SOME CLASSICAL EXPANSIONS FOR KNOP-SAHI AND MACDONALD POLYNOMIALS

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ABSTRACT: In recent simultaneous work, Knop and Sahi introduced a non-homogeneous non-symmetric polynomial $G_\alpha(x;q,t)$ whose highest homogeneous component gives the non-symmetric Macdonald polynomial $E_\alpha(x;q,t)$. Macdonald shows that for any composition α that rearranges to a partition λ, an appropriate Hecke algebra symmetrization of E_α yields the Macdonald polynomial $P_\lambda(x;q,t)$. In the original papers all these polynomials are only shown to exist. No explicit expressions are given relating them to the more classical bases. Our basic discovery here is that $G_\alpha(x;q,t)$ appears to have surprisingly elegant expansions in terms of the polynomials $Z_\alpha(x_1,\ldots,x_n;q) = \prod_{i=1}^n (x_i;q)_{\alpha_i}$. In this paper we present the first results obtained in the problem of determining the connection coefficients relating these bases. In particular we give a solution to the problem of two variables. Our proofs rely on the theory of basic hypergeometric series and reveal a deep connection between this classical subject and the theory of Macdonald polynomials.

Introduction

The Macdonald basis $\{P_\lambda(x;q,t)\}_\lambda$ has recently become an intensive subject of study as a result of the many difficult conjectures that surround it. Its importance in the development of symmetric function theory is now widely recognized. In addition to specializing to several fundamental bases, (such as the Schur, the Hall-Littlewood, the Zonal, the Jack) its has been conjectured to occur in a natural way [1] in representation theory and in some problems of particle mechanics [8]. One of the difficulties encountered in its study is the absence of explicit formulas expressing $P_\lambda(x;q,t)$ in terms of more familiar bases. In fact, the connection coefficients relating a rescaled version of $\{P_\lambda(x;q,t)\}_\lambda$ to the modified Schur basis $\{S_\lambda[X(1-t);q,t]\}_\lambda$ have only recently been shown to be polynomial functions of q,t ([2],[4],[5],[7]). Macdonald in [12] shows that each $P_\lambda(x;q,t)$ decomposes into a sum of homogeneous non-symmetric polynomials $E_\alpha(x;q,t)$ indexed by compositions. More precisely, if α is any composition that rearranges to λ then

$$P_\lambda \;=\; \sum_{\sigma\in S_n} t^{-length(\sigma)}\, T_\sigma\, E_\alpha$$

where T_σ is an appropriately defined Hecke algebra operator. Since the E_α are triangularly related to the monomial basis $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$, they form themselves a basis for the polynomials in $x_1,x_2,\ldots,x_n$. In view of the fundamental nature of the polynomials $P_\lambda(x;q,t)$ and their central place in symmetric function theory, it is reasonable to assume that the E_α should also play a central role in the study of polynomials in several variables. It is shown in [12] that in fact the E_α themselves are but a special case of families of orthogonal polynomials associated to root systems; the E_α being associated the root system A_n. As for the P_λ, the E_α have only been shown to exist. They have also been characterized as eigenfunctions of certain operators and may be computed explicitly only through algorithms derived from the recursions they satisfy.

A breakthrough in the study of Macdonald polynomials is the simultaneous discovery by Knop [5] and Sahi [14] of two closely related families $\{R_\lambda(x;q,t)\}_\lambda$ and $\{G_\alpha(x;q,t)\}_\alpha$ respectively indexed by partitions and compositions, whose highest homogeneous components yield $\{P_\lambda(x;q,t)\}_\lambda$ and $\{E_\alpha(x;q,t)\}_\alpha$ respectively. Knop and Sahi show that R_λ may be also be obtained as the Hecke algebra symmetrization of G_α for any composition α that rearranges to λ. What is remarkable about these new polynomials is that they may be chararacterized by very elementary vanishing properties which not only yield simple algorithms for their construction but allow a quick and simple derivation of several heretofore difficult and apparently deep properties of the Macdonald polynomials. The work of Knop and Sahi brings to evidence that the G_α may be used as natural building blocks in the construction of Macdonald polynomials, yet in a very concrete sense, the $G_\lambda(x;q,t)$ are considerably easier to study than E_α and P_λ. This given we have set ourselves the task of finding some classical basis in terms of which the $G_\lambda(x;q,t)$ may be given explicit, closed form expansions. Our discovery is that a most natural candidate to this effect appears to be the basis

$$Z_\alpha(x_1,x_2,\ldots,x_n;q)=\prod_{i=1}^{n}(1-x_i)(1-qx_i)\cdots(1-q^{\alpha_i-1}x_i)\ . \tag{I.1}$$

In fact, as we shall see, the various properties of the polynomials $G_\alpha(x;q,t)$ established by Knop and Sahi, when expressed in terms of the connection coefficients relating the bases $G_\alpha(x;q,t)$ and $\{Z_\alpha(x_1,x_2,\ldots,x_n;q)\}_\alpha$, encode some of the less elementary identities [3] of the theory of basic hypergeometric series. To see how all this comes about and to state our results we need to review the definitions and some of the characterizing properties of the Knop-Sahi polynomials.

We recall that by a composition we mean a vector $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ with non negative integral components. For convenience we set

$$|\alpha|\ =\ \alpha_1+\alpha_2+\cdots+\alpha_n\ .$$

We shall sometime express that $|\alpha|=m$ by saying that α *is a composition of* m.

