Séminaire Lotharingien de Combinatoire S36h (24pp.)

A COMBINATORIAL BIJECTION BETWEEN STANDARD YOUNG TABLEAUX AND REDUCED WORDS OF GRASSMANNIAN PERMUTATIONS

RUDOLF WINKEL

Abstract: For every partition λ we construct a very simple combinatorial bijection between the set of standard Young tableaux of shape λ and the set of reduced words for the Grassmannian permutation $\pi(\lambda)$ associated to λ. The basic tools in setting up this bijection are partial orders on the respective sets. These partial orders are interesting in their own right, and we give some first results about them: (1) the poset of standard tableaux for an arbitrary shape D is isomorphic to an order ideal in left weak Bruhat order, (2) for hook shapes the Poincaré polynomial is the q-binomial coefficient, (3) for general Ferrer shapes a recursion formula for the Poincaré polynomials is given, (4) the poset of reduced words for a Grassmannian permutation is anti-isomorphic to the poset of reduced words for its "conjugate" and inverse permutation, (5) for the Grassmannian and dominant permutation associated to a hook shape the respective posets of reduced words are isomorphic.

A *diagram* D is a finite set $\{(i,j) \in \mathbb{Z} \times \mathbb{Z}\}$, where the index i of the rows is increasing from top to bottom and the index j of the columns from left to right. We depict a diagram D as a set of unit squares or boxes in the plane with center points in D and call $N \equiv N(D) := |\{(i,j) \mid i,j \in \mathbb{Z}\}|$ the *weight* of D. Usually we are interested only in the *shape of* D, i.e. the equivalence class of all diagrams congruent under translations in the $\mathbb{Z} \times \mathbb{Z}$ plane; it will cause no confusion to denote the shape of D by D, too, but it may be convenient to characterize the shape of D by a certain representative: a *Ferrer diagram* is a diagram of the form $\{(i,j) \mid 1 \leq j \leq \lambda_i\}$ for some partition $\lambda \equiv \lambda_1 \dots \lambda_s \vdash N$ with $\lambda_1 \geq \dots \geq \lambda_s \geq 1$ and length $l(\lambda) := s$.

A numbering of the boxes of a diagram D with natural numbers is called a *tableau*. And a numbering using the set $\{1, \dots, N\}$, such that the numbers are strictly increasing in rows and columns, is called a *standard tableau*. The sets of all tableaux and standard tableaux of a given shape D are denoted by $T(D)$ and $ST(D)$, respectively, where in case of $D = \lambda$ it is customary to use the notation $SYT(\lambda)$ (Y for 'Young') instead of $ST(\lambda)$. The number $f^\lambda := |SYT(\lambda)|$ can be computed by the famous hook formula ([M, S]).

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For every diagram D the *conjugate diagram* D' is defined as $D' := \{(j,i) \mid (i,j) \in D\}$, and for any tableau $\eta \in T(D)$ the conjugate tableau $\eta' \in T(D')$ is defined in the obvious way. Note that the conjugate of a standard tableau is again standard.

Let D be an arbitrary diagram; then we call

$$D^- := \bigcup_{(i,j)\in D} \{(i',j') \mid i' \le i,\ j' \le j\} \tag{0.1}$$

the *shadow* of D and

$$\partial^- D := \{(i,j) \mid (i-1,j-1) \in D^-\} \setminus D^- \tag{0.2}$$

the *(shadow) boundary* of D. Two boxes in $\mathbb{Z} \times \mathbb{Z}$ are called *adjacent* iff they have a common side. For $k \in \mathbb{N}$ and any $(i,j) \in D$ with $(i,j+1) \in \partial^- D$ we say to *move upwards by k steps from* (i,j) to some box $(i_k,j_k) \in \partial^- D$ iff there is a chain $(i_1,j_1) := (i,j+1),\ldots,(i_k,j_k)$ of k different adjacent boxes in $\partial^- D$, such that $(i_{\nu+1},j_{\nu+1}) = (i_\nu, j_\nu+1)$ or $(i_{\nu+1},j_{\nu+1}) = (i_\nu - 1, j_\nu)$ for $\nu = 1,\ldots,k-1$. Similarly for any $(i,j) \in D$ with $(i+1,j) \in \partial^- D$ and $k \in \mathbb{N}$ we say to *move downwards by k steps from* (i,j) to some box $(i_k,j_k) \in \partial^- D$ iff there is a chain $(i_1,j_1) := (i,j+1),\ldots,(i_k,j_k)$ of k different adjacent boxes in $\partial^- D$, such that $(i_{\nu+1},j_{\nu+1}) = (i_\nu+1, j_\nu)$ or $(i_{\nu+1},j_{\nu+1}) = (i_\nu, j_\nu - 1)$ for $\nu = 1,\ldots,k-1$.

Turning now to the symmetric groups S_n on n 'letters' $1,\ldots,n$ it is well known that every permutation $\pi \equiv \pi(1)\ldots\pi(n) \in S_n$ can be generated as a composition of the *elementary transpositions* $\sigma_i = (i, i{+}1)$ for $i = 1,\ldots,n{-}1$, which obey the relations

$$\text{(i) } \sigma_i^2 = id, \quad \text{(ii) } \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1, \quad \text{(iii) } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\ .$$

If for some $\pi \in S_n$ the number p in $\pi = \sigma_{a_1}\ldots\sigma_{a_p}$ is minimal, then $\sigma_{a_1}\ldots\sigma_{a_p}$ is called a *reduced word* for π, $a = a_1\ldots a_p$ a *reduced sequence* for π, and $l(\pi) := p$ the *length* of π. The set of all reduced sequences of π is denoted by $R(\pi)$ and its cardinality by $r(\pi) := |R(\pi)|$. Moreover $a^* \equiv a_1^*\ldots a_p^* \equiv a_p\ldots a_1$ is the *reversed sequence* of a.

The *left (weak Bruhat) order* on S_n is defined as the transitive closure of the following covering relation:

$$[\ \pi' \text{ covers } \pi\ ] \quad :\Longleftrightarrow \quad \pi' = \sigma_i\pi\ , \quad l(\pi') = l(\pi) + 1\ .$$

The permutation $\pi(\lambda) \in S_n$ $(n := s + \lambda_1)$ associated to a partition λ of length s is defined as

$$\pi(\lambda) := \lambda_s + 1 \quad \lambda_{s-1} + 2 \quad \ldots \quad \lambda_1 + s \quad 1\ 2\ 3\ \ldots \quad , \tag{0.3}$$

where the ellipsis 1 2 3 ... means that all letters $1,\ldots,n$ not occurring on the first s places are appended in increasing order. The $\pi(\lambda)$ defined such are *Grassmannian permutations*, i.e. permutations with a unique descent: $\pi(i) > \pi(i+1)$ for exactly one $i \in \{1,\ldots,n-1\}$. In fact one can obtain all Grassmannian permutations in this way, if one admits $\lambda_s \geq 0$ instead of $\lambda_s \geq 1$.

In [S1] R. P. Stanley has proven that for all permutations π the number of reduced words $r(\pi)$ can be expressed as

$$r(\pi) = \sum_{\lambda \vdash l(\pi)} a_{\pi\lambda} f^{\lambda} \quad , \tag{0.4}$$

where, moreover, by the results of Edelman and Greene ([EG]) the coefficients $a_{\pi\lambda}$ are nonnegative integers; under special circumstances specified in [S1, Cor.4.2] the sum contains exactly one or two summands f^{λ} with coefficients equal to 1. The main goal of the present paper is to give a combinatorial bijection proving the following

Theorem. *Let $\lambda \vdash N$ be a partition and $\pi(\lambda)$ its associated Grassmannian permutation, then*

$$r(\pi(\lambda)) = f^{\lambda} \quad . \tag{0.5}$$

It is possible to derive equality (0.5) purely algebraically from [S1, Cor.4.2] using some results of [W1, W2], and we include such a proof for the sake of completeness, but a combinatorial understanding of the relation between the sets $SYT(\lambda)$ and $R(\pi(\lambda))$ clearly demands a combinatorial construction as provided by our bijection. In fact we prove the stronger result that for every partition λ the partial orders on the sets $SYT(\lambda)$ and $R(\pi(\lambda))$ are isomorphic (cf. Prop.4.1).

Proof. (*algebraic*) Let $L(\pi)$ and $K(\pi)$ denote certain *codes* of $\pi \equiv \pi(\lambda) \in S_n$ (cf. [W1,W2]), and let $\lambda(M)$ be the partition built up from some finite set or sequence of natural numbers M; let $\lambda'(M)$ be the conjugate partition of $\lambda(M)$ and $\omega_n := n \dots 1$ the permutation of maximal length in S_n.

