 \end{cases}$$

Proof. We use the notation of Proposition 2.2 and its proof. Assuming that $\nu_2(\langle\lambda\rangle(1)) = \left[\frac{n-s}{2}\right] + 1$ holds, one immediately obtains $k \leq 2$ by Proposition 2.2. More precisely, for $k = 0$ $s = 1$, for $k = 1$ one has $s = 2$, and for $k = 2$ $s = 2$ or 3. Checking the inequalities of the proof of Proposition 2.2, one finds that $\rho \in \mathcal{M}_0$

has to be satisfied. Moreover, $s = 2$ and $k = 2$ only occurs for $n = 3$, $\lambda = (3)$, and the other cases are equivalent to r being a 2-power and $k \leq 2$. This gives the sets on the right hand side above.

Conversely, in all these cases the required equality holds.

Theorem 2.6 *Let $n \in \mathbb{N}$, $s = s(n)$.*

(a) If $\lambda \in \mathcal{D}^+(n)$, then

$$\nu_2(\langle\lambda\rangle(1)) \geq \left[\frac{n-s+1}{2}\right].$$

(b) If $\lambda \in \mathcal{D}^-(n)$, then

$$\nu_2(\langle\lambda\rangle)(1)) \geq \left[\frac{n-s}{2}\right].$$

Proof. Let

$$\lambda = \sum_{i\geq 0} 2^i \lambda_i \text{ with } \lambda_i \in (\mathcal{D} \cap \mathcal{O})(|\lambda_i|) \text{ or } \lambda_i = \emptyset.$$

Then

$$n = \sum_{i \geq 0} 2^i |\lambda_i|$$

and

$$\bar{d}_2(\lambda) = \sum_i \bar{d}_2(\lambda_i) + (n - \sum_i |\lambda_i|)$$

where $\bar{d}_2(\lambda) = \nu_2(\overline{H}_\lambda)$ [9, 7.7] or [8, 4.3]. Notice also that $\ell(\lambda) = \sum_i \ell(\lambda_i) \equiv \sum_i |\lambda_i|$ (mod 2), since the partitions λ_i have only odd parts.

Furthermore, by [9, 7.8] or [8, 4.4]

$$\alpha(\lambda) = \sum_i \alpha(\lambda_i) + \left[\frac{1}{2}(n - \sum_i |\lambda_i|)\right].$$

Now by the Bar Formula

$$\nu_2(\langle\lambda\rangle(1)) = n - s + \alpha(\lambda) - \bar{d}_2(\lambda).$$

(a) If $\lambda \in \mathcal{D}^+(n)$, then $n \equiv \sum |\lambda_i| \, (mod\, 2)$, and we obtain

$$\nu_2(\langle\lambda\rangle(1)) = n - s + \sum_i (\alpha(\lambda_i) - \bar{d}_2(\lambda_i)) - \frac{1}{2}(n - \sum_i |\lambda_i|).$$

By Corollary 2.3

$$\alpha(\lambda_i) - \bar{d}_2(\lambda_i) \geq \left[\frac{|\lambda_i| - s(\lambda_i)}{2}\right] + 1 - (|\lambda_i| - s(\lambda_i))$$

if $|\lambda_i| > 1$. Hence

$$\nu_2(\langle\lambda\rangle(1)) \geq \frac{n}{2} - s + \frac{1}{2}|\{|\lambda_i| = 1\}| + \sum_{\substack{i \\ |\lambda_i|>1}} \left(\frac{|\lambda_i|}{2} - \left[\frac{|\lambda_i| - s(\lambda_i) + 1}{2}\right] + 1\right)$$

Since $s \leq \sum_i s(\lambda_i)$, we thus obtain

$$\begin{aligned}
\nu_2(\langle\lambda\rangle(1)) \quad &\geq \quad \frac{n-s}{2} + \sum_{\substack{i \\ |\lambda_i|>1}} \left(\frac{|\lambda_i| - s(\lambda_i)}{2} - \left[\frac{|\lambda_i| - s(\lambda_i) + 1}{2}\right] + 1\right) \\
&\geq \quad \frac{n-s}{2} + \frac{1}{2}|\{|\lambda_i| > 1 \mid |\lambda_i| \not\equiv s(\lambda_i)\,(mod\,2)\}| \\
&\qquad + |\{|\lambda_i| > 1 \mid |\lambda_i| \equiv s(\lambda_i)\,(mod\,2)\}|
\end{aligned}$$

Now, if there is no contribution from some $|\lambda_i| > 1$, then λ is the partition corresponding to the 2-adic decomposition of n, and hence $s \equiv n\,(mod\,2)$ as $\lambda \in \mathcal{D}^+(n)$, so $\left[\frac{n-s+1}{2}\right] = \frac{n-s}{2}$.
Thus in any case $\nu_2(\langle\lambda\rangle(1)) \geq \left[\frac{n-s+1}{2}\right]$.