We shall also denote by α^* the partition obtained by rearranging the components (parts) of α in weakly decreasing order. If α has distinct parts then each α_i occupies a well defined position $k_i=k_i(\alpha)$ in α^*. By this we mean that $\alpha_i=\alpha^*_{k_i}$. We can extend $k_i(\alpha)$ to the case when α has equal parts, breaking ties by considering equal parts as decreasing from left to right. In other words if we label the parts of α from by decreasing size and from left to right then $k_i(\alpha)$ gives the label of α_i. Here and after we refer to $k_i(\alpha)$ as the *position* of α_i in α^*. Since Knop and Sahi use slightly different notations we will not be able to make our notation consistent to both papers. We shall try as much as possible to adhere to Sahi's notation here. This means that we will have to translate to Sahi notation some of those ingredients and results that are in Knop's and not in Sahi's. This given, we recall that Sahi associates to each composition α a vector of monomials $\overline{\alpha}$ by setting

$$\overline{\alpha}_i\ =\ q^{-\alpha_i}t^{-n+k_i(\alpha)} \tag{I.2}$$

In [14] Sahi shows that if α is a composition of m then in the linear span of the monomials $\{x^\beta\}_{|\beta|\leq m}$ there exists a unique polynomial $G_\alpha(x;q,t)$ which satisfies the following two conditions

$$\begin{aligned} a) \quad G_\alpha(\overline{\beta};q,t) &= 0 \qquad \text{for all } |\beta| \leq |\alpha| \text{ and } \beta \neq \alpha \,, \\ b) \quad G_\alpha(\overline{\alpha};q,t) &= 1 \,. \end{aligned} \tag{I.3}$$

Moreover it is shown that the coefficient of x^α in G_α doesnot vanish. We shall express this by writing

$$c) \quad G_\alpha(x;q,t)\,|_{x^\alpha} \neq 0 \tag{I.4}$$

The uniqueness part of the Knop-Sahi result is relatively easy to show, yet uniqueness permits the immediate derivation of a number of surprising identities and recursions. Some of these are given by Sahi in [14] and others only in Knop [5]. We shall state them here in our present notation. For the sake of completeness, we shall give some of the proofs as we need them in later sections. To begin with we have the following immediate recursion.

Property I.1 *If $r = min\{\alpha_i \,:\, i = 1..n\,\} > 0$ then setting $\gamma_i = \alpha_i - r$ we have*

$$G_\alpha(x;q,t) \doteq \prod_{i=1}^{n} (x_i;q)_r \;\; G_\gamma(q^r x;q,t) \tag{I.5}$$

where the symbol "$\doteq$" is to represent equality up to a scalar factor.

Property I.2 *If $\alpha_n > 0$ then*

$$G_\alpha(x;q,t) \;\; \doteq \;\; (1-x_n)\, G_{(\alpha_n-1,\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}(qx_n, x_1, x_2, \ldots, x_{n-1};q,t) \;\; . \tag{I.6}$$

For $1 \leq i \leq n-1$ let $s_i = (i, i+1)$ denote the transposition that interchanges x_i and x_{i+1} and set [9]

$$T_{s_i} \;=\; s_i \;+\; \frac{(1-t)}{x_i - x_{i+1}}\, x_i\,(1-s_i) \;\; . \tag{I.7}$$

It is well known that the operators T_{s_i} generate a faithful representation of the Hecke algebra of S_n in the space of polynomials in $x_1, x_2, \ldots, x_n$. Indeed, it is easily verified (nowadays using symbolic manipulation software) that we have

$$\begin{aligned} a) \quad T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} \qquad\qquad \text{- for} \quad |i-j| > 1 \;\;, \\ b) \quad T_{s_i}T_{s_{i+1}}T_{s_i} &= T_{s_{i+1}}T_{s_i}T_{s_{i+1}} \qquad \text{for} \quad i = 1\ldots n-1, \\ c) \quad t\,T_{s_i}^{-1} &= s_i \;+\; \frac{(1-t)}{x_i - x_{i+1}}\,(1-s_i)\,x_{i+1} \;\; . \end{aligned} \tag{I.8}$$

This permits us to extend the definition of T to all permutations $\sigma \in S_n$ by setting for any reduced expression $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$

$$T_\sigma \;=\; T_{s_{i_1}}T_{s_{i_2}}\cdots T_{s_{i_k}} \;\; . \tag{I.9}$$

Property I.3

$$\begin{cases} (a) \quad G_\alpha(x;q,t) \;=\; T_{s_i}\, G_\alpha(x;q,t) \;=\; s_i\, G_\alpha(x;q,t) & \text{if } \alpha_i = \alpha_{i+1} \\ (b) \quad G_{s_i\alpha}(x;q,t) \doteq (1 - \frac{\overline{\alpha}_{i+1}}{\overline{\alpha}_i})\, T_{s_i} G_\alpha(x;q,t) + (t-1)\, G_\alpha(x;q,t) & \text{if } \alpha_i \neq \alpha_{i+1} \;. \end{cases}$$

It should be apparent that these three properties may be combined into a recursive algorithm for computing the polynomials G_α starting from the initial condition

$$G_{(o,o,\ldots,o)}(x;q,t) \;=\; 1 \;. \qquad \text{I.10}$$

This permits the rapid computation of extensive tables. Now the particular nature of the right hand side of I.5 suggested that the basis defined in I.1 may turn out to be a natural tool for the study of these polynomials. Our preliminary results confirm this possibility. The basic identity in the case of two variables may be stated as follows:

Theorem I.1 *Setting $x_1 = x$ and $x_2 = y$ we have for $m \geq 1$:*

$$G_{(m,0)}(x,y;q,t) = \frac{(-1)^m q^{\binom{m+1}{2}}}{(t;q)_{m+1}} \sum_{0\leq j+k\leq m} \frac{t^{k+j} q^{j(k+1)} (t;q)_{m-k}\, (t;q)_{m+1-j}\, (x;q)_k\, (y;q)_j}{(q;q)_k\, (q;q)_j\, (q;q)_{m-k-j}} \qquad \text{I.11}$$

Note that Property I.1, in the two variable case, may be rewritten as

$$G_{(a,b)}(x,y;q,t) \;=\; \begin{cases} (x;q)_a (y;q)_a\, G_{(0,b-a)}(q^a x, q^a y; q,t) & \text{if } a \leq b \\ (x;q)_b (y;q)_b\, G_{(a-b,0)}(q^b x, q^b y; q,t) & \text{if } b < a \end{cases} \qquad \text{I.12}$$

In the same notation, Property I.2 applied to the composition $\alpha = (0,b)$ gives:

$$G_{(0,b)}(x,y;q,t) \;=\; (1-y) G_{(b-1,0)}(qy,x) \quad \text{for} \quad b > 0 \qquad \text{I.13}$$

Thus we see that the appropriate combination of I.11, I.12 and I.13 yields entirely explicit formulas for the coefficients connecting the bases $\{G_{(a,b)}(x,y;q,t)\}_{a,b\geq 0}$ and $\{(x;q)_a (y;q)_b\}_{a,b\geq 0}$.