Now [S1, Cor.4.2] asserts in our terminology that

$$\lambda'(L(\pi)) = \lambda(K(\pi)) \Longrightarrow r(\pi) = f^{\lambda(K(\pi))} \quad .$$

But for $\pi \equiv \pi(\lambda)$ it follows from [W1, Sec.2, Prop.4.9] that

$$\lambda(K(\pi)) = \lambda(L(\omega_n \pi \omega_n)) = \lambda(L(\pi(\lambda'))) = \lambda'(L(\pi)) = \lambda' \ ,$$

and therefore $r(\pi) = f^{\lambda'} = f^{\lambda}$. □

We describe next for a given partition λ a combinatorial rule, which sets up a **bijection** Ψ between the sets $R(\pi(\lambda))$ of reduced words and $SYT(\lambda)$ of standard tableaux:

For $a \in R(\pi(\lambda))$ the corresponding $\zeta := \Psi(a) \in SYT(\lambda)$ is found as follows: for a_1^* draw a box numbered with 1; assume that for some $\nu \in \{1, \dots, p-1\}$ a tableau D containing the numbers $1, \dots, \nu$ has been constructed, then the next box to be added to D is found by

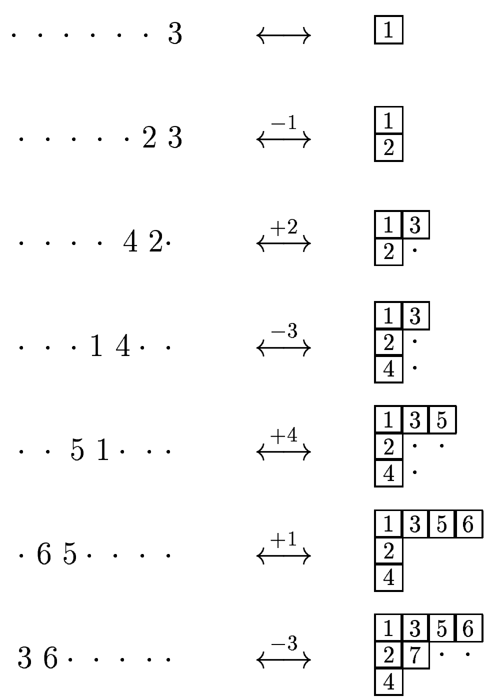
moving upwards by k_ν steps from the box $\nu \in D$ iff $k_\nu := a^*_{\nu+1} - a^*_\nu > 0$,
moving downwards by $-k_\nu$ steps from the box $\nu \in D$ iff $k_\nu < 0$.

The new box is numbered with $\nu + 1$.

For $\zeta \in SYT(\lambda)$ the corresponding $a := \Psi^{-1}(\zeta) \in R(\pi(\lambda))$ is found by the obvious reversal of the above process, where $a^*_1 := l(\lambda)$.

The combinatorial rule for Ψ described above will be called the **μ-rule**, where μ stands for 'movements'.

Example 0.1. Take $\lambda = 4\ 2\ 1$. Then the associated Grassmannian permutation is $\pi \equiv \pi(\lambda) = 2471356 \in S_7$ of length 7. An elementary calculation shows that $a = 3651423 \in R(\pi)$. The above bijection Ψ then yields the following correspondences, where instead of using a^* (which gives convenient indices in the description of Ψ) we simply read a backwards. Boundary boxes used in moving up or down are marked by a dot.



Historical remark. The combinatorics of tableaux and its relationship with other combinatorial and algebraic structures has been thoroughly investigated by Marcel-Paul Schützenberger (1920 - 1996). Therefore the link between reduced decompositions and tableaux was first described by A. Lascoux and Marcel-Paul Schützenberger in [LS] in terms of 'nilplactic classes': the set $R(\pi)$ is partitioned into nilplactic classes, where each element inside a given nilplactic class is characterized by its 'Q-symbol' in the sense of the usual Robinson-Schenstedt-Knuth correspondence. Taking into consideration that for Grassmannian permutations (and more generally vexillary permutations) the set $R(\pi)$ consists only of one nilplactic class, this correspondence can be used to set up a bijection between reduced compositions and standard Young tableaux. This line of reasoning has been generalized by V. Reiner and M. Shimozono in [RS1] eliminating the use of the Q-symbol.

In [FGRS, Thm.2.4] a bijection is described between the set $R(\pi)$ and the set $BL(D)$ of 'injective balanced labelings' of the diagrams $D \equiv D(\pi)$ associated with a permutation π. In the case of the Grassmannian permutations $\pi(\lambda)$ this bijection is as a mapping almost identical to our bijection (see Rem.4.3 below), but its description is far more complicated. In [FGRS, Thm.2.6] the 'balanced column strict labelings' of diagrams $D(\pi)$ are used to rewrite the Billey-Jockusch-Stanley (BJS) formula for Schubert polynomials (cf. [FS]). In fact our bijection originates from work on Schubert polynomials, too, namely the observation that Schubert polynomials generalize Schur polynomials in the same way as the reduced words appearing in the BJS formula generalize the standard Young tableaux appearing in the combinatorial definition of Schur polynomials. This will be made precise in a future paper about 'Schubert functions', where we generalize the '$\tau P x$' or 'Baxter sequence' formulas for Schur functions (cf. [T, W3], see also Sec.2 below).

Once the μ-rule for the bijection Ψ is established, it turns out that one can simplify it further to a δ-rule: let the Ferrer diagram of a partition λ be given as in the first paragraph, then the *diagonal tableau* δ_λ *of shape* λ is the numbering of boxes $(i,j) \in \lambda$ with $\delta_\lambda(i,j) := l(\lambda) + j - i$, i.e. the entries of δ_λ are constant on the 'diagonals' $j - i =$constant . It will be convenient to count the *positions* of the numbers in a diagonal of some δ_λ by $1, 2, 3, \ldots$ as their row or column index increases.

For example:

$$\delta_{421} \quad = \quad \begin{array}{|c|c|c|c|} \hline 3 & 4 & 5 & 6 \\ \hline 2 & 3 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array} \qquad ,$$

where the 1^{st} 3 is in the place $(1,1)$, and the 2^{nd} 3 is in the place $(2,2)$.

Let now $a \in R(\pi(\lambda))$ and $\zeta \in SYT(\lambda)$ correspond via Ψ, then the δ**-rule** for Ψ (δ stands for 'diagonal') can be described as follows:

Assume that ζ is given; then the entry a^*_ν of $a := \Psi^{-1}(\zeta)$ is $a^*_\nu = l(\lambda)+j-i$, where (i,j) is the box containing ν in ζ, or, alternatively, a^*_ν is the entry in δ_λ standing in the the same place as ν in ζ.

Assume on the other hand that $a \in R(\pi(\lambda))$ is given; then the entry ν in $\zeta := \Psi(a)$ is placed in the same place as the number a^*_ν in δ_λ, where the position of the number a^*_ν in δ_λ is chosen to be $|\{j \mid a^*_j = a^*_\nu,\ 1 \le j \le \nu\}|$.

The reader may convince himself that the δ-rule immediately yields the correspondence

$$3\,6\,5\,1\,4\,2\,3 \quad \overset{\delta_{421}}{\longleftrightarrow} \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \quad .$$

We now briefly describe the plan of the paper: In Section 1 we introduce a natural partial order on the set $ST(D)$ of standard tableaux for an arbitrary shape D and show that this poset is isomorphic to an order ideal in weak left Bruhat order. Moreover, for hook shapes the Poincaré polynomial is shown to be the q-binomial coefficient, and for general Ferrer shapes a recursion formula for the Poincaré polynomials is given. Section 2 contains a description of the polynomials and functions associated by G.P. Thomas to every shape D, and addresses the question of the symmetry of these functions. In Section 3 we introduce a partial order on the set $R(\pi)$ of reduced words of a permutation π; we show that the elements of $R(\pi(\lambda))$ are generated using only the relations of the type (ii) above, and that for the Grassmannian and dominant permutation associated to a hook shape the respective posets of reduced words are isomorphic. In Section 4 we prove the validity of the μ-rule and the δ-rule for the bijection Ψ. Finally in Section 5 we show that the permutations π, $\omega\pi\omega$, π^{-1} and $\omega\pi^{-1}\omega$ ($\omega \equiv \omega_n := n \ldots 1$ for $\pi \in S_n$) have (anti-)isomorphic posets of reduced words in the Grassmannian case, and formulate a conjecture, which extends the Main Theorem to the case of signed permutations. Especially in Sections 1, 2, and 3 there are many open questions, which we believe to be worthy of further study.

1. The partial order on the set $ST(D)$

Let D be any diagram as defined in the introduction, where for our purpose it is adequate to assume that D doesn't contain empty rows and columns. The numbering of D with $1, 2, 3, \ldots$ in (Latin) reading order, i.e. inside the rows from left to right and the rows from top to bottom, is called the *row (order) tableau of shape* D, which will be denoted by $\zeta_0(D) \in ST(D)$. Similarly the *column (order) tableau of shape* D is defined as $\zeta_1(D) := (\zeta_0(D'))' \in ST(D)$ using conjugation.