(b) If $\lambda \in \mathcal{D}^-(n)$, then $n \not\equiv \sum_i |\lambda_i|\,(mod\,2)$ and we have

$$\begin{aligned}
\nu_2(\langle\lambda\rangle(1)) \quad &= \quad n - s + \sum_i (\alpha(\lambda_i) - \bar{d}_2(\lambda_i)) - \frac{1}{2}(n - \sum_i |\lambda_i| + 1) \\
&\geq \quad \frac{n-s-1}{2} + \frac{1}{2}|\{|\lambda_i| > 1 \mid |\lambda_i| \not\equiv s(\lambda_i)\,(mod\,2)\}| \\
&\qquad + |\{|\lambda_i| > 1 \mid |\lambda_i| \equiv s(\lambda_i)\,(mod\,2)\}|
\end{aligned}$$

by similar reasoning as in (a).
Again, if there is no contribution from some $|\lambda_i| > 1$, then λ corresponds to the 2-adic decomposition of n, and thus $\frac{n-s-1}{2} = \left[\frac{n-s}{2}\right]$ as $\lambda \in \mathcal{D}^-(n)$.
Hence $\nu_2(\langle\lambda\rangle(1)) \geq \left[\frac{n-s}{2}\right]$ for all $\lambda \in \mathcal{D}^-(n)$.

For the following we have to introduce some further notations. For $n \in \mathbb{N}$ we set

$$\begin{aligned}
\bar{\mathcal{M}}_0(n) &= \{\lambda \in \mathcal{D}(n) \mid \nu_2(\langle\lambda\rangle(1)) = \left\lceil \frac{n-s(n)}{2} \right\rceil\} \\
\bar{\mathcal{M}}_0^+(n) &= \{\lambda \in \mathcal{D}^+(n) \mid \nu_2(\langle\lambda\rangle(1)) = \left\lceil \frac{n-s(n)+1}{2} \right\rceil\} \\
\bar{\mathcal{M}}_0^-(n) &= \{\lambda \in \mathcal{D}^-(n) \mid \nu_2(\langle\lambda\rangle(1)) = \left\lceil \frac{n-s(n)}{2} \right\rceil\} \\
\bar{m}_0(n) &= |\bar{\mathcal{M}}_0(n)| \\
\bar{m}_0^+(n) &= |\bar{\mathcal{M}}_0^+(n)| \\
\bar{m}_0^-(n) &= |\bar{\mathcal{M}}_0^-(n)|
\end{aligned}$$

Attention $\bar{\mathcal{M}}_0^+(n)$ is <u>not</u> the set $\bar{\mathcal{M}}_0(n) \cap \mathcal{D}^+(n)$!

Furthermore, if $n = \sum_{i=1}^{s} 2^{k_i}$, $k_1 > k_2 > \ldots > k_s$, is the 2-adic decomposition of n, we let $\delta_2(n) = (2^{k_1}, \ldots, 2^{k_s}) \in \mathcal{D}(n)$ denote the corresponding partition of n.

Theorem 2.7 *Let $n \in \mathbb{N}$, $s = s(n)$, and let ε be a sign. We set*

$$\begin{aligned}
\mathcal{D}_0^\varepsilon(n) = \{\lambda = \sum_{i\geq 0} 2^i \lambda_i \in \mathcal{D}^\varepsilon(n) \mid \exists! i_0 : |\lambda_{i_0}| > 1; \textit{ and for this } \lambda_{i_0} \textit{ we have:} \\
s(\lambda_{i_0}) \leq 2, \lambda_{i_0} \in \bar{\mathcal{M}}_1,\ s = |\{\lambda_i \neq \emptyset\}| + s(\lambda_{i_0}) - 1\}
\end{aligned}$$

Then we have

$$\begin{aligned}
\bar{\mathcal{M}}_0^+(n) &= \begin{cases} \{\delta_2(n)\} & \textit{if } n \equiv s \,(\textit{mod}\, 2) \\ \mathcal{D}_0^+(n) & \textit{if } n \not\equiv s \,(\textit{mod}\, 2) \end{cases} \\
\bar{\mathcal{M}}_0^-(n) &= \begin{cases} \mathcal{D}_0^-(n) & \textit{if } n \equiv s \,(\textit{mod}\, 2) \\ \{\delta_2(n)\} \cup \mathcal{D}_0^-(n) & \textit{if } n \not\equiv s \,(\textit{mod}\, 2) \end{cases} \\
\bar{\mathcal{M}}_0(n) &= \{\delta_2(n)\} \cup \mathcal{D}_0^-(n)
\end{aligned}$$

Proof. This follows from the proof of the previous Theorem and Proposition 2.5.

Remark 2.8 By different methods, A. Wagner [12] has shown that for a field F of characteristic $\neq 2$, the degree of any projective representation of S_n over F is divisible by $2^{\left[\frac{n-s(n)}{2}\right]}$. He has also noticed that the complex representation labelled by the 2-adic decomposition of n is divisible by exactly this 2-power.