Finally, let us denote by ϕ the linear operator that sends a polynomial $Q = Q(x_1, x_2, \ldots, x_n)$ into the polynomial

$$\phi Q \;=\; Q(qx_n, x_1, x_2, \ldots, x_{n-1}) \;. \qquad \text{I.14}$$

This given, following Knop we set for $1 \leq i \leq n-1$

$$\Xi_i \;=\; \frac{1}{x_i} \;+\; \frac{1}{x_i}\, T_{s_i} T_{s_{i+1}} \cdots T_{s_{n-1}}\, (x_n - 1)\, \phi\, T_{s_1} T_{s_2} \cdots T_{s_{i-1}} \qquad \text{I.15}$$

Translating theorem 3.6 of [6] to the present notation we can state that

Property I.4

$$\Xi_i\, G_\alpha \;=\; \frac{1}{\overline{\alpha}_i}\, G_\alpha \qquad \text{for } i = 1, \ldots, n-1 \qquad \text{I.16}$$

This remarkable result not only shows that the operators Ξ_i form a commuting family, but also constitute an independent characterization of the Knop-Sahi polynomials.

This paper is divided into three sections. In the first section we rederive the identities I.5, I.6 , I.10 and I.16. We also include a simple proof of the existence and uniqueness part of the Knop-Sahi result. In the second section, we use the vanishing property of $G_{(m,o)}$ to give a short proof of Theorem I.1. Our formula can be used to give explicit expansions for the two variable case of the Macdonald polynomials E_α and P_λ. In section two we show that formula 4.9 [11], which gives $P_{(m)}$, can be derived from Theorem I.1.

Section three includes the exploration of an alternate path for proving Theorem I.1 based on the characterization of the G_α as eigenfunctions of the operators Ξ_i. In following this path we were able to determine explicit expressions for the entries of the matrix expressing the action of Ξ_1 on the basis $\{(x;q)_a(y;q)_b\}_{a,b\geq 0}$.

It is remarkable that consistency of these expressions, Theorem I.1 and Property I.4, as well as the computations carried out in section two, rest on some of the deeper identities in the theory of basic hypergeometric series. In particular, a crucial role is played by the summation formula for the well-poised ${}_6\Phi_5$.

1. The basic identities

Since we are using a slightly different notation than in the original papers, for convenience of the reader we shall rederive here the results of Knop and Sahi expressed by Properties I.1-I.4. Here as in [14] and [5],[6] all these properties are derived from the vanishing conditions I.3 a) using the uniqueness portion of the Knop-Sahi existence theorem. In each case we show that "dot"-equality holds by showing that the right-hand side has the same vanishing properties as the the left-hand side. In order not to be unduly repetitious, in each of the following two proofs, α and β will be a generic pair of compositions satisfying

$$\beta \neq \alpha \qquad \text{and} \qquad |\beta| \leq |\alpha|$$

Proof of Property I.1 Set $r' = min\{\beta_i \,:\, 1 \leq i \leq n\}$. Assume first that $r' < r$ and let j be the rightmost index such that $\beta_j = r'$. Then $k_j(\beta) = n$ and the definition in I.2 gives that $\overline{\beta}_j = q^{-r'}$. Consequently

$$(\overline{\beta}_j;q)_r \;=\; 0$$

and this forces the vanishing of one of the factors in right-hand side of I.5.

Assume next that $r' \geq r$ and set $\delta = (\delta_1, \ldots, \delta_n)$ with $\delta_i = \beta_i - r$. Note that since the positions of β_i in β^* and δ_i in δ^* are the same we must have

$$\overline{\beta}_i = q^{-\beta_i} t^{-n+k_i(\beta)} \;=\; q^{-r} q^{-\delta_i} t^{-n+k_i(\delta)} \;=\; q^{-r}\,\overline{\delta}_i \;.$$

This given, the definition of G_γ and the fact that $\delta \neq \gamma$ if and only if $\beta \neq \alpha$ yield

$$G_\gamma(q^r\overline{\beta}) \;=\; 0 \;.$$

The proof is completed by noting that the right-hand side of I.5 does not vanish at $x = \overline{\alpha}$.

For convenience let $\mathbf{R}$ and $\mathbf{\Delta}_q$ be the two "affine" rotation operators defined by setting for any composition $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$

$$\mathbf{R}\gamma = (\gamma_n - 1, \gamma_1, \ldots, \gamma_{n-1}) \qquad \text{and} \qquad \mathbf{\Delta}_q\gamma = (q\,\gamma_n\ ,\ \gamma_1, \ldots, \gamma_{n-1})\ .$$

Note that if $\gamma_n > 0$ then $\mathbf{R}\gamma$ is also a composition. Moreover it is easy to see that the position of γ_n in γ^* is exactly the same as the position of $\gamma_n - 1$ in $(\mathbf{R}\gamma)^*$. In other words we have $k_n(\gamma) = k_1(\mathbf{R}\gamma)$ and thus

$$\overline{(\mathbf{R}\gamma)}_1 \ = \ q^{-\gamma_n+1}t^{-n+k_n(\gamma)} \ = \ \overline{\gamma}_n q\ .$$

Since we trivially have that $k_i(\mathbf{R}\gamma) = k_{i-1}(\gamma)$ for $i = 2, \ldots, n$ we immediately deduce that

$$\overline{\mathbf{R}\gamma} \ = \ \mathbf{\Delta}_q\overline{\gamma}\ . \tag{1.1}$$

Proof of Property I.2 If $\beta_n = 0$ then $\overline{\beta}_n = q^{-o}t^{-n+n} = 1$ this gives $(1 - \overline{\beta}_n) = 0$ which causes the vanishing of the first factor in the right-hand side of I.6.