For a fixed $\zeta \in ST(D)$ let $i(\nu)$ and $j(\nu)$ denote the row and column index, respectively, of the box with entry ν in ζ. If $j(\nu) \neq j(\nu+1)$ for some $\zeta \in ST(D)$ and $\nu \in \{1, \ldots, |D|-1\}$, it will be possible to interchanges the entries ν and $\nu+1$ of ζ by an *elementary transposition* τ_ν. The *partial order on* $ST(D)$ or the *poset* $PST(D)$ is defined as the transitive closure of the following covering relation on $ST(D)$:

$$[\ \zeta' \text{ (ν-)covers } \zeta\ ] :\Longleftrightarrow\ \zeta' = \tau_\nu\zeta,\ i(\nu) < i(\nu+1), \text{ and } j(\nu) \neq j(\nu+1).$$

Moreover we introduce the *set of inversions of* ζ

$$I(\zeta) := \{(\nu,\nu') \mid \nu < \nu',\ i(\nu) > i(\nu'),\ j(\nu) \neq j(\nu')\ \}\ ,$$

and the *in* and *out set of* ζ

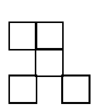
$$\begin{aligned} In(\zeta) &:= \{\nu \mid i(\nu) > i(\nu+1),\ j(\nu) \neq j(\nu+1)\ \}\ , \\ Out(\zeta) &:= \{\nu \mid i(\nu) < i(\nu+1),\ j(\nu) \neq j(\nu+1)\ \}\ . \end{aligned}$$

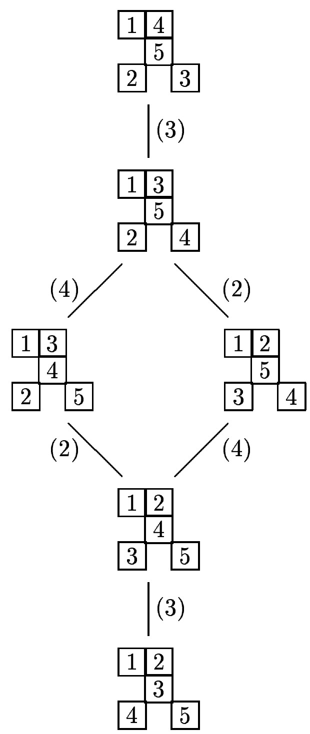
It will be convenient to label the edges in the Hasse diagram of $PST(D)$ with the index of the elementary transposition used, i.e. the labels of edges coming from below to some ζ or going up from some ζ are taken from $In(\zeta)$ and $Out(\zeta)$, respectively.

Remark 1.1. The above partial order can be 'enriched' to a *strong* order, if one allows not only elementary transpositions τ_ν but general transpositions $\tau_{\nu,\nu'}$ with $1 \leq \nu < \nu' \leq |D| - 1$. Clearly $PST(D)$ embeds into $ST(D)$ equipped with strong order, but we will not pursue this further in the present paper.

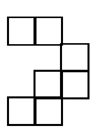
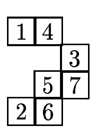
Proposition 1.2. *For any diagram D the poset $PST(D)$ is ranked by $i(\zeta) := |I(\zeta)|$. The unique minimal or bottom element is ζ_0.*

Proof. It is not hard to see that $In(\zeta_0) = \emptyset$, and that for all $\zeta \neq \zeta_0$ there exist at least one ζ' covered by ζ, i.e. $In(\zeta) \neq \emptyset$. If ζ' ν-covers ζ, then $I(\zeta')$ is the same as $I(\zeta)$ with ν and $\nu+1$ interchanged in the pairs and $(\nu,\nu+1)$ in addition. Hence: $i(\zeta') = i(\zeta) + 1 \iff \zeta'$ covers ζ, which proves the assertion on the rank. $\square$

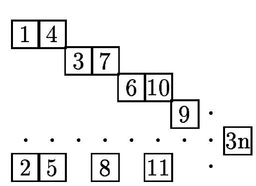
Example 1.3. We draw the (labeled) Hasse diagram of $PST(D)$ for the shape $D =$  :



The above example especially shows that ζ_1 is not necessarily maximal, namely, $r(\zeta_1) = 2$ and $Out(\zeta_1) = \{4\} \neq \emptyset$. In general an element $\zeta \in ST(D)$ for some shape D is maximal iff $Out(\zeta) = \emptyset$, and it is an interesting question to *find an algorithm, which yields directly all maximal ζ for a given shape D without enumerating the whole set $ST(D)$.*

For the shape  one has two maximal elements: the column tableau ζ_1 and  . In fact as the next example shows the number of maximal standard tableaux for an arbitrary shape D may be arbitrarily large.

Example 1.4. Let $\zeta(n)$ be the following standard tableau:



Clearly $Out(\zeta(n)) = \emptyset$, hence $\zeta(n)$ is maximal. It is now possible to interchange the "diagonal elements" $3, 6, 9, \ldots, 3n$ in such a way that the resulting tableau is again standard and maximal. For this purpose we write $[3, 6, 9, \ldots, 3n]$ for $\zeta(n)$ indicating only the "linear order" of the "diagonal elements".

For $n = 2$ the entries 3 and 6 can be interchanged or in abbreviated notation: $[3, 6]$ and $[6, 3]$ are maximal elements. For $n = 3$ one has $[3, 6, 9]$ and $[6, 3, 9]$ as "embeddings" from the former case, and in addition $[6, 9, 3]$ and $[3, 9, 6]$. For $n = 4$ the "new" maximal elements are $[6, 9, 12, 3]$, $[3, 9, 12, 6]$, and $[3, 6, 12, 9]$. The pattern is clear now, and we can conclude that the shape of $\zeta(n)$ admits at least $1 + 2 + \cdots + (n-1)$ maximal standard tableau.

For special classes of shapes D much more can be said. We are interested especially in Ferrer diagrams λ, *skew Ferrer diagrams* λ/μ, i.e. the shape of λ without the boxes of μ for some partition $\mu \subset \lambda$, and *shifted Ferrer diagrams*. The latter is defined for all partitions $\lambda^\nabla \equiv \lambda_1 \ldots \lambda_s$ with *distinct* parts $\lambda_1 > \cdots > \lambda_s$ as $\{(i,j) \mid i \le j \le \lambda_i + i,\ 1 \le i \le s\}$. The associated sets of standard diagrams $ST(D)$ are denoted by $SYT(\lambda)$, $SYT(\lambda/\mu)$, and $SST(\lambda^\nabla)$, respectively. Moreover, for an arbitrary shape D let $w(\zeta)$ be the word (or permutation) obtained from $\zeta \in ST(D)$ by reading the entries in row order.

Proposition 1.5. *In case of D being an ordinary, skew, or shifted Ferrer diagram the poset $ST(D)$ is a ranked lattice with ζ_0 as bottom (or unique minimal) element and ζ_1 as top (or unique maximal) element. Moreover the posets $PST(D)$ and $PST(D')$ are anti-isomorphic under conjugation.*

In the skew case — including the ordinary case as $\mu = \emptyset$ — the poset $PSYT(\lambda/\mu)$ is isomorphic to a lower left interval $[id,\ w(\lambda/\mu)]$, i.e. an interval of the form $[id, \pi]$ with respect to left order of $S_{|D|}$.

Proof. Prop.1.1 already implies that $PST(D)$ is a ranked poset with bottom element ζ_0. Observe now that under conjugation in and out sets interchange, i.e. for $\zeta \in ST(D)$ one has $In(\zeta') = Out(\zeta)$ and $Out(\zeta') = In(\zeta)$, because for all $\nu, \nu+1$ with $i(\nu) \ne i(\nu+1),\ j(\nu) \ne j(\nu+1)$ one has necessarily that $i(\nu) < i(\nu+1) \Longleftrightarrow j(\nu) > j(\nu+1)$; otherwise by standardness there would be a natural number properly between ν and $\nu + 1$! Therefore the posets $PST(D)$ and $PST(D')$ are anti-isomorphic under conjugation. (Compare ζ_1 of Ex.1.3 for an other behavior under conjugation.)

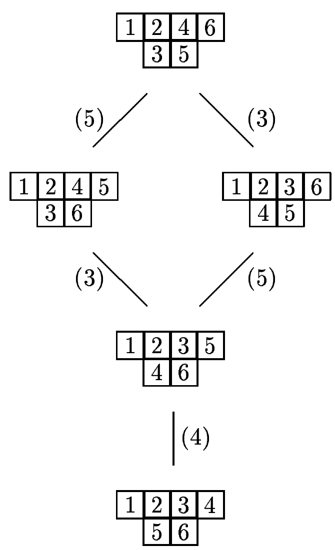
By Prop.1.1 again ζ_1' is the bottom element of $PST(D')$ and consequently ζ_1 is the top element of $PST(D)$.

The assertion on $PSYT(\lambda/\mu)$ being isomorphic to a lower left interval of $S_{|D|}$ is a simple consequence of the fact that $w(\zeta_0) = id$ and for all $\zeta, \overline{\zeta} \in SYT(\lambda/\mu)$ the relation $\overline{\zeta} = \tau_\nu \zeta$ implies $w(\overline{\zeta}) = \sigma_\nu w(\zeta)$. Hence by well known results on left order $PSYT(\lambda/\mu)$ is a lattice. □

Remark 1.6. The posets $PSYT(\lambda/\mu)$ have been considered already in the paper [BW] of Björner and Wachs. They have actually described a general criterion for a partial order on the set of linear extensions of a fixed poset to be isomorphic to a left interval of some S_n ([BW, Thm.7.6]). [BW] contains as Fig.1 a picture of $PSYT(321/1)$, which is isomorphic to $PSYT(321)$.

Remark 1.7. A shape D is called *reflexive* iff $PST(D)$ and $PST(D')$ are anti-isomorphic under conjugation. The different types of Ferrer diagrams are reflexive by the preceding proposition. Naturally one wants to know a simple geometric criterion for a shape D to be reflexive. Is it true that D is reflexive iff $\zeta_1 \in ST(D)$ is the unique maximal element of $PST(D)$?