3 Heights of spin characters in 2-blocks

We now want to study the height of irreducible spin characters in their 2-blocks. The relationship between the 2-combinatorics for $\mathrm{dbl}(\lambda)$ and the $\bar{4}$-combinatorics for λ in described in detail in [1]. As in § 1, we denote by $\rho = \lambda^{(\bar{4})}$ the $\bar{4}$-quotient of λ, say $\rho = (i^{2m_i+\varepsilon_i})$ with $\varepsilon_i \in \{0,1\}$, and we set $\rho_o = (i^{m_i})$ and $\rho_e = (i^{\varepsilon_i})$.
Let λ_o resp. λ_e denote the partition consisting of all odd resp. even parts of λ. Then $\lambda_e = 2\rho_e$ and $\rho_o = \mu(\lambda_o)$ in the notation of [9, 7.11]. Furthermore, the spin character $\langle\lambda\rangle$ belongs to a 2-block of weight $w = w(\lambda) = 2|\rho_o| + |\rho_e|$.
Finally, we define

$$\bar{h}(\lambda) = h(\langle\lambda\rangle)$$

to be the height of the spin character $\langle\lambda\rangle$ in its 2-block of $\tilde{S}_n$.

We now have:

Theorem 3.1 *Let $\lambda \in \mathcal{D}(n)$, $w = w(\lambda)$, ρ_o, ρ_e as defined above. Then*

$$\bar{h}(\lambda) = \nu_2([\rho_o](1)) + \nu_2(\langle\rho_e\rangle(1)) + \nu_2\binom{w}{|\rho_e|} + 2|\rho_o| + \left[\frac{|\rho_e|}{2}\right] + \gamma(\rho_e)$$

where

$$\gamma(\rho_e) = \begin{cases} 1 & \text{if } |\rho_e| \text{ odd and } \rho_e \in \mathcal{D}^- \\ 0 & \text{otherwise} \end{cases}$$

Proof. As a 2-block of $\tilde{S}_n$ of weight w is of defect $\nu_2(2\cdot(2w)!)$, we have

$$\begin{aligned} \bar{h}(\lambda) &= \nu_2(\langle\lambda\rangle(1)) - \nu_2(2\cdot n!) + \nu_2(2\cdot(2w)!) \\ &= \nu_2((2w)!) - \bar{d}_2(\lambda) + \alpha(\lambda) \end{aligned}$$

With λ_o defined as above, we have by [9, 7.5 and 7.6] or [8, 4.1 and 4.2]:

$$\bar{d}_2(\lambda) = \bar{d}_2(\lambda_o) + |\rho_e| + \bar{d}_2(\rho_e)$$

Furthermore,

$$\begin{aligned}
\alpha(\lambda) &= \left[\frac{n-l(\lambda)}{2}\right] = \left[\frac{|\lambda_o|+|\lambda_e|-l(\lambda_o)-l(\lambda_e)}{2}\right] \\
&= \alpha(\lambda_o)+\alpha(\rho_o)+\left[\frac{|\rho_e|}{2}\right]+\gamma(\rho_e)
\end{aligned}$$

as is easily checked.
As

$$\nu_2((2w)!) = 2w - s(w) = w + \nu_2\binom{w}{|\rho_e|} + \nu_2((2|\rho_o|)!) + \nu_2(|\rho_e|!)\,,$$

we thus obtain

$$\begin{aligned}
\bar{h}(\lambda) &= \nu_2((2|\rho_o|)!) - \bar{d}_2(\lambda_o) + \alpha(\lambda_o) + \nu_2(|\rho_e|!) - \bar{d}_2(\rho_e) + \alpha(\rho_e)\nu_2\binom{w}{|\rho_e|} + w \\
&\quad -|\rho_e| + \left[\frac{|\rho_e|}{2}\right] + \gamma(\rho_e)
\end{aligned}$$

By [9, 7.12] we know

$$\bar{d}_2(\lambda_o) - \alpha(\lambda_o) = |\rho_o| + d_2(\rho_o),$$

hence

$$\begin{aligned}
\bar{h}(\lambda) &= \nu_2((2|\rho_o|)!) - |\rho_o| - d_2(\rho_o) + \nu_2(\langle\rho_e\rangle(1)) + \nu_2\binom{w}{|\rho_e|} + 2|\rho_o| + \left[\frac{|\rho_e|}{2}\right] + \gamma(\rho_e) \\
&= \nu_2(|\rho_o|!) - d_2(\rho_o) + \nu_2(\langle\rho_e\rangle(1)) + \nu_2\binom{w}{|\rho_e|} + 2|\rho_o| + \left[\frac{|\rho_e|}{2}\right] + \gamma(\rho_e) \\
&= \nu_2([\rho_o]\,(1)) + \nu_2(\langle\rho_e\rangle(1)) + \nu_2\binom{w}{|\rho_e|} + 2|\rho_o| + \left[\frac{|\rho_e|}{2}\right] + \gamma(\rho_e),
\end{aligned}$$

proving the assertion.