On the other hand if $\beta_n > 0$ then 1.1 gives that

$$G_{\mathbf{R}\alpha}(\mathbf{\Delta}_q\overline{\beta}) \ = \ G_{\mathbf{R}\alpha}(\overline{\mathbf{R}\beta}) \ = \ 0\ ,$$

(since $\mathbf{R}\beta \neq \mathbf{R}\alpha$ if and only if $\beta \neq \alpha$). This causes the vanishing of the second factor in the right-hand side of I.6.

The proof is completed by noting that the right-hand side of I.6 does not vanish at $x = \overline{\alpha}$.

Proof of Property I.3 We start by proving formula b). So suppose that $\alpha_i \neq \alpha_{i+1}$ and set $m = \alpha_{i+1}/\alpha_i$. Substituting I.7 in the right-hand side of b) we derive that

$$\begin{aligned} RHS = \ &(1-m)\left\{s_iG_\alpha + (1-t)\frac{x_i}{x_i - x_{i+1}}(1-s_i)G_\alpha\right\} \ + \ (t-1)G_\alpha \ = \\ = \ &(1-m)\left\{1 - (1-t)\frac{x_i}{x_i - x_{i+1}}\right\}s_iG_\alpha \ + \ (1-t)\Big\{(1-m)\frac{x_i}{x_i - x_{i+1}} - 1\Big\}G_\alpha\ . \end{aligned}$$

Thus formula b) is equivalent to the identity

$$G_{s_i\alpha}(x;q,t) \ \doteq \ (1-m)\left\{\frac{tx_i - x_{i+1}}{x_i - x_{i+1}}\right\}s_iG_\alpha \ + \ (1-t)\Big\{\frac{x_{i+1} - mx_i}{x_i - x_{i+1}}\Big\}G_\alpha\ . \tag{1.2}$$

For convenience let R_1 and R_2 respectively denote the two summands on the right-hand side of 1.2. To prove Property I.3 we must test the vanishing of $R_1 + R_2$ for all β satisfying $|\beta| \leq |\alpha|$ and $\beta \neq s_i\alpha$. Remarkably, it develops that R_1 and R_2 individually vanish for all such β's! We shall work with each separately.

R_1 :

(1) If $\beta_i \neq \beta_{i+1}$ then the relations $k_i(\beta) = k_{i+1}(s_i\beta)$, $k_{i+1}(\beta) = k_i(s_i\beta)$ and $k_j(\beta) = k_j(s_i\beta)$ for $j \neq i, i+1$ give that $s_i\overline{\beta} = \overline{s_i\beta}$. Thus, since $s_i\beta \neq \alpha$ and $|s_i\beta| = |\beta| \leq |\alpha$ we get that

$$s_iG_\alpha(\overline{\beta}) \ = \ G_\alpha(s_i\overline{\beta}) \ = \ G_\alpha(\overline{s_i\beta}) \ = \ 0\ .$$

(2) If $\beta_i = \beta_{i+1}$ then $k_{i+1}(\beta) = k_i(\beta) + 1$ gives $\overline{\beta}_{i+1} = q^{\beta_i}t^{-n+k_i(\beta)+12} = t\overline{\beta}_i$ and in this case it is the factor $tx_i - x_{i+1}$ that forces the vanishing of R_1.

R_2 :

(1) If $\beta \neq \alpha$ then the factor G_α itself vanishes for $x = \overline{\beta}$. Note that, since $\alpha_i \neq \alpha_{i+1}$ forces $s_i\alpha \neq \alpha$, this factor will vanish even for $\beta = s_i\alpha$.

(2) If $\beta = \alpha$ then the vanishing of R_2 is simply due to the factor $x_{i+1} - mx_i$ which for $x = \overline{\alpha}$ reduces to $\overline{\alpha}_i - \frac{\overline{\alpha}_{i+1}}{\overline{\alpha}_i}\overline{\alpha}_i$.

To complete the proof of (b) we need only check that $R_1 + R_2$ does not vanish for $x = s_i\alpha$. However, since we noted that R_2 vanishes there, we need only show that R_1 doesn't vanish. But this is immediate since $(1-m)(t\overline{\alpha}_{i+1} - \overline{\alpha}_i) \neq 0$ and $s_iG_\alpha(s_i\overline{\alpha}) = G_\alpha(\overline{\alpha}) = 1$.

To prove (a) we start by noting that for any polynomial Q we have

$$\begin{aligned} T_{s_i}Q \;&=\; s_iQ \;+\; \frac{(1-t)\,x_i}{x_i - x_{i+1}}\,(Q - s_iQ) \;=\; \big(1 - \frac{(1-t)\,x_i}{x_i - x_{i+1}}\big)\,s_iQ \;+\; \frac{(1-t)\,x_i}{x_i - x_{i+1}}\,Q \\ &=\; \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}\,Q(s_ix) \;+\; (1-t)\frac{x_i}{x_i - x_{i+1}}\,Q(x) \end{aligned} \qquad 1.3$$

Using the second form of $T_{s_i}Q$ we immediately see that $T_{s_i}Q = Q$ implies $s_iQ = Q$. Thus we need only show the first of the equalities in (a). To this end note that for $\beta \neq \alpha$ and $|\beta| \leq |\alpha|$ the third equality in 1.3 for $Q = G_\alpha$ and $x = \overline{\beta}$ yields that

$$(T_siG_\alpha)(\overline{\beta}) \;=\; \frac{t\overline{\beta}_i - \overline{\beta}_{i+1}}{\overline{\beta}_i - \overline{\beta}_{i+1}}\,G_\alpha(s_i\overline{\beta}) \qquad 1.4$$

now if $\beta_i \neq \beta_{i+1}$ then $s_i\overline{\beta} = \overline{s_i\beta}$ and $G_\alpha(s_i\overline{\beta})$ must vanish since $\alpha_i = \alpha_{i+1}$ gives $s_i\beta \neq \alpha$. On the other hand if $\beta_i = \beta_{i+1}$ then $\overline{\beta}_{i+1} = t\overline{\beta}_i$ (as we have seen). This forces the vanishing of the first factor in 1.3. Thus T_siG_α has the same vanishing properties as G_α. This given, we only need to compute its value at $x = \overline{\alpha}$. But here, we have $t\overline{\alpha}_i = \overline{\alpha}_{i+1}$ and $G_\alpha(\overline{\alpha}) = 1$ which combined with the third equality in 1.3 yields that

$$(T_{s_i}G_\alpha)(\overline{\alpha}) \;=\; \frac{t\overline{\alpha}_i - \overline{\alpha}_{i+1}}{\overline{\alpha}_i - \overline{\alpha}_{i+1}}\,Q(s_i\overline{\alpha}) \;+\; (1-t)\,\frac{\overline{\alpha}_i}{\overline{\alpha}_i - \overline{\alpha}_{i+1}}\,Q(\overline{\alpha}) \;=\; 1$$

This completes the proof of Property I.3.