Example 1.8. As an illustration of the preceding theorem and for later use we depict $PSST(42)$:



Proposition 1.9. *For an arbitrary shape D the poset $PST(D)$ embeds into a lower left interval of $S_{|D|}$ as an order ideal generated by the maximal elements of $PST(D)$.*

Proof. Björner and Wachs ([BW, Rem.7.3]) have defined the set $\widetilde{ST(D)}$ of 'standard tableaux' of an arbitrary shape D by the condition that the numbering with $1, \ldots, |D|$ increases in rows and columns, but in columns not necessarily across gaps! Therefore $ST(D) \subset \widetilde{ST(D)}$ and an appropriate $\zeta \in ST(D)$ may be covered by $\widetilde{\tau}_\nu(\zeta)$, where $\widetilde{\tau}_\nu$ is the 'elementary transposition across a gap' of entries ν and $\nu + 1$ in the same column. Clearly there can be no element of $ST(D)$ above an element of $\widetilde{ST(D)} \setminus ST(D)$, whence the assertion follows from the fact stated in [BW, Rem.7.3] that the poset $\widetilde{ST(D)}$ is isomorphic to a lower left interval. $\square$

The following questions require some more research: define two shapes D and $\overline{D}$ to be *order equivalent* iff they generate isomorphic posets of standard tableaux, in signs:

$$D \sim \overline{D} :\Longleftrightarrow PST(D) \cong PST(\overline{D}) \ .$$

Describe the equivalence class of order equivalent shapes by a simple and natural representant! And in view of Prop.1.9: given any order ideal in the left order of some S_n, is there a shape D with $PST(D)$ isomorphic to this order ideal?

Let S be an arbitrary (finite) poset with rank function i. Then the *rank generating* or *Poincaré polynomial* of S is given by

$$P_S(q) := \sum_{s \in S} q^{i(s)} \ .$$

It would be nice to have an explicit formula for $P_{PST(D)}(q)$ at least in case of $D = \lambda$, i.e. for $P_\lambda(q) := P_{PSYT(\lambda)}(q)$. Such a formula would both refine the the hook formula ($f^\lambda = P_\lambda(1)$) and explain via inversions the occurrence of hooks. But: "A general formula ... does unfortunately not seem to exist." ([BW, p.31]) For the moment we must be content with the following two results:

Proposition 1.10. (Recursion formula) *For a partition $\lambda \equiv \lambda_1 \dots \lambda_s \vdash N$ let $1 \leq i_1 < \dots < i_\nu < \dots < i_r \leq s$ be the indices with $\lambda_{i_\nu} > \lambda_{i_\nu+1}$ ($\lambda_{s+1} := 0$) and*

$$\lambda_{(i_\nu)} := \lambda_1 \dots \lambda_{i_\nu-1}(\lambda_{i_\nu} - 1)\lambda_{i_\nu+1} \dots \lambda_s \ ,$$

i.e. the $\lambda_{(i_\nu)}$ are exactly the partitions covered by λ in the Young lattice. Then:

$$P_\lambda(q) = \sum_{\nu=1}^{r} q^{\lambda_{i_\nu+1}+\dots+\lambda_s} P_{\lambda_{(i_\nu)}}(q) \quad .$$

Proof. Observe that for any $\zeta \in SYT(\lambda)$ the entry N can occur only at the end of some of the rows i_ν. In other words: ζ without the box N is an element of $SYT(\lambda_{(i_\nu)})$ and adding it causes exactly $\lambda_{i_\nu+1} + \dots + \lambda_s$ new inversions, whence the formula. □

Let $\binom{n}{k}_q := \frac{(n)!_q}{(k)!_q (n-k)!_q}$, where $(n)!_q := (1)_q(2)_q \dots (n)_q$ and $(n)_q := 1 + q + \dots + q^{n-1}$. Let moreover the (n,k)-*hook* be the partition $\lambda = (n+1-k)\ 1^k \vdash n+1$ with leg length k and arm length $n-k$, then by the hook formula $f^\lambda = \binom{n}{k}$, which is refined by

Proposition 1.11. *With the above notation the Poincaré polynomial for the (n,k)-hook λ is given by*

$$P_\lambda(q) = \binom{n}{k}_q \quad .$$

Proof. Every $\zeta \in SYT(\lambda)$ is uniquely determined by the set $I \equiv I(\zeta) = \{i_1, \dots, i_k\}$ of numbers $1 < i_1 < \cdots < i_k \leq n+1$ appearing in its leg or equivalently by the partition $\mu \equiv \mu_1 \dots \mu_k$ with parts $\mu_\nu = |\{i \mid i > i_\nu\} \cap (\{1, \dots, n+1\} \setminus I)|$. Clearly $n - k \geq \mu_1 \geq \cdots \geq \mu_k \geq 0$ and $r(\zeta) = |\mu|$. In other words, there is a bijection between $SYT(\lambda)$ and the set of partitions with at most k parts of size at most $n-k$, such that the rank in $SYT(\lambda)$ is equal to the weight of the associated partition. But the weight generating polynomial for exactly this set of partitions is $\binom{n}{k}_q$ by [S2, Prop.1.3.19]. $\square$

For general λ the Poincaré polynomial $P_\lambda(q)$ is unfortunately not given by the q-hook formula!

Example 1.12. For $\lambda = 3\ 2\ 1$ one computes $P_\lambda(q) = q^3 P_{221}(q) + qP_{311}(q) + P_{32}(q)$ with $P_{221}(q) = qP_{211}(q) + P_{22}(q) = q\binom{3}{2}_q + P_{21}(q) = q(3)_q + (2)_q = 1 + 2q + q^2 + q^3$, $P_{311}(q) = \binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$, and $P_{32}(q) = q^2 P_{22}(q) + P_{31}(q) = q^2(2)_q + (3)_q = 1 + q + 2q^2 + q^3$. Therefore: $P_{321}(q) = 1 + 2q + 3q^2 + 4q^3 + 3q^4 + 2q^5 + q^6$.

2. D-polynomials and D-functions

In this section we discuss the polynomials and (graded) functions associated by G.P. Thomas to an arbitrary shape D (see [T, W3]). The set $SST(D)$ of *semistandard tableaux of shape* D is the set of all numberings of D with entries increasing strictly in columns from top to bottom and weakly increasing ('=' allowed) in rows from left to right; for every $m \in \mathbb{N}$ the subsets of $SST(D)$ with entries at most m and maximal entry $= m$ are denoted $SST_{(m)}(D)$ and $SST_{[m]}(D)$, respectively. To every $\eta \in SST(D)$ there is associated the monomial x^η, which is the product over all (commuting) x_ν, ν being an entry of η. It is possible now to define the *D-polynomial in* $x_1, \dots, x_m$ as

$$s_D^{(m)}(x) := \sum_{\eta \in SST_{(m)}(D)} x^\eta \ ,$$

the *D-function in* $x_1, x_2, x_3, \dots$ as

$$s_D(x) := \sum_{\eta \in SST(D)} x^\eta \ ,$$

and the *graded D-function in* $x_1, x_2, x_3, \dots$ as

$$s_{[D]}(x) := (s_D^{[1]}(x), s_D^{[2]}(x), s_D^{[3]}(x), \dots)$$

with $s_D^{[m]}(x) := s_D^{(m)}(x) - s_D^{(m-1)}(x)$. The latter is an element of the $\mathbb{Z}$-algebra (componentwise operations)

$$A_{\mathbb{Z}}(x) := (\ R[x_1],\ R[x_1, x_2],\ R[x_1, x_2, x_3],\ \dots) \quad .$$

Note that in case of D being a Ferrer tableau λ these definitions specialize to the Schur polynomials and the (graded) Schur function. Clearly the knowledge about $s_{[D]}(x)$ implies knowledge about every $s_D^{(m)}(x)$ and $s_D(x)$. It comes therefore as a pleasant surprise that $s_{[D]}(x)$ can be constructed with ease from the finite set $ST(D)$ alone:

Proposition 2.1. ([T1, Thm.5.2]) *For every shape D and $\zeta \in ST(D)$ let*

$$B(\zeta)(x) := xB_{|D|-1}x \dots B_1 x$$

denote a sequence of multiplication operators $x \equiv (x_1, x_2, x_3, \dots)$ and shift operators $B_\nu \in \{P, S\}$ on $A_{\mathbb{Z}}(x)$, where $B_\nu = S$ iff $i(\nu) < i(\nu+1)$ in ζ, $S := \tau P$, $P := \sum_{\nu=0}^{\infty} \tau^\nu$, and τ is the left shift on $A_{\mathbb{Z}}(x)$ defined by $\tau(a_1, a_2, a_3, \dots) := (0, a_1, a_2, \dots)$ for every $a = (a_1, a_2, \dots) \in A_{\mathbb{Z}}(x)$. Then

$$s_{[D]}(x) = \sum_{\zeta \in ST(D)} B(\zeta)(x) \ .$$

This formula for the graded D-function behaves nicely under conjugation of diagrams, namely:

Proposition 2.2. ([T1, Thm.7.2]) *Let D' be the shape conjugate to D and $B'(\zeta)$ the same as $B(\zeta)$ in Prop.2.1 above, but with $P \leftrightarrow S$ (P interchanged with S); then*

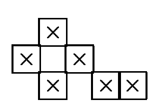
$$s_{[D']}(x) = \sum_{\zeta \in ST(D)} B'(\zeta)(x) \ .$$

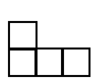
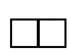
Remark 2.3. Using the results of [W3] it is not hard to generalize Hall-Littlewood, Jack, and Macdonald polynomials and functions and their 'super' variants for arbitrary diagrams.