The main point of this formula is that it does not depend on the 2-core of the 2-block, but only on the $\bar{4}$-quotient of λ. Thus in conjunction with the corresponding result for 2-blocks of S_n, it implies the following reduction result:

Theorem 3.2 *Let $\widetilde{B}$ be a 2-block of $\widetilde{S}_n$ of weight w, and let $\widetilde{B}_0$ be the principal 2-block of $\widetilde{S}_{2w}$.*
Then

$$k_i(\widetilde{B}) = k_i(\widetilde{B}_0) \ \text{ for all } \ i \in \mathbb{N}_0.$$

Based on the results of the previous section, we now want to investigate the spin characters of minimal height in a 2-block.

Theorem 3.3 *Let $n \in \mathbb{N}$, $\lambda \in \mathcal{D}(n)$, $w = w(\lambda)$, $s = s(w)$.*

(a) If $\lambda \in \mathcal{D}^+(n)$, then

$$\bar{h}(\lambda) \geq \left[\frac{2w - s + 1}{2}\right].$$

(b) If $\lambda \in \mathcal{D}^-(n)$, then

$$\bar{h}(\lambda) \geq \left[\frac{2w - s}{2}\right].$$

(c) If λ has $\bar{4}$-quotient $\rho \leftrightarrow (\rho_o, \rho_e) = (\emptyset, \delta_2(w))$, then $\bar{h}(\lambda) = \left[\frac{2w-s}{2}\right]$. In this case, $\lambda \in \mathcal{D}^{\varepsilon(s)}(n)$, where

$$\varepsilon(s) = \begin{cases} + & \text{if } s \text{ is even} \\ - & \text{if } s \text{ is odd.} \end{cases}$$

Proof. We use the same notation as before, so by Theorem 3.1 we have

$$\begin{aligned} \bar{h}(\lambda) &= \nu_2([\rho_o](1)) + \nu_2(\langle\rho_e\rangle(1)) + \nu_2\binom{w}{|\rho_e|} + 2|\rho_o| + \left[\frac{|\rho_e|}{2}\right] + \gamma(\rho_e) \\ &\geq \nu_2(\langle\rho_e\rangle(1)) + s(\rho_e) + s(\rho_o) - s(w) + 2|\rho_o| + \left[\frac{|\rho_e|}{2}\right] + \gamma(\rho_e) \end{aligned}$$

(a) By [1, 3.3], $\lambda \in \mathcal{D}^+(n)$ if and only if w is even and $\rho_e \in \mathcal{D}^+$, or w is odd and $\rho_e \in \mathcal{D}^-$. Using $w \equiv |\rho_e| \pmod 2$, we obtain in the first case by Theorem 2.6:

$$\bar{h}(\lambda) \geq \left[\frac{|\rho_e| - s(\rho_e) + 1}{2}\right] + \delta + s(\rho_e) + s(\rho_o) - s + 2|\rho_o| + \frac{|\rho_e|}{2}$$

where

$$\delta = \begin{cases} 1 & \text{if } \rho_e \notin \bar{\mathcal{M}}_0^+ \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\bar{h}(\lambda) \geq \left[\frac{2|\rho_e| + s(\rho_e) + 2s(\rho_o) - 2s + 4|\rho_o| + 1}{2}\right]$$

$$\begin{aligned}
&= \left\lceil \frac{2w + s(\rho_e) + 2s(\rho_o) - 2s + 1}{2} \right\rceil \\
&\geq \left\lceil \frac{2w + s(\rho_o) - s + 1}{2} \right\rceil \qquad (\text{ since } s \leq s(\rho_o) + s(\rho_e)) \\
&\geq \left\lceil \frac{2w - s + 1}{2} \right\rceil
\end{aligned}$$

In the second case, $|\rho_e|$ is odd and $\rho_e \in \mathcal{D}^-$, so $\gamma(\rho_e) = 1$ and hence

$$\bar{h}(\lambda) \geq \left\lceil \frac{|\rho_e| - s(\rho_e)}{2} \right\rceil + \delta + s(\rho_e) + s(\rho_o) - s + 2|\rho_o| + \frac{|\rho_e| - 1}{2} + 1$$

with

$$\delta = \begin{cases} 1 & \text{if } \rho_e \notin \bar{\mathcal{M}}_0^- \\ 0 & \text{otherwise} \end{cases}$$

Similarly as above we get this time

$$\bar{h}(\lambda) \geq \left\lceil \frac{2w + s(\rho_o) - s - 1}{2} \right\rceil + 1 \geq \left\lceil \frac{2w - s + 1}{2} \right\rceil.$$

(b) Again we use [1, 3.3], and consider first the case where $w \equiv |\rho_e| \pmod 2$ is even and $\rho_e \in \mathcal{D}^-$. Here, similarly as above,

$$\begin{aligned}
\bar{h}(\lambda) &\geq \left\lceil \frac{|\rho_e| - s(\rho_e)}{2} \right\rceil + \delta + s(\rho_o) + s(\rho_e) - s + 2|\rho_o| + \frac{|\rho_e|}{2} \\
&\geq \left\lceil \frac{2w - s}{2} \right\rceil
\end{aligned}$$

In the second case, $|\rho_e|$ is odd and $\rho_e \in \mathcal{D}^+$, so

$$\begin{aligned}
\bar{h}(\lambda) &\geq \left\lceil \frac{|\rho_e| - s(\rho_e) + 1}{2} \right\rceil + \delta + s(\rho_e) + s(\rho_o) - s + 2|\rho_o| + \frac{|\rho_e| - 1}{2} \\
&\geq \left\lceil \frac{2w - s}{2} \right\rceil
\end{aligned}$$

as before.