Before we can proceed to the proof of Property I.4 we need some preliminary observations. We shall begin by rederiving Knop's beautiful result that all the operators Ξ_i send polynomials into polynomials. To this end, we note that T_{s_i} and $T_{s_i}^{-1}$ may be given the alternate forms:

$$\begin{aligned} &a) \quad T_{s_i} \;=\; t\,s_i \;+\; \frac{(1-t)}{x_i - x_{i+1}}(1 - s_i)\,x_i \;, \\ &b) \quad T_{s_i}^{-1} \;=\; s_i \;+\; \frac{(1-t)}{t}\,\frac{x_{i+1}}{x_i - x_{i+1}}(1 - s_i) \;. \end{aligned} \qquad 1.5$$

This given, using 1.5 b) and then 1.5 a) we derive that for any polynomial Q we have

$$\begin{aligned} \frac{t}{x_{i+1}}T_{s_i}^{-1}Q \;&=\; t\,s_i\,\frac{1}{x_i}Q \;+\; \frac{(1-t)}{x_i - x_{i+1}}(1 - s_i)\,Q \\ &=\; \Big(t\,s_i \;+\; \frac{(1-t)}{x_i - x_{i+1}}(1 - s_i)x_i\Big)\frac{1}{x_i}Q \\ &=\; T_{s_i}\frac{1}{x_i}\,Q \;. \end{aligned}$$

Recalling the definition of in I.15 and using this relation we obtain for $i < n$:

$$
\begin{aligned}
t\,\Xi_{i+1}T_{s_i}^{-1}\,Q \;&=\; \frac{t}{x_{i+1}}T_{s_i}^{-1}\,Q \;+\; \frac{t}{x_{i+1}}T_{s_i}^{-1}T_{s_i}T_{s_{i+1}}\cdots T_{s_{n-1}}(x_n-1)\phi T_{s_1}\cdots T_{s_i}T_{s_i}^{-1}Q \\
&=\; T_{s_i}\,\frac{Q}{x_i} \;+\; T_{s_i}\frac{1}{x_i}T_{s_i}\cdots T_{s_{n-1}}(x_n-1)\phi T_{s_1}\cdots T_{s_{i-1}}\,Q \\
&=\; T_{s_i}\Xi_i\,Q
\end{aligned}
$$

Equivalently, for any polynomial Q we have

$$
\Xi_i Q \;=\; t\;T_{s_i}^{-1}\,\Xi_{i+1}\,T_{s_i}^{-1}\,Q\;. \tag{1.6}
$$

Iterating this relation we finally obtain that

$$
\Xi_i Q \;=\; t^{n-i}\;T_{s_i}^{-1}\cdots T_{s_{n-1}}^{-1}\Xi_n\,T_{s_{n-1}}^{-1}\cdots T_{s_i}^{-1} \tag{1.7}
$$

This shows that if Ξ_n sends polynomials into polynomials the same will be true for all the other Ξ_i. To prove the result for Ξ_n we follow Knop and write

$$
\Xi_n \;=\; \phi\,T_{s_1}\cdots T_{s_{n-1}} \;+\; \frac{1}{x_n}\big(I-\phi\,T_{s_1}\cdots T_{s_{n-1}}\big)\;.
$$

Now the first term in this decomposition is clearly a polynomial operator. It develops that the same is true for the second term for the simple reason that it is a sum of divided difference operators. To see this we note that we may write

$$
\phi \;=\; \tau_n\,s_{n-1}\cdots s_2 s_1
$$

where τ_n is the operator which replaces x_n by qx_n . This gives that

$$
\begin{aligned}
Q-\phi\,T_{s_1}\cdots T_{s_{n-1}}Q \;&=\; Q-\phi\,s_1\cdots s_{n-1}Q \;+\; \phi\,(s_1\cdots s_{n-1} \;-\; T_{s_1}\cdots T_{s_{n-1}})\,Q \\
&=\; Q-\tau_n Q \;+\; \sum_{i=1}^{n-1}\phi\,s_1\cdots s_{i-1}\,\big(s_i-T_{s_i}\big)\,T_{s_{i+1}}\cdots T_{s_{n-1}}\,Q \\
&=\; Q-\tau_n Q \;+\; (t-1)\sum_{i=1}^{n-1}\tau_n\,s_{n-1}\cdots s_i\,\frac{x_i}{x_i-x_{i+1}}(1-s_i)\,T_{s_{i+1}}\cdots T_{s_{n-1}}\,Q \\
&=\; Q-\tau_n Q \;+\; (t-1)\sum_{i=1}^{n-1}\tau_n\,\frac{x_n}{x_n-x_i}\big(1-(i,n)\big)\,s_{n-1}\cdots s_i\,T_{s_{i+1}}\cdots T_{s_{n-1}}\,Q\;\;,
\end{aligned}
$$

where (i,n) denotes the transposition which interchanges x_i with x_n.

In summary we have

$$
\Xi_n \;=\; \phi\,T_{s_1}\cdots T_{s_{n-1}} \;+\; \Delta_n \;+\; q\,(t-1)\sum_{i=1}^{n-1}\tau_n\,D_{(i,n)}\,s_{n-1}\cdots s_i\,T_{s_{i+1}}\cdots T_{s_{n-1}}\;\;,
$$

where Δ_n denotes the n^{th} q-derivative operator and $D_{i,j}$ is the divided difference operator acting on the pair (x_i, x_n) .