A natural question is: for which diagrams D are the associated polynomials $s_D^{(m)}(x)$ and functions $s_D(x)$ *symmetric*? We conjecture that either all $s_D^{(m)}(x)$ (and hence $s_D(x)$) are symmetric or no one.

Let $\emptyset \neq C \subset D$ and

$$\tilde{C} := \bigcup_{(i,j) \in C} \{(i', j') \in \mathbb{Z}^2 \mid i' = i \text{ or } j' = j\} \ ;$$

then C is called a *component* of D iff $\tilde{C} \cap (D \setminus C) = \emptyset$. Clearly every diagram D is the disjoint union of its components $C_1, \dots, C_r$; D is called *connected* iff $r = 1$. Eliminating empty rows and columns from the diagrams of the components gives the *irreducible components of* D and D as the *direct sum* of its irreducible components, e.g. the shape $D =$  has the *irreducible decomposition*:

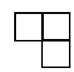
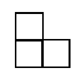
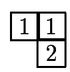
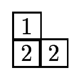
$D \quad \cong$  $\oplus$ .

Proposition 2.4. *Let $D \cong \bigoplus_{\nu=1}^{r} D_\nu$ be a irreducible decomposition of D. Then:*
a) D determines its irreducible decomposition up to permutation of components;
b) 'having the same irreducible components' is an equivalence relation on diagrams;
c) there is a bijection:

$$SST(D) \xrightarrow{bij.} \bigoplus_{\nu=1}^{r} SST(D_\nu) = (SST(D_1), \ldots, SST(D_r)) \ ;$$

d) $s_D^{(m)}(x) = \prod_{\nu=1}^{r} s_{D_\nu}^{(m)}(x)$, and $s_D^{(m)}(x)$ is symmetric, if all polynomials $s_{D_\nu}^{(m)}(x)$ are symmetric.

Proof. a) and b) are trivial, and the bijection of c) is given as the mapping between boxes of D and boxes in the irreducible decomposition, which preserves the relative order of boxes and the entries. The first equality of d) is immediate from c), and the 'if' part is obvious. $\square$

The reversal of d): "$s_D^{(m)}(x)$ is symmetric, then all polynomials $s_{D_\nu}^{(m)}(x)$ are symmetric" is in general not true. For example let $D \cong$  $\oplus$  . Then $s_D^{(2)}(x) = x_1^3 x_2^3$ is symmetric:  $\oplus$  , but the polynomials in x_1, x_2 of the irreducible parts are not symmetric. The same is true for all other $m > 2$. Is there a general geometric reason for this phenomenon? Is it connected with a "theory" of plane puzzles? Is there a representation theoretic meaning of D-polynomials, D-functions, and irreducible decompositions for arbitrary shapes D ? How is it related to the Specht, Schur, and flagged Schur modules associated to an arbitrary shape D (cf. e.g. [RS2, Sec.2]) ?

We conjecture, that the D-polynomials are symmetric for a connected shape D iff D is a (skew) Ferrer shape λ/μ.

3. The partial order on the set $R(\pi)$

Recall that $R(\pi)$ is the set of reduced words or sequences of a finite permutation $\pi \in S_n$. $R(\pi)$ can be made into a *graph* $GR(\pi)$ with the vertex set $R(\pi)$ as follows: two 'vertices' $a, b \in R(\pi)$ are *adjacent* iff a can be transformed into b by exactly one application of the relation (ii) or (iii). (Note that reduced words correspond to chains in weak (left or right Bruhat) order, and that two chains, which differ by one application of the relation (ii) or (iii), enclose a square or a hexagon, respectively, in their Hasse diagram; therefore:) Transformations of reduced sequences using relations (ii) or (iii) will be called applications of the *square rule* or *hexagon rule*, respectively.

Proposition 3.1. *The graph $R(\pi)$ is connected, i.e. for all $a, b \in R(\pi)$ there is a chain of pairwise adjacent elements of $GR(\pi)$ connecting a and b.*

Proof. The result has been proved first in the general framework of finite Coxeter groups by J. Tits (1968) (cf. [B]). A proof in the present special case of symmetric groups can be found for example in [W3, Prop.1.2]. □

The graph $GR(\pi)$ is called *hexagon free* iff no adjacency originates from the hexagon rule.

Lemmma 3.2. *The graph $GR(\pi)$ is hexagon free iff one (and consequently all) $a \in R(\pi)$ has the following property*

$$(*) \qquad \forall m \in \mathbb{N}: \quad a \equiv \ldots m\ a'\ m\ \ldots \text{ and } m \text{ not a letter of } a' \\ \implies \quad m+1, m-1 \text{ letters of } a'.$$

Proof. Assume first that $a \equiv \ldots m\ a'\ m\ \ldots$ with a' containing both $m+1$ and $m-1$, but not m. Assume furthermore that both $m+1$ and $m-1$ occur with multiplicity one; then by application of the square rule the subword '$m\ a'\ m$' of a can be transformed into '$b\ m\ m+1\ m-1\ m$' (with b not containing $m-1, m, m+1$). If on the other hand $m-1$ (or $m+1$) occurs more than one time in a', then consider again $(*)$ with $m-1$ (or $m+1$) in place of m. In any case it is not possible to generate a subword $m\ m-1\ m$ or $m\ m+1\ m$ of a for some $m \in \mathbb{N}$, whence the hexagon rule can not be applied.

To prove the other direction we assume that there is some m, such that $a \equiv \ldots m\ a'\ m\ \ldots$ with m and $m+1$ not in a'; without restriction of generality we can assume further that no letter $\geq m$ is contained in a'. Then a' contains $m-1$, otherwise a would not be reduced. If $m-1$ occurs with multiplicity one, then '$m\ a'\ m$' can be transformed into '$b\ m\ m-1\ m$', and the hexagon rule can be applied. If on the other hand $m-1$ occurs more than one time in a', then $a' \equiv \ldots m-1\ a''\ m-1\ \ldots$ with $m-1$ and m not in a'', and the argument applies again. □

The *Lehmer code* $L(\pi)$ of some $\pi \in S_n$ is an element of the set

$$\mathbf{L}_n := \{\ [l_{n-1} \ldots l_0] \mid 0 \leq l_{n-i} \leq n-i \text{ for } i = 1, \ldots, n\}$$

defined by $l_{n-i}(\pi) := |\{j \mid i < j,\ \pi i > \pi j\}|$, e.g. $L(361542) = [240210]$. L is a bijection between S_n and $\mathbf{L}_n$, and, moreover, there is an easy way to compute a reduced word for π via its Lehmer code (cf. [W1, W2]):

$\Phi(L(\pi)) := \Phi(l_{n-1}) \ldots \Phi(l_{n-i}) \ldots \Phi(l_0)$ with $\Phi(l_{n-i}) := (l_{n-i}+i-1) \ldots (i+1)\ i$ for $l_{n-i} > 0$ and $\Phi(0) := \emptyset$ is a reduced sequence for π, e.g. $\Phi(L(361542)) = \Phi([240210]) = 21\ 5432\ 54\ 5 \in R(361542)$. We use the notation $a^{(0)} \equiv a^{(0)}(\pi)$ for the unique reduced sequence $\Phi(L(\pi))$.

Proposition 3.3. *$GR(\pi(\lambda))$ is hexagon free for every partition λ.*

Proof. Observe that for $\lambda \equiv \lambda_1 \ldots \lambda_s$ by definition of $\pi(\lambda)$ one has $L(\pi(\lambda)) = [\lambda_s \ldots \lambda_1\, 0 \ldots 0]$ (with λ_1 zeros). Therefore (vertical bars included for clarity):

$$\Phi(L(\pi(\lambda))) = \lambda_s \ldots 1 \mid (\lambda_{s-1}+1) \ldots 2 \mid \; \ldots \mid (\lambda_1 + s - 1) \ldots s$$

is reduced for $\pi(\lambda)$ and easily seen to obey $(*)$ of Lemma 3.2 . □

Remark 3.4. Grassmannian permutations are special cases of 321-avoiding permutations, i.e. permutations π with the property that for no $i < j < k$ one has $\pi(i) > \pi(j) > \pi(k)$, and while finishing the present paper we recognized [BJS, Thm.2.1], which says that 321-avoiding permutations are exactly the hexagon free permutations.