(c) The first assertion is easily checked using the formula given in Theorem 3.1. The second one is immediate from the fact that the number of even parts in λ

equals the number of parts of $\rho_e = \delta_2(w)$.

Again, by going through the sequence of inequalities in the proof above, we can describe the set of spin characters of minimal height in a 2-block in detail. First we need some further definitions. Let $\widetilde{B}$ be a 2-block of $\widetilde{S}_n$ of weight w. Then we set

$$\begin{aligned}
\bar{\mathcal{K}}_0(\widetilde{B}) &= \{\lambda \in \mathcal{D}(n) \mid w(\lambda) = w,\, \bar{h}(\lambda) = \left[\frac{2w - s(w)}{2}\right]\} \\
\bar{\mathcal{K}}_0^+(\widetilde{B}) &= \{\lambda \in \mathcal{D}^+(n) \mid w(\lambda) = w,\, \bar{h}(\lambda) = \left[\frac{2w - s(w) + 1}{2}\right]\} \\
\bar{\mathcal{K}}_0^-(\widetilde{B}) &= \bar{\mathcal{K}}(\widetilde{B}) \cap \mathcal{D}^-(n) \\
\bar{k}_0(\widetilde{B}) &= |\bar{\mathcal{K}}_0(\widetilde{B})|\,,\, \bar{k}_0^+(\widetilde{B}) = |\bar{\mathcal{K}}_0^+(\widetilde{B})|\,,\, \bar{k}_0^-(\widetilde{B}) = |\bar{\mathcal{K}}_0^-(\widetilde{B})|
\end{aligned}$$

Furthermore, for a sign ε we let

$$\bar{\mathcal{M}}_0^{\widetilde{\varepsilon}}(n) = \begin{cases} \bar{\mathcal{M}}_0^{\varepsilon}(n) & \text{if } n \text{ is even} \\ \bar{\mathcal{M}}_0^{-\varepsilon}(n) & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 3.4 *Let $\widetilde{B}$ be a 2-block of $\widetilde{S}_n$ of weight $w = \sum_{i=1}^s 2^{w_i}$, $w_1 > w_2 > \ldots > w_s$, and let ε be a sign. Set*

$$\begin{aligned}
\mathcal{D}_0^{\varepsilon}(n,w) &= \{\lambda \in \mathcal{D}^{\varepsilon}(n) \mid w = w(\lambda),\, \lambda_e = 2\rho_e,\, \rho_e \in \bar{\mathcal{M}}_0^{\widetilde{\varepsilon}}(w)\} \\
\mathcal{D}_1^{\varepsilon}(n,w) &= \{\lambda \in \mathcal{D}^{\varepsilon}(n) \mid w = w(\lambda),\, \lambda^{(4)} \leftrightarrow (\rho_o, \rho_e),\, \exists w_i > 0 : \rho_o \in \mathcal{M}_0(2^{w_i - 1})\,, \\
&\qquad \rho_e \in \bar{\mathcal{M}}_0^{\widetilde{\varepsilon}}(w - 2^{w_i})\}
\end{aligned}$$

Then we have:

$$\bar{\mathcal{K}}_0^{\varepsilon}(\widetilde{B}) = \begin{cases} \mathcal{D}_0^{\varepsilon}(n,w) & \textit{if } \varepsilon(s) = \varepsilon \\ \mathcal{D}_0^{\varepsilon}(n,w) \cup \mathcal{D}_1^{\varepsilon}(n,w) & \textit{if } \varepsilon(s) \neq \varepsilon \end{cases}$$

and

$$\bar{\mathcal{K}}_0(\widetilde{B}) = \begin{cases} \mathcal{D}_0^+(n,w) \cup \mathcal{D}_0^-(n,w) \cup \mathcal{D}_1^-(n,w) & \textit{if } s \textit{ is even} \\ \mathcal{D}_0^-(n,w) & \textit{if } s \textit{ is odd} \end{cases}$$

Proof. This follows by a careful analysis of the inequalities in the proof of the preceding Theorem. We omit the details.

Remark 3.5 (i) By definition, $\bar{\mathcal{K}}_0^-(\tilde{B}) \subseteq \bar{\mathcal{K}}_0(\tilde{B})$, but note that $\bar{\mathcal{K}}_0^+(\tilde{B}) \subseteq \bar{\mathcal{K}}_0(\tilde{B})$ if and only if $s(w)$ is even.
(ii) If $\varepsilon(w) = \varepsilon(s) = \varepsilon$, then $\bar{\mathcal{K}}_0^\varepsilon(\tilde{B}) = \{\lambda = \kappa + 2\delta_2(w)\}$, where κ is the $\bar{4}$-core of the spin characters in $\tilde{B}$.
(iii) If w is odd and $s(w)$ even, then note that in the $\mathcal{D}_1^-$ contribution of $\mathcal{K}_0(\tilde{B})$ above, for any $w_i > 0$ the partition $\rho_e = \delta_2(w - 2^{w_i})$ is the only element in $\tilde{\mathcal{M}}_0^-(w - 2^{w_i}) = \mathcal{M}_0^+(w - 2^{w_i})$.