Our next observations reveal a remarkable property of the operator $(x_n - 1)\phi$. To this end note that the inverse of the operator $\mathbf{R}$ defined in the introduction is obtained by setting

$$\mathbf{R}^{-1}\,(\gamma_1, \gamma_2, \ldots, \gamma_n) \;=\; (\gamma_2, \ldots, \gamma_n, \gamma_1 + 1)\ .$$

With this notation, we may rewrite Property I.2 by stating that

$$(x_n - 1)\,\phi\, G_\gamma(x;q,t) \;\doteq\; G_{\mathbf{R}^{-1}\gamma}(x;q,t) \qquad 1.8$$

Now let $\mathcal{G}_m$ denote the linear span of the collection of polynomials $\{\, G_\gamma \,\}_{|\gamma|=m}$. Since the collection $\{G_\alpha\}_\alpha$ is a polynomial basis, we may view 1.8 as defining a linear operator $\mathcal{R}^{-1}$ which sends $\mathcal{G}_m$ into the subspace $\mathcal{R}^{-1}\mathcal{G}_m$ of $\mathcal{G}_{m+1}$ spanned by the collection $\{G_{\mathbf{R}^{-1}\gamma}\}_{|\gamma|=m}$. Keeping all this in mind we are in a position to give

The Proof of Property I.4 Let α be a given composition. The definition in I.15 may now be written as

$$\Xi_i\, G_\alpha \;=\; \frac{1}{x_i} G_\alpha \;+\; \frac{1}{x_i}\, T_{s_i} \cdots T_{s_{n-1}} \mathcal{R}^{-1}\, T_{s_1} \cdots T_{s_{i-1}}\, G_\alpha\ . \qquad 1.9$$

Since Property I.3 implies in particular that each T_{s_i} leaves $\mathcal{G}_m$ invariant we see that the polynomial

$$T_{s_i} \cdots T_{s_{n-1}} \mathcal{R}^{-1}\, T_{s_1} \cdots T_{s_{i-1}}\, G_\alpha$$

will lie in the space $\mathcal{G}_{|\alpha|+1}$. In particular it follows that the second term in 1.9 will necessarily vanish for all $|\beta| \leq |\alpha|$. This immediately gives that the right hand side of 1.9 vanishes for all

$$|\beta| \leq |\alpha|\ ,\ \ \beta \neq \alpha\ .$$

Moreover, evaluating both sides of 1.9 at $\overline{\alpha}$ gives

$$\Xi_i\, G_\alpha(\overline{\alpha};q,t) \;=\; \frac{1}{\overline{\alpha}_i}\ .$$

This establishes I.16 and completes our proof.

Our treatment here would be completely self contained where it not for the fact that we have made repeated use of the uniqueness part of the Knop-Sahi existence theorem. For sake of completeness, we shall terminate this section with a simple proof of this result. To this end, for a given integers $n, m > 0$ let us denote by $\mathcal{B}_m(n)$ the collection of all n-component compositions of a number $\leq m$. In symbols

$$\mathcal{B}_m(n) \;=\; \{\ \alpha\ :\ |\alpha| \leq m\ \}\ .$$

For n and m being fixed, let

$$\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)}, \qquad 1.10$$

denote the elements of $\mathcal{B}_m(n)$ in some fixed total order. This given, the existence of the Sahi polynomial G_α for $|\alpha| = m$ depends on being able to construct coefficients $c_j(q,t)$ such that

$$G_\alpha(\overline{\alpha}^{(i)}) \;=\; \sum_{j=1}^{N} c_j(q,t)\,\left[\overline{\alpha}^{(i)}\right]^{\alpha^{(j)}} \;=\; \begin{cases} 0 & \text{if } \alpha^{(i)} \neq \alpha \ , \\ 1 & \text{if } \alpha^{(i)} = \alpha \ . \end{cases}$$

We can thus see that existence and uniqueness is assured at once for all compositions of m by proving that the matrix

$$\| \left[\overline{\alpha}^{(i)}\right]^{\alpha^{(j)}} \|_{i,j=1..N} \qquad 1.11$$

has non vanishing determinant. Now it develops that this is but a very special case of a result which may be stated as follows.

Proposition 1.1 *Let $\alpha^{(i)}, \delta^{(i)}$ for $i = 1..N$ be n-vectors with non-negative integral components. then the polynomial*

$$P(q,t) \;=\; det\, \|\, q^{(\alpha^{(i)},\alpha^{(j)})}\; t^{(\alpha^{(i)},\delta^{(j)})}\, \|_{i,j=1..N} \qquad 1.12$$

cannot vanish identically.

Proof We shall follow closely the argument used by Macdonald in the proof a similar result (see p. 334 of [11]). We note first that we may write

$$P(q,t) \;=\; \sum_{\sigma \in S_N} sign(\sigma)\;\; q^{\;\sum_{i=1}^N (\alpha^{(i)},\alpha^{(\sigma_i)})}\;\; t^{\;\sum_{i=1}^N (\alpha^{(i)},\delta^{(\sigma_i)})} \ .$$

Now, the simple inequality $ab \leq (a^2+b^2)/2$ valid for any two numbers $a, b \geq 0$ immediately implies that

$$\sum_{i=1}^N \left(\alpha^{(i)},\alpha^{(\sigma_i)}\right) \;\leq\; \frac{1}{2}\sum_{i=1}^N \left(\alpha^{(i)},\alpha^{(i)}\right) \;+\; \frac{1}{2}\sum_{i=1}^N \left(\alpha^{(\sigma_i)},\alpha^{(\sigma_i)}\right) \;=\; \sum_{i=1}^N \left(\alpha^{(i)},\alpha^{(i)}\right) \ . \qquad 1.13$$

However, since $ab = (a^2+b^2)/2$ only if $a = b$ we see that equality can hold true in 1.13 only if $\alpha^{(\sigma_i)} = \alpha^{(i)}$ for $i = 1, ..N$. This means that the term of highest q-degree in $P(q,t)$ can only come from the identity permutation. Since its coefficient is

$$t^{\;\sum_{i=1}^N (\alpha^{(i)},\delta^{(i)})} \not\equiv 0$$

the same must hold true for $P(q,t)$ itself.