Proposition 3.5. *Let $\pi \in S_n$ and $\omega \equiv \omega_n := n \ldots 1 \in S_n$. Then the graphs $GR(\pi)$, $GR(\pi^{-1})$, $GR(\omega\pi\omega)$ are all isomorphic. Consequently:*

$$r(\pi) = r(\pi^{-1}) = r(\omega\pi\omega) \ .$$

Proof. For a (not necessarily reduced) sequence $a \equiv a_1 \ldots a_r$ generating π let $a^* := a_r \ldots a_1$ be the reversed sequence and $a' := (n - a_1) \ldots (n - a_r)$ the *conjugate* sequence. Clearly: 'a generates π' iff 'a^* generates π^{-1}', and [W3, Prop.3.2] says: 'a generates π' iff 'a' generates $\omega\pi\omega$'. Now under the operations of reversal and conjugation the relations of type (ii) and (iii) transform to relations of type (ii) and (iii), respectively, and therefore these operations are bijections between the sets $R(\pi)$, $R(\pi^{-1})$ and the sets $R(\pi)$, $R(\omega\pi\omega)$, which preserve adjacency. □

For arbitrary finite permutation $\pi \in S_n$ we define the ranked *poset $PR(\pi)$ of reduced words for π* as follows: take $a^{(0)}(\pi)$ as the bottom element (of rank zero) and the elements of rank $s > 0$ as the reduced sequences for π, which have distance s to $a^{(0)}$ in $GR(\pi)$. (The *distance* between a and b in a connected graph G is defined as the minimal number of edges in a path connecting a and b.) We write $i(a)$ for the rank of a. The partial order on $GR(\pi)$ is defined as the transitive closure of the following covering relation:

$$a \text{ covers } b :\Longleftrightarrow a \text{ adjacent to } b \text{ and } i(a) > i(b) \ .$$

(Of course then $i(a) = i(b) + 1$.) We label the edge from a to b of the Hasse diagram of $PR(\pi)$ by (ν), if a^*_ν and $a^*_{\nu+1}$ are interchanged by the square rule, and by $[\nu]$, if $a^*_{\nu-1}, a^*_\nu$ and $a^*_{\nu+1}$ are changed by the hexagon rule. (Recall that a^* is the reverse of a.)

Example 3.6. We draw the labeled Hasse diagram for $\pi = 52314 \in S_5$. $L(\pi) = [41100]$ and $a^{(0)}(\pi) = 432123$.

$$\begin{tikzcd}
& 124321 \arrow[d, dash, "(4)"'] & & \\
& 142321 \arrow[dl, dash, "(5)"'] \arrow[dr, dash, "{[3]}"] & & \\
412321 \arrow[dr, dash, "{[3]}"'] & & 143231 \arrow[dl, dash, "(5)"] \arrow[dr, dash, "(1)"] & \\
& 413231 \arrow[dl, dash, "(4)"'] \arrow[dr, dash, "(1)"] & & 143213 \arrow[dl, dash, "(5)"] \\
431231 \arrow[dr, dash, "(1)"'] & & 413213 \arrow[dl, dash, "(4)"] & \\
& 431213 \arrow[d, dash, "{[3]}"'] & & \\
& 432123 & &
\end{tikzcd}$$

Remark 3.7. The posets $PR(\pi)$, $PR(\pi^{-1})$, $PR(\omega\pi\omega)$ are in general neither isomorphic nor anti-isomorphic, because the respective $a^{(0)}$'s and $a^{(1)}$'s are in general unrelated. But see Prop.5.1 below for the Grassmannian case $\pi \equiv \pi(\lambda)$.

In general it is hard to determine the Poincaré polynomial $P_\pi(q)$ of $PR(\pi)$, but some special statements are possible:

By Prop.4.1 below the posets $PR(\pi(\lambda))$ and $PSYT(\lambda)$ are isomorphic for every partition λ, and therefore one has $P_{\pi(\lambda)}(q) = P_\lambda(q)$. To every partition λ there is associated not only a Grassmannian permutation $\pi(\lambda)$, but also a *dominant* permutation $\overline{\pi}(\lambda)$, which is defined by

$$\overline{\pi}(\lambda) := L^{-1}[\lambda_1\lambda_2 \ldots \lambda_s\ 0 \ldots 0]$$

with a sufficiently large number of zeros. Note that $\pi = 52314$ from the Ex.3.6 is in fact $\overline{\pi}(411)$.

Proposition 3.8. $r(\overline{\pi}(\lambda)) = f^\lambda = r(\pi(\lambda))$ *for all partitions* λ.

Proof. We proceed as in the algebraic proof of the Main Theorem:

$$\lambda'(L(\overline{\pi}(\lambda))) := (\lambda(L(\overline{\pi}(\lambda))))' = \lambda' = \lambda(L(\overline{\pi}(\lambda')))$$

and on the other hand $\lambda(K(\overline{\pi}(\lambda))) = \lambda(L(\overline{\pi}(\lambda)^{-1}))$. Using the operator $\overrightarrow{E}$ from [W1] one sees $\overrightarrow{E}(L(\overline{\pi}(\lambda))) = L(\overline{\pi}(\lambda'))$. Together with $\overrightarrow{E}(L(\pi)) = L(\pi^{-1})$ for all π (cf. [W1, Prop.4.5]) this yields $L(\overline{\pi}(\lambda)^{-1})) = L(\overline{\pi}(\lambda'))$ and $\lambda'(L(\overline{\pi}(\lambda))) = \lambda(K(\overline{\pi}(\lambda)))$, which implies the assertion by [S1, Cor.4.2]. $\square$

The above Ex.3.6 shows that $P_{\overline{\pi}(411)}(q) = \binom{5}{2}_q = P_{\pi(411)}(q)$. This is no accident:

Proposition 3.9. *Let λ be a (n,k)-hook. Then $PR(\overline{\pi}(\lambda)) \cong PR(\pi(\lambda))$ and consequently $P_{\overline{\pi}(\lambda)} = \binom{n}{k}_q = P_{\pi(\lambda)}$.*

Proof. Let $\lambda = (n+1-k)\ 1^k$. In the Young lattice λ covers the partitions $\lambda_a = (n-k)\ 1^k$ and $\lambda_l = (n+1-k)\ 1^{k-1}$ with shortened arm length and leg length, respectively. Then

$$\begin{aligned}
\Phi(L(\pi(\lambda))) &= 1\ldots(k-1)\ k \quad (n+2)\ (n+1)\ldots(k+1)\ ,\\
\Phi(L(\pi(\lambda_l))) &= 1\ldots(k-1) \quad (n+2)\ (n+1)\ldots(k+1)\ ,\\
\Phi(L(\pi(\lambda_a))) &= 1\ldots(k-1)\ k \quad (n+1)\ldots(k+1)\ ,
\end{aligned}$$

and

$$\begin{aligned}
\Phi(L(\overline{\pi}(\lambda))) &= (n-k+1)\ (n-k)\ldots 1\ 2\ldots k\ (k+1)\ ,\\
\Phi(L(\overline{\pi}(\lambda_l))) &= (n-k+1)\ (n-k)\ldots 1\ 2\ldots k\ ,\\
\Phi(L(\overline{\pi}(\lambda_a))) &= (n-k)\ldots 1\ 2\ldots k\ (k+1)\ .
\end{aligned}$$

Every $a \in R(\pi(\lambda))$ begins either with 1 or with $n+2$, since by elementary transpositions none of the numbers $2,\ldots,k$ can move left to 1, and none of the numbers $n+1,\ldots,k+1$ left to $n+1$. Now the subposet of $a = 1\ldots$ in $PR(\pi(\lambda))$ is obviously isomorphic to $PR(\pi(\lambda_l))$, which by induction hypothesis is isomorphic to $PR(\overline{\pi}(\lambda_l))$, and the subposet of $a = (n+2)\ldots$ in $PR(\pi(\lambda))$ is isomorphic to $PR(\pi(\lambda_a)) \cong PR(\overline{\pi}(\lambda_a))$. In other words: $PR(\pi(\lambda))$ is a 'stack' of $PR(\pi(\lambda_a))$ above $PR(\pi(\lambda_l))$ with 'covering edges' given by the interchange of 1 and $n+2$.

Similarly every $a \in R(\overline{\pi}(\lambda))$ ends either with $k+1$ or 1, the $a = \ldots(k+1)$ forming a subposet of $PR(\overline{\pi}(\lambda))$ isomorphic to $PR(\overline{\pi}(\lambda_l))$ and the $a = \ldots 1$ a subposet isomorphic to $PR(\overline{\pi}(\lambda_a))$. They are 'stacked' upon each other in $PR(\overline{\pi}(\lambda))$ along the 'covering edges' given by the interchange of $k+1$ and 1 (i.e. the edges labeled by (1), see e.g. Ex.3.6). Now the observation that $n+1$ stands in position $k+1$ from left in $\Phi(L(\pi(\lambda)))$ and 1 stands in position $k+1$ from right in $\Phi(L(\overline{\pi}(\lambda)))$, both needing the same number of steps to the border position, completes the proof. □

Notice that this gives a nice illustration of the recursion formula for q-binomial coefficients: $\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q$.

In general one has $P_{\overline{\pi}(\lambda)} \neq P_{\pi(\lambda)}$, for example $P_{\pi(321)} = 1+2q+3q^2+4q^3+3q^4+2q^5+q^6$ (Ex.1.12 and Prop.4.1), but a computation of $PR(4321)$ shows that $P_{\overline{\pi}(321)} = 1+2q+2q^2+3q^3+3q^4+2q^5+2q^6+q^7$.