Corollary 3.6 *Let $\tilde{B}$ be a 2-block of $\tilde{S}_n$ of weight $w = \sum_{i=1}^s 2^{w_i}$, $w_1 > w_2 > \ldots > w_s$.*

(i) If w and $s(w) = s$ are both even, then

$$\begin{aligned}
\bar{k}_0^+(\tilde{B}) &= \bar{m}_0^+(w) = 1 \\
\bar{k}_0^-(\tilde{B}) &= \bar{m}_0^-(w) + \sum_{i=1}^s \bar{m}_0^-(w - 2^{w_i})2^{w_i - 1} = \bar{m}_0^-(w) + \sum_{i=1}^s \bar{m}_0(w - 2^{w_i})2^{w_i-1} \\
\bar{k}_0(\tilde{B}) &= \bar{k}_0^+(\tilde{B}) + \bar{k}_0^-(\tilde{B})
\end{aligned}$$

(ii) If w is even, $s(w) = s$ odd, then

$$\begin{aligned}
\bar{k}_0^+(\tilde{B}) &= \bar{m}_0^+(w) + \sum_{i=1}^s \bar{m}_0^+(w - 2^{w_i})2^{w_i-1} = \bar{m}_0^+(w) + \frac{w}{2} \\
\bar{k}_0^-(\tilde{B}) &= \bar{m}_0^-(w) = \bar{m}_0(w) = \bar{k}_0(\tilde{B})
\end{aligned}$$

(iii) If w is odd, $s(w) = s$ even, then

$$\begin{aligned}
\bar{k}_0^+(\tilde{B}) &= \bar{m}_0^-(w) = \bar{m}_0(w) \\
\bar{k}_0^-(\tilde{B}) &= \bar{m}_0^+(w) + \sum_{i=1}^{s-1} 2^{w_i-1} = \bar{m}_0^+(w^-) + \frac{w-1}{2} \\
\bar{k}_0(\tilde{B}) &= \bar{m}_0^+(w) + \bar{m}_0^-(w) + \frac{w-1}{2}
\end{aligned}$$

(iv) If w and $s(w) = s$ are both odd, then

$$\begin{aligned}\bar{k}_0^+(\tilde{B}) &= \bar{m}_0^-(w) + \sum_{i=1}^{s-1} \bar{m}_0^-(w - 1^{w_i})2^{w_i-1} = \bar{m}_0^-(w) + \sum_{i=1}^{s-1} \bar{m}_0(w - 2^{w_i})2^{w_i-1} \\ \bar{k}_0^-(\tilde{B}) &= \bar{m}_0^+(w) = 1 = \bar{k}_0(\tilde{B})\end{aligned}$$

Proof. This follows from the preceding Theorem, Theorem 2.8 and Macdonald's Theorem 2.1.

Before proceeding, let us look at some examples to illustrate the results above.

Examples 3.7 (i) Let $\tilde{B}$ be the 2-block of weight $w = 5$ in $\tilde{S}_{13}$. Then the minimal spin character height is $\left[\frac{2w-s(w)}{2}\right] = 4$.
To compute $\bar{\mathcal{K}}_0(\tilde{B})$ we need $\bar{\mathcal{M}}_0^{\varepsilon}(5)$ and $\mathcal{M}_0(2)$, which are easy to calculate:

$$\begin{aligned}\bar{\mathcal{M}}_0^+(5) &= \{\lambda \in \mathcal{D}^+(5) \mid \nu_2(\langle\lambda\rangle(1)) = 2\} &= \{(5)\} \\ \bar{\mathcal{M}}_0^-(5) &= \{\lambda \in \mathcal{D}^-(5) \mid \nu_2(\langle\lambda\rangle(1)) = 1\} &= \{(4,1)\} \\ \mathcal{M}_0(2) &= \{(2),(1^2)\}\end{aligned}$$

Hence $\bar{\mathcal{K}}_0^+(\tilde{B}) = \{(8,3,2)\}$, $\bar{\mathcal{K}}_0^-(\tilde{B}) = \{(10,3),(11,2),(7,3,2,1)\}$,
here $\bar{\mathcal{K}}_0(\tilde{B}) = \bar{\mathcal{K}}_0^+(\tilde{B}) \cup \bar{\mathcal{K}}_0^-(\tilde{B})$.