To derive the non vanishing of the determinant of our matrix we only need to observe that for any n component compositions α and β we have

$$\overline{\beta}^{\alpha} \;=\; q^{\,-\sum_{i=1}^n \alpha_i\beta_i}\;\; t^{\,-\sum_{i=1}^n \alpha_i(n-k_i)} \;=\; q^{\,-(\alpha,\beta)}\;\; t^{\,-(\alpha,\delta(\beta))}$$

where we are letting $\delta(\beta)$ denote the vector with components $\delta_i(\beta) = n - k_i(\beta)$ (for $i = 1..n$). Thus our determinant is simply given by $P(1/q, 1/t)$ when the $\alpha^{(i)}$'s are as given in 1.10 and $\delta^{(i)} = \delta(\alpha^{(i)})$.

This completes our proof of existence and uniqueness of the Sahi polynomials.

Remark 1.1 We should note that the non vanishing of our determinant, also shows that a polynomial in $x_1, x_2, \ldots, x_n$ which is of degree $\leq m$ is completely determined by its values at the points $\alpha^{(1)}, \ldots, \alpha^{(N)}$. It is then not difficult to derive from this fact that the Knop-Sahi polynomials do form a basis for the polynomials in $x_1, x_2, \ldots, x_n$.

2. The two variable case and basic hypergeometric series.

The theory of hypergeometric series comes into play in our first proof of Theorem I.1.

Proof of Theorem I.1 For convenience let $K_m(x, y; q, t)$ denote the expression on the right-hand side of I.11. That is, let us set for a moment

$$K_m(x, y; q, t) = \frac{(-1)^m\, q^{\binom{m+1}{2}}}{(t;q)_{m+1}} \sum_{0 \leq j+k \leq m} \frac{t^{k+j} q^{j(k+1)} (t;q)_{m-k}\, (t;q)_{m+1-j}\, (x;q)_k\, (y;q)_j}{(q;q)_k\, (q;q)_j\, (q;q)_{m-k-j}} \quad . \tag{2.1}$$

We recall that $G_{(m,0)}(x, y; q, t)$ may be characterized as the unique polynomial of degree $\leq m$ which satisfies I.3 and I.4 for $\alpha = (m, 0)$. Since $K_m(x, y; q, t)$ is clearly a polynomial of degree $\leq m$, to show I.11 we need only verify that $K_m(x, y; q, t)$ satisfies these two conditions. We start by verifying I.3 a).

Transformations will be applied to 2.1 which reduce $K_m(x, y; q, t)$ to an expression that clearly vanishes for $(x, y) = \overline{\beta}$ when $\beta = (a, b)$ with $a < b$. Similar transformations will make obvious the vanishing when $a \geq b$.

Note that when $a < b$ we have

$$\overline{\beta} = (\frac{1}{q^a}, \frac{1}{q^b\, t})\ .$$

Thus it will first be shown that

$$K_m(1/q^a, 1/q^b t; q, t) \;=\; 0 \qquad \text{when} \quad a < b \quad \text{and} \quad a + b \leq m. \tag{2.2}$$

The property of q-shifted factorials [3]

$$(a;q)_{n-k} = \frac{(a;q)_n}{(q^{1-n}/a;q)_k} (\frac{-q}{a})^k\, q^{\binom{k}{2}-nk} \tag{2.3}$$

will be used to modify several terms in 2.1. Namely we have

$$\begin{aligned}
&a) \quad (t;q)_{m-k} = \frac{(t;q)_m}{(q^{1-m}/t;q)_k} (\frac{-q}{t})^k\, q^{\binom{k}{2}-mk} \\
&b) \quad (t;q)_{m+1-j} = \frac{(t;q)_{m+1}}{(q^{-m}/t;q)_j} (\frac{-q}{t})^j\, q^{\binom{j}{2}-(m+1)j} \\
&c) \quad (q;q)_{m-k-j} = \frac{(q;q)_{m-k}}{(q^{-m+k};q)_j}\, (-1)^j\, q^{\binom{j}{2}-(m-k)j} \\
&d) \quad (q;q)_{m-k} = \frac{(q;q)_m}{(q^{-m};q)_k}\, (-1)^k\, q^{\binom{k}{2}-mk}
\end{aligned} \tag{2.4}$$

Substituting a) b) and c) (combined with d)) in 2.1 transforms it to

$$K_m(x,y;q,t)=\frac{(-1)^m q^{\binom{m+1}{2}}(t;q)_m}{(q;q)_m}\sum_{k\geq 0}^{m}\frac{q^k\ (x;q)_k\ (q^{-m};q)_k}{(q;q)_k\ (q^{1-m}/t;q)_k}\sum_{j\geq 0}^{m-k}\frac{q^j(y;q)_j\ (q^{-m+k};q)_j}{(q;q)_j\ (q^{-m}/t;q)_j}\ . \qquad 2.5$$

The sum over j can be viewed as one going from 0 to ∞ by virtue of $(q^{-m+k};q)_j$ vanishing for all $j>m-k$. This allows the use of a ${}_2\phi_1$ summation identity, (II.7 in [3]),

$${}_2\phi_1(a,q^{-n};c;q,q)=\frac{(c/a;q)_n}{(c;q)_n}\ a^n \qquad 2.6$$

with $a=y$, $q^{-n}=q^{-(m-k)}$ and $c=q^{-m}/t$. The dependence on j is nicely eliminated and the double sum becomes

$$\frac{(-1)^m q^{\binom{m+1}{2}}(t;q)_m}{(q;q)_m}\sum_{k\geq 0}^{m}\frac{q^k\ (x;q)_k\ (q^{-m};q)_k}{(q;q)_k\ (q^{1-m}/t;q)_k}\frac{(q^{-m}/ty;q)_{m-k}\ y^{m-k}}{(q^{-m}/t;q)_{m-k}}\ .$$

Evaluation at $x=1/q^a$ and $y=1/q^bt$ yields

$$\frac{(-1)^m q^{\binom{m+1}{2}}(t;q)_m}{(q;q)_m}\sum_{k\geq 0}^{m}\frac{q^k\ (q^{-a};q)_k\ (q^{-m};q)_k}{(q;q)_k\ (q^{1-m}/t;q)_k}\frac{(q^{-m+b};q)_{m-k}\ (q^{-b}/t)^{m-k}}{(q^{-m}/t;q)_{m-k}} \qquad 2.7$$

Each term in this sum vanishes individually! This can be seen by assuming the contradiction. Suppose that the k^{th} term in the sum does not vanish. This implies that

$$i)\quad (q^{-a};q)_k\neq 0 \qquad \text{AND} \qquad ii)\quad (q^{-m+b};q)_{m-k}\neq 0\ .$$

However, $i)\Rightarrow a\geq k$ and $ii)\Rightarrow k>b$. But this contradicts our initial stipulation that $a<b$. Therefore 2.5 clearly vanishes as indicated, which implies 2.1 as desired.