4. Proof of the μ- and δ-rule

We first resume all results, which are important for the proof of the μ- and δ-rule for the bijection Ψ between the sets $R(\pi(\lambda))$ and $SYT(\lambda)$: the partially ordered set $PSYT(\lambda)$ has as bottom element the row tableau ζ_0 and the covering relations are given by elementary transpositions τ_ν; the partially ordered set $PR(\pi(\lambda))$ has as bottom element the reduced sequence $a^{(0)}$ and the covering relations are given by transpositions according to the square rule. Using now that

$$a^{(0)} \equiv \Phi(L(\pi(\lambda))) = \lambda_s \dots 1 \mid \lambda_{s-1}+1 \dots 2 \mid \dots \mid (\lambda_1+s-1)\dots s ,$$

it is an easy matter to see from the description of the μ- or δ-rule that $\Psi(a^{(0)}) = \zeta_0$. The lemma below now shows that going up one step in either poset is compatible with Ψ, and therefore we have in fact proved not only Ψ to be a bijection or $r(\pi(\lambda)) = f^\lambda$, but:

Proposition 4.1. *For every partition λ the posets $PR(\pi(\lambda))$ and $PSYT(\lambda)$ are isomorphic, and consequently $PR(\pi(\lambda))$ is a lattice with bottom element $a^{(0)} \equiv \Phi(L(\pi(\lambda)))$ and a top element named $a^{(1)}$, which can be computed as*

$$a^{(1)}(\pi(\lambda)) := \Psi^{-1}(\zeta_1(\pi(\lambda))) .$$

Lemmma 4.2. *For a given partition λ assume that $\zeta \in SYT(\lambda)$ has been computed from some $a \in R(\pi(\lambda))$ in accordance with the μ- or δ-rule for Ψ. Assume further that $k_\nu := a^*_{\nu+1} - a^*_\nu \geq 2$ so that $\tau_\nu a$ is reduced for $\pi(\lambda)$, too. Then $\tilde\zeta := \Psi(\tau_\nu a) \in SYT(\lambda)$ coincides with ζ except for the entries ν and $\nu+1$ being interchanged.*

Proof. We give a proof that applies simultaneously to the description of Ψ by both the μ- and the δ-rule. Let $D_{\nu-1}$ denote the Ferrer diagram generated by Ψ applied to $a^*_1, \dots, a^*_{\nu-1}$, and $D_\nu \equiv D_{\nu-1} \cup \{(i_\nu, j_\nu)\}$ the Ferrer diagram generated by Ψ applied to $a^*_1, \dots, a^*_\nu$, etc. . By hypothesis we can assume that $a^*_{\nu-1} = l(\lambda) + j_{\nu-1} - i_{\nu-1}$, and $a^*_\nu = l(\lambda) + j_\nu - i_\nu$, etc. . Note that for the boxes of $\partial^- D_{\nu-1}$ one has in the diagonal tableau of shape λ the gapless ascending chain of natural numbers:

$$\dots, \delta_\lambda(i_{\nu-1}+1, j_{\nu-1}) = a^*_{\nu-1} - 1, \quad \delta_\lambda(i_{\nu-1}+1, j_{\nu-1}+1) = a^*_{\nu-1},$$
$$\delta_\lambda(i_{\nu-1}, j_{\nu-1}+1) = a^*_{\nu-1} + 1, \dots .$$

Note moreover that (i_ν, j_ν) and $(i_{\nu+1}, j_{\nu+1})$ are 'corner boxes' in $D_{\nu+1}$, since D_ν and $D_{\nu+1}$ are Ferrer diagrams and $k_\nu \geq 2$.

Therefore applying Ψ to $(\tau_\nu a)^* \equiv \dots a^*_{\nu-1}, a^*_{\nu+1}, a^*_\nu, a^*_{\nu+2}, \dots$ yields the shapes $\dots$, $\tilde D_{\nu-1} = D_{\nu-1}$, $\tilde D_\nu = D_{\nu+1} \setminus \{(i_\nu, j_\nu)\}$, $\tilde D_{\nu+1} = D_{\nu+1}, \dots$, and the necessary upward and downward moves for the steps from $\nu-1$ to ν, and from ν to $\nu+1$ can be considered as taking place in $\partial^- D_{\nu-1}$. In other words: the difference between the standard tableaux of shape $\tilde D_{\nu+1} = D_{\nu+1}$ generated so far is that the entries ν and $\nu+1$ are interchanged.

To complete the proof of the lemma it remains to be shown that $\tilde D_{\nu+2} = D_{\nu+2}$ and that $\nu+2$ is in the same place in both cases. Observe that the movement from box $\nu+1$ to $\nu+2$ in $D_{\nu+1}$ it starts from box $(i_\nu+1, j_\nu+1)$ with $\delta_\lambda(i_\nu+1, j_\nu+1) = a^*_{\nu+1}$, and in $\tilde D_{\nu+1}$ starts from box (i_ν, j_ν) with

$\delta_\lambda(i_\nu, j_\nu) = a^*_\nu$. In case of $a^*_{\nu+2}$ being between a^*_ν and $a^*_{\nu+1}$ the respective movements can be considered again as taking place in $\partial^- D_{\nu-1}$ as if there were no boxes (i_ν, j_ν) and $(i_\nu+1, j_\nu+1)$ placed.

In case of $a^*_{\nu+2}$ not being between a^*_ν and $a^*_{\nu+1}$ one of the movements, say the one, which builds up $D_{\nu+2}$ from $D_{\nu+1}$ is as in $\partial^- D_{\nu-1}$; the other movement faces the obstacle of (i_ν, j_ν) placed in $\partial^- D_{\nu-1}$, but (i_ν, j_ν) is a 'corner box', and therefore it is possible to circumvent it by moving trough the positions

$$\ldots, (i_\nu+1, j_\nu), (i_\nu+1, j_\nu+1), (i_\nu, j_\nu+1), \ldots \text{ instead of}$$
$$\ldots, (i_\nu+1, j_\nu), (i_\nu, j_\nu), (i_\nu, j_\nu+1), \ldots \quad .$$

□

Remark 4.3. For $\zeta \in SYT(\lambda)$ let ζ^* denote the *reversal of* ζ, which is obtained from ζ by replacing every entry ν by $|\lambda| - \nu + 1$. Following the lines of the proof given in this section it is not hard to show that the bijection defined in [FGRS, Sec.2] between $R(\pi)$ and the set $BL(D)$ of 'injective balanced labelings of the diagram of π' in the Grassmannian case $\pi \equiv \pi(\lambda)$ specializes to our Ψ followed by the operation of reversal.

Indeed the set $BL(\pi(\lambda))$ is the element wise reversal of $SYT(\lambda)$, the labeled diagram associated to $a^{(0)}(\pi(\lambda))$ by the FGRS-rule is ζ^*_0, and a transposition in $R(\pi(\lambda))$ has under this rule the same effect as a transposition in Lemma 4.2 above.

But the FGRS-rule is much more complicated then our μ- and the δ-rules, and this is necessarily so, because on one hand the FGRS-rule applies to arbitrary permutations π, and on the other hand balanced labelings are intrinsically more complicated than standard numberings.

5. The 'Fourfold Way' & signed permutations

In this section we consider the relations between the finite permutations π, π^{-1}, $\omega\pi\omega$ and $\omega\pi^{-1}\omega$ for any $n \in \mathbb{N}$, $\pi \in S_n$ and $\omega \equiv \omega_n := n \ldots 1 \in S_n$; as diagram (note that $\omega^{-1} = \omega$):

$$\begin{array}{ccc} \pi & \longrightarrow & \pi^{-1} \\ \downarrow & & \downarrow \\ \omega\pi\omega & \longrightarrow & \omega\pi^{-1}\omega \end{array} \quad . \tag{5.1}$$

This is the *'Fourfold Way'*.

It is well known that the length of π is invariant under inversion $\pi \mapsto \pi^{-1}$ and *conjugation* $\pi \mapsto \omega\pi\omega$ (see e.g. [W1, Lemma 2.1]), and in Prop.3.5 we have seen that the graphs $GR(\pi)$, $GR(\pi^{-1})$, $GR(\omega\pi\omega)$ are isomorphic, whence $r(\pi) = r(\pi^{-1}) = r(\omega\pi\omega)$. This has been a consequence of

$$\begin{array}{ccc} a \in R(\pi) & \longrightarrow & a^* \in R(\pi^{-1}) \\ \downarrow & & \downarrow \\ a' \in R(\omega\pi\omega) & \longrightarrow & (a')^* = (a^*)' \in R(\omega\pi^{-1}\omega) \end{array} \tag{5.2}$$

using the operations of *reversion* $a \mapsto a^*$ and *conjugation* $a \mapsto a'$ defined in Sec.3 . We have introduced there moreover the Lehmer code $L(\pi)$ of a permutation and a mapping $\Phi^L \equiv \Phi$, which associates a 'standard reduced sequence' $a^{(0)}(\pi) := \Phi^L(L(\pi)) \in R(\pi)$ to every Lehmer code. In [W1, W2] we have introduced and investigated in fact four codes and four mappings from codes to reduced sequences related by

$$\begin{array}{ccc} L(\pi) & \longrightarrow & H(\pi) = L(\pi^{-1}) \\ \downarrow & & \downarrow \\ K(\pi) = L(\omega\pi\omega) & \longrightarrow & G(\pi) = L(\omega\pi^{-1}\omega) \end{array} \tag{5.3}$$

with $\Phi^L(L(\pi)),\ \Phi^H(H(\pi)),\ \Phi^K(K(\pi)),\ \Phi^G(G(\pi)) \in R(\pi)$. This enables us to show

Proposition 5.1. *For every partition λ the poset $PR(\pi(\lambda))$ is isomorphic to $PR(\pi(\lambda')^{-1})$ and anti-isomorphic to $PR(\pi(\lambda)^{-1})$ and $PR(\pi(\lambda'))$. In fact (using the operations of reversion (*) and of conjugation ($'$) on words from Prop.3.5) one has*

$$a^{(0)}(\pi(\lambda')) = [a^{(1)}(\pi(\lambda))]' \text{ and } a^{(0)}(\pi(\lambda)^{-1}) = [a^{(1)}(\pi(\lambda))]^* \ . \tag{5.4}$$