(ii) Let $\tilde{B}$ be the 2-block of weight $w = 6$ in $\tilde{S}_{15}$. Here the minimal spin character height is $\left[\frac{2w-s(w)}{2}\right] = 5$.
We first compute:

$$\begin{aligned}\bar{\mathcal{M}}_0^+(6) &= \{(4,2)\}, & \bar{\mathcal{M}}_0^-(6) &= \{(6),(3,2,1)\} \\ \mathcal{M}_0(1) &= \{(1)\}, & \mathcal{M}_0(2) &= \{(2),(1^2)\} \\ \bar{\mathcal{M}}_0^-(2) &= \{(2)\}, & \bar{\mathcal{M}}_0^-(4) &= \{(4)\}\end{aligned}$$

With this we obtain

$$\begin{aligned}\bar{\mathcal{K}}_0^+(\tilde{B}) &= \{(8,4,3)\} \\ \bar{\mathcal{K}}_0^-(\tilde{B}) &= \{(12,3),(6,4,3,2),(8,7),(11,4),(7,4,3,1)\}\end{aligned}$$

Again, $\bar{\mathcal{K}}_0(\tilde{B}) = \bar{\mathcal{K}}_0^+(\tilde{B}) \cup \bar{\mathcal{K}}_0^-(\tilde{B})$.

We want to conclude this section by considering spin characters of maximal height in their 2-block. We have the following upper bound for the height:

Theorem 3.8 *Let $\lambda \in \mathcal{D}(n)$, $w = w(\lambda)$, $s = s(w)$. Then*

$$\bar{h}(\lambda) \leq \left[\frac{3w-2s}{2}\right].$$

Proof. Using the notation introduced before, we have by Theorem 3.1

$$\begin{aligned}\bar{h}(\lambda) &= \nu_2([\rho_o](1)) + \nu_2(\langle\rho_e\rangle(1)) + \gamma(\rho_e) + 2|\rho_o| + \nu_2\binom{w}{|\rho_e|} + \left[\frac{|\rho_e|}{2}\right] \\ &\leq |\rho_o| - s(\rho_o) + |\rho_e| - s(\rho_e) + 2|\rho_o| + s(\rho_o) + s(\rho_e) - s + \left[\frac{|\rho_e|}{2}\right],\end{aligned}$$

since $\gamma(\rho_e) = 0$ if ρ_e is a $\bar{4}$-core. As $w = 2|\rho_o| + |\rho_e|$, this gives

$$\bar{h}(\lambda) \leq w + \left[\frac{w}{2}\right] - s = \left[\frac{3w-2s}{2}\right].$$

Keeping the notation from above, we can describe explicity for which $\lambda \in \mathcal{D}(n)$ the bound above is attained:

Theorem 3.9 *Let $\lambda \in \mathcal{D}(n)$, $w = w(\lambda)$, $s = s(w)$, $\varepsilon = \varepsilon(w)$. Then $\bar{h}(\lambda) = \left[\frac{3w-2s}{2}\right]$ if and only if ρ_o is a 2-core and one of the following holds:*

(i) ρ_e is a $\bar{4}$-core (in this case, $\lambda \in \mathcal{D}^\varepsilon(n)$).

(ii) w is odd and $\rho_e = (4k+1, 4(k-1)+1, \ldots, 5, 2, 1)$ or $\rho_e = (4k+3, 4(k-1)+3, \ldots, 7, 3, 2)$ for some $k \in \mathbb{N}_0$ (in this case, $\lambda \in \mathcal{D}^+(n)$).

Proof. That ρ_o has to be a 2-core is immediate from the inequality in the proof above.
Now $\nu_2(\langle\rho_e\rangle(1)) + \gamma(\rho_e) = |\rho_e| - s(\rho_e)$ if and only if ρ_e is a $\bar{4}$-core or $\nu_2(\langle\rho_e\rangle(1)) = |\rho_e| - s(\rho_e) - 1$ and $\gamma(\rho_e) = 1$. By [8, p. 245] this happens exactly in the cases stated in (ii) above.
The assertions on the parity of λ follow easily from the fact that the number of even parts in λ is the length of ρ_e.
It is clear that the conditions on ρ_o, ρ_e given above lead to a partition λ with $\bar{h}(\lambda) = \left[\frac{3w-2s}{2}\right]$.

Corollary 3.10 *Let $\tilde{B}$ be a 2-block of $\tilde{S}_n$ of weight w, $s = s(w)$. Then $\tilde{B}$ contains a spin character of height $\left[\frac{3w-2s}{2}\right]$ if and only if $w = 2\Delta_0 + \Delta_1$ or w is odd and $w = 2\Delta_0 + \Delta_1 + 2$, where Δ_0, Δ_1 are triangular numbers.*
More precisely, $\tilde{B}$ contains a non-selfassociate spin character of height $\left[\frac{3w-2s}{2}\right]$ if and only if w is odd and of the form $w = 2\Delta_0 + \Delta_1$, where Δ_0, Δ_1 triangular numbers. If this is the case, the number of pairs of non-selfassociate spin characters in $\tilde{B}$ of height $\left[\frac{3w-2s}{2}\right]$ equals the number of decompositions of w as $w = 2\Delta_0 + \Delta_1$ with Δ_0, Δ_1 triangular numbers.

Examples 3.11 (i) $w = 4$ is the smallest weight that can not be written in the form above; in such a block the maximal spin height is 4.

(ii) For $w = 7$ we have $7 = 2\cdot 3 + 1 = 2\cdot 1 + 3 + 2$, leading to $\lambda = (9,3,2,1) \in \mathcal{D}^-(15)$ and $\lambda = (6,5,4) \in \mathcal{D}^+(15)$ as the partitions of maximal spin height $\left[\frac{3w-2s}{2}\right] = 7$ in a 2-block of weight 7 in $\tilde{S}_{15}$.

Remark 3.12 Note that by the bounds obtained in this section, there is only an interval of length $\left[\frac{w}{2}\right] - \left[\frac{s}{2}\right]$ for possible spin character heights in a 2-block of weight w.

4 Applications

Using the results of [1] and the results of the preceding sections we want to prove that the following conjectures hold for the 2-blocks of $\tilde{S}_n$ (see [9]); below, B is always a p-block of the finite group G and $\delta(B)$ is its defect group.

Conjecture 4.1 *(Brauer) $k(B) \leq |\delta(B)|$.*

Conjecture 4.2 *(Brauer's Height 0 Conjecture) $k(B) = k_0(B)$ if and only if $\delta(B)$ is abelian.*

Conjecture 4.3 *(Olsson) $k_0(B) \leq |\delta(B) : \delta(B)'|$*

All these conjectures are known to hold for the p-blocks of $\tilde{S}_n$ if p is an odd prime (see [9]). For dealing with the case $p = 2$, we first recall a result from [1]:

Theorem 4.4 *Let B be a 2-block of S_n of weight w, and let $\tilde{B}$ be the 2-block of $\tilde{S}_n$ containing B. Then*

$$k(\tilde{B}) = k(B) + p(w) + \tilde{p}^-(w) = k(2,w) + p(w) + \tilde{p}^-(w)$$

where $k(2,w)$ is the number of 2-quotients of weight w, i.e. the number of pairs of partitions (λ_0, λ_1) with $|\lambda_0| + |\lambda_1| = w$.

Corollary 4.5 *Let $\tilde{B}$ be a 2-block of $\tilde{S}_n$, then $k(\tilde{B}) \leq |\delta(\tilde{B})|$.*

Proof. Let $B \subseteq \tilde{B}$ be the 2-block of S_n contained in $\tilde{B}$, and let w be its weight. It is known that $k(B) = k(2,w) \leq |\delta(B)|$ (see [9]). Hence by the theorem above we obtain

$$k(\tilde{B}) = k(B) + p(w) + \tilde{p}^-(w) \leq 2k(2,w) \leq 2|\delta(B)| = |\delta(\tilde{B})|\,.$$

With the same notations as above, we know by Theorem 3.3 that the irreducible spin characters in $\tilde{B}$ are of height $\geq \left[\frac{2w-s(w)}{2}\right]$, and that there always is an irreducible spin character of exactly this height in $\tilde{B}$. Hence there are irreducible spin characters of height 0 in $\tilde{B}$ if and only if $\left[\frac{2w-s(w)}{2}\right] = 0$, i.e. exactly if $w \leq 1$. This immediately implies

Corollary 4.6 *Brauer's Height 0 Conjecture holds for all 2-blocks of $\tilde{S}_n$, for all $n \in \mathbb{N}$.*

Corollary 4.7 *Olsson's Conjecture holds for all 2-blocks of $\tilde{S}_n$, for all $n \in \mathbb{N}$.*

Recently, Robinson [10] has put forward a new conjecture on the height of an irreducible character:

Conjecture 4.8 *(Robinson) Let B be a p-block of the finite group G, $D = \delta(B)$ its defect group and χ an irreducible character in B. If D is non-abelian, then*

$$h(\chi) < log_p|D : Z(D)|\,.$$

We provide further evidence for this conjecture by proving

Theorem 4.9 *Robinson's Conjecture holds for all p-blocks of $\tilde{S}_n$, for all primes p and all $n \in \mathbb{N}$.*

Proof. Let $\tilde{B}$ be a p-block of $\tilde{S}_n$ of weight w, and let D be its defect group. If p is an odd prime, then the height of an irreducible character χ in $\tilde{B}$ satisfies

$$h(\chi) \leq \frac{w - \sum a_i}{p-1},$$

where $w = \sum a_i p^i$ is the p-adic decomposition of w [9, 11.9 and 13.8]. The defect group D of $\tilde{B}$ is isomorphic to a Sylow p-subgroup of $\tilde{S}_{pw}$, which is non-abelian if and only if $w \geq 3$. Hence the inequality above holds.
If $p = 2$, then the height of an ordinary irreducible character in $\tilde{B}$ is bounded by $w - s(w)$ [9, 11.9], and the height of an irreducible spin character in $\tilde{B}$ is bounded by $\left[\frac{3w-2s(w)}{2}\right]$ by Theorem 3.8. On the other hand, by [12] we have $|Z(D)| = 2$ if D is non-abelian, so $\log_2|D : Z(D)| = \nu_2((2w)!) = 2w - s(w)$. Hence if D is non-abelian, which is exactly the case if $w \geq 2$, then again the inequality above is satisfied for the irreducible characters in $\tilde{B}$.

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