Proof. First we recall from [W1] that the operation of conjugation of permutations $\pi \mapsto \pi^{-1}$ generalizes the conjugation of Ferrer diagrams, i.e. for $\pi = \pi(\lambda)$ the diagram (5.1) reads:

$$\begin{array}{ccc} \pi(\lambda) & \longrightarrow & \pi(\lambda)^{-1} \\ \downarrow & & \downarrow \\ \pi(\lambda') = \omega\pi(\lambda)\omega & \longrightarrow & \pi(\lambda')^{-1} = \omega\pi(\lambda)^{-1}\omega \end{array} \quad . \tag{5.5}$$

We will show now that for the reduced sequences $a^{(0)}(\pi(\lambda)) := \Phi^L(L(\pi)(\lambda))$ and $a^{(1)}(\pi(\lambda)) := \Psi^{-1}(\zeta_1(\pi(\lambda)))$ this means:

$$\begin{array}{ccc} a^{(0)}(\pi(\lambda)) & \xrightarrow{(a)} & a^{(0)}(\pi(\lambda)^{-1}) = [a^{(1)}(\pi(\lambda))]^* \\ {\scriptstyle (b)}\downarrow & & \downarrow \\ a^{(0)}(\pi(\lambda')) = [a^{(1)}(\pi(\lambda))]' & \longrightarrow & a^{(0)}(\pi(\lambda')^{-1}) = [a^{(0)}(\pi(\lambda))']^* \end{array} \quad , \tag{5.6}$$

which by definition of the partial order on $GR(\pi)$ immediately implies the remaining assertions.

Since conjugation of reduced sequences changes the signs of the $k_\nu := a^*_{\nu+1} - a^*_\nu$ the μ-rule for Ψ implies

$$(5.7) \qquad \forall a \in R(\pi(\lambda)) : \quad \Psi(a') = [\Psi(a)]' \ .$$

Hence one computes

$$\Psi[a^{(1)}(\pi(\lambda))] = \zeta_1(\pi(\lambda)) = [\zeta_0(\pi(\lambda'))]' = [\Psi(a_{(0)}(\pi(\lambda')))]' = \Psi[a^{(0)}(\pi(\lambda'))']$$

implying (b).

In the proof of (a) we use for the equalities (α) and (β) below the results [W2, Prop.2.4] and [W2, Prop.3.1], respectively, where (α) is true for all π and (β) only in the Grassmannian case $\pi \equiv \pi(\lambda)$.

$$\begin{aligned}
&[a^{(1)}(\pi(\lambda))]^* \overset{(b)}{=} ([a^{(0)}(\pi(\lambda'))]')^* = ([\Phi^L(L(\pi(\lambda')))]')^* = \\
&\quad ([\Phi^L(K(\pi(\lambda)))]')^* \overset{(\alpha)}{=} [\Phi^K(K(\pi(\lambda)))]^* \overset{(\beta)}{=} [\Phi^H(H(\pi(\lambda)))]^* = \\
&\quad [\Phi^H(H(\omega\pi(\lambda')\omega))]^* = [\Phi^H(K(\pi(\lambda')^{-1}))]^* \overset{(\alpha)}{=} [\Phi^L(K(\pi(\lambda')^{-1}))] = \\
&\qquad\qquad [\Phi^L(L(\pi(\lambda)^{-1}))] = [a^{(0)}(\pi(\lambda)^{-1})] \quad .
\end{aligned}$$

Having shown (a) and (b) it is easy to calculate

$$a^{(0)}(\pi(\lambda')^{-1}) \overset{(\alpha)}{=} a^{(1)}(\pi(\lambda'))^* \overset{(\beta)}{=} [a^{(0)}(\pi(\lambda))']^* \quad ,$$

which finishes the proof. □

Remark 5.2. It is interesting to observe that a similar 'Fourfold way' exists for the Robinson-Schenstedt-Knuth correspondence between permutations and standard bitableau: $\pi \overset{RSK}{\mapsto} (P,Q)$. As shown first by Schützenberger, if $\pi \overset{RSK}{\mapsto} (P,Q)$, then $\pi^{-1} \overset{RSK}{\mapsto} (Q,P)$ and $\pi^r := \pi\omega \overset{RSK}{\mapsto} (P', ev(Q'))$, where '$ev$' stands for the operation of evacuation (cf. [S, Thm.3.8.6, Thm.3.4.3]). Since $\omega\pi\omega = (((\pi^{-1})^r)^{-1})^r$ it is easily seen that $\omega\pi\omega \overset{RSK}{\mapsto} ((ev(P'))', ev(Q))$. In this paper we have not discussed the operation $\pi \mapsto \pi^r$, because $l(\pi^r) = l(\omega) - l(\pi)$.

We finally formulate a conjecture, which is the analog of the Main Theorem for signed permutations or Coxeter groups of type B.

Let $\lambda^\nabla \equiv \lambda_1 \ldots \lambda_s$ be a partition with distinct parts $\lambda_1 > \cdots > \lambda_s > 0$ and $SST(\lambda^\nabla)$ the poset of standard shifted tableaux of shape λ^∇ as discussed in Sec.1 . As in the case of ordinary permutations we have introduced in [W2] codes $\bar{G}$ and a mapping $\bar{\Phi} \equiv \bar{\Phi}^G$ with the property that $\bar{\Phi}\bar{G}(\bar{\pi}) \in R(\bar{\pi})$. permutations $\bar{\pi}$. To be more specific:

To the above λ^∇ is associated the code $\overline{\lambda_1 \ldots \lambda_s\, 0 \ldots 0}$ with exactly $\gamma := n - s$ zeros, where n is the smallest number, such that $\lambda_1 \le 2n-1, \lambda_2 \le 2n-3, \ldots$; its "signed Grassmannian permutation" $\bar{\pi}(\lambda^\nabla) := \bar{G}^{-1}(\overline{\lambda_1 \ldots \lambda_s\, 0 \ldots 0})$;

and a certain reduced sequence

$$a^{(0)}(\bar{\pi}(\lambda^{\triangledown})) := \bar{\Phi}\bar{G}(\bar{\pi}(\lambda^{\triangledown})) := \bar{\Phi}(\lambda_s)\ldots\ \bar{\Phi}(\lambda_1)$$

with

$$\bar{\Phi}(\lambda_\nu) := \begin{cases} (n-\nu)\ldots\ (n-\nu+1-\lambda_\nu) & \text{if } 0 < \lambda_\nu \le n-\nu, \\ (n-\nu)\ldots\ 1\ 0\ 1\ \ldots\ (\lambda_\nu - n + \nu - 1) & \text{if } n-\nu < \lambda_\nu. \end{cases}$$

For example $\lambda^{\triangledown} = 651$ gives $s = 3$, $n = 4$, $\gamma = 1$, the code $\overline{6510}$, and $a^{(0)}(\lambda^{\triangledown}) = 1|21012|321012 \in R(\bar{\pi}(\lambda^{\triangledown}))$, where we have included vertical bars for clarity.

Of course one uses $a^{(0)}(\bar{\pi}(\lambda^{\triangledown}))$ as the bottom element of the partially ordered set $PR(\bar{\pi}(\lambda^{\triangledown}))$. For example let $\lambda^{\triangledown} = 42$; one gets $s = 2$, $n = 3$, $\gamma = 1$, $a^{(0)}(\lambda^{\triangledown}) = 10|2101$, and the poset $PR(\bar{\pi}(\lambda^{\triangledown}))$:

$$\begin{array}{c} 210210 \\ | \\ 212010 \\ | \\ 121010 \\ | \\ 120101 \\ | \\ 102101 \end{array}$$

Conjecture: For every partition with distinct parts $\lambda^{\triangledown}$ one has:

$$f^{\lambda}_{\triangledown} = r(\bar{\pi}(\lambda^{\triangledown}))\ , \tag{5.8}$$

where $r(\bar{\pi}(\lambda^{\triangledown})) := |R(\bar{\pi}(\lambda^{\triangledown}))|$ and $f^{\lambda}_{\triangledown} := |SST(\lambda^{\triangledown})|$. (The latter quantity can be computed by the 'shifted hook formula' ([S, Ex.3.12.4])). □

Unfortunately a simple bijection $\overline{\Psi}$ analogous to the case of ordinary permutations does not exist, because (1) the posets $PR(\bar{\pi}(\lambda^{\triangledown}))$ and $PSST(\lambda^{\triangledown})$ are in general not isomorphic and (2) the natural candidate for such a bijection is not "stable under inclusion". What we mean by this is easily explained for the concrete example above in connection with Ex.1.8: for $\lambda^{\triangledown} = 31$ the set of reduced words is $\{0101, 1010\}$ and there are exactly two standard shifted tableaux of this shape, both occurring twice as subtableau in $SST(42)$; but the words 0101 and 1010 occur both only once as initial segments of the $a^* \in R(\bar{\pi}(42))$.

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Institut für Reine und Angewandte Mathematik, RWTH Aachen, D-52056 Aachen, Germany, *papers/preprints*: http://www.iram.rwth-aachen.de/ ~winkel/

E-mail address: `winkel@iram.rwth-aachen.de`