Note that when $\beta=(a,b)$ with $a\geq b$ we have

$$\overline{\beta}\ =\ (\frac{1}{q^at},\frac{1}{q^b})$$

So we must show next that

$$K_m(\frac{1}{q^at},\frac{1}{q^b};q,t)\ =\ 0 \qquad \text{when} \quad m>a\geq b \quad \text{and} \quad a+b\leq m. \qquad 2.8$$

In this case we use $a)$, $b)$ of 2.4 and the combination of

$$d)\quad (q;q)_{m-j-k}=\frac{(q;q)_{m-j}}{(q^{-m+j};q)_k}\,(-1)^k\,q^{\binom{k}{2}-(m-j)k}\ ,$$

$$e)\quad (q;q)_{m-j}=\frac{(q;q)_m}{(q^{-m};q)_j}\,(-1)^j\,q^{\binom{j}{2}-mj}\ .$$

With these modifications formula 2.1 becomes

$$K_m(x,y;q,t) \;=\; \frac{(-1)^m\, q^{\binom{m+1}{2}}\,(t;q)_m}{(q;q)_m} \sum_{j\geq 0}^{m} \frac{q^j\,(y;q)_j\,(q^{-m};q)_j}{(q;q)_j\,(q^{-m}/t;q)_j} \sum_{k\geq 0}^{m-j} \frac{q^k\,(x;q)_k\,(q^{-m+j};q)_k}{(q;q)_k\,(q^{1-m}/t;q)_k}\ .$$

Letting $a=x,\ q^{-n}=q^{-(m-j)}$ and $c=q^{-m}/t$, 2.6 can again be applied to eliminate the dependence on k giving

$$K_m(x,y;q,t) \;=\; \frac{(-1)^m\, q^{\binom{m+1}{2}}\,(t;q)_m}{(q;q)_m} \sum_{j\geq 0}^{m} \frac{q^j\,(y;q)_j\,(q^{-m};q)_j}{(q;q)_j\,(q^{-m}/t;q)_j}\, \frac{(q^{1-m}/tx;q)_{m-j}\,x^{m-j}}{(q^{1-m}/t;q)_{m-j}}\ . \qquad 2.9$$

Evaluation at $x=1/q^a t$ and $y=1/q^b$ then yields

$$\frac{(-1)^m\, q^{\binom{m+1}{2}}\,(t;q)_m}{(q;q)_m} \sum_{j\geq 0}^{m} \frac{q^j\,(q^{-b};q)_j\,(q^{-m};q)_j}{(q;q)_j\,(q^{-m}/t;q)_j}\, \frac{(q^{1-m+a};q)_{m-j}\,(q^{-a}/t)^{m-j}}{(q^{1-m}/t;q)_{m-j}}\ .$$

Proceding as before, let us assume that the j^{th} term in the sum does not vanish. We must then have

$$i)\quad (q^{a+1-m};q)_{m-j}\neq 0 \quad \text{and} \quad ii)\quad (q^{-b};q)_j\neq 0\ .$$

But then $i)\ \Rightarrow\ a<j$ and $ii)\Rightarrow b\geq j$ which contradict our initial assumption that $a\geq b$. This proves 2.8 as desired. We have thus verified that K_m vanishes for all $\overline{\beta}$ when $\beta=(a,b)\neq(m,0)$ and $|\beta|\leq|\alpha|$.

We must next check condition I.3.b. We must then evaluate K_m at $\overline{\alpha}$ when $\alpha=(m,0)$. For this purpose we can use formula 2.9 which was shown to be equivalent to 2.1. Now, $\overline{\alpha}=(m,0)$ implies

$$\overline{\alpha}=(\frac{1}{q^m\,t},1)\ .$$

Substituting $x=\frac{1}{q^m}$ and $y=1$ in 2.9 produces

$$\frac{(-1)^m\, q^{\binom{m+1}{2}}\,(t;q)_m}{(q;q)_m} \sum_{j\geq 0}^{m} \frac{q^j\,(1;q)_j\,(q^{-m};q)_j\,(q;q)_{m-j}\,(q^{-m}/t)^{m-j}}{(q;q)_j\,(q^{-m}/t;q)_j(q^{1-m}/t;q)_{m-j}}\ .$$

Note that the occurence of $(1;q)_j$ in the numerator forces all the terms to vanish except the $j=0$ term. Formula 2.9 reduces to

$$\frac{(-1)^m\, q^{\binom{m+1}{2}}\,(t;q)_m\,(q;q)_m\,(q^{-m}/t)^m}{(q;q)_m\,(q^{1-m}/t;q)_m}\ .$$

The q-shifted factorial property, (I.8 in [3])

$$(a\,q^{-n};q)_n=(\frac{q}{a};q)_n\,(-\frac{a}{q})^n\,q^{-\binom{n}{2}}\ , \qquad 2.10$$

converts the term $(q^{1-m}/t;q)_m$ to $(t;q)_m\,(-\frac{1}{t})^m\,q^{-\binom{m}{2}}$ beautifully cancelling all the terms proving that the expression 2.9 evaluates to 1 as desired! This completes our proof of Theorem I.1.

Remark 2.1 We should also mention that in the Sahi paper the condition $G_\alpha(x;q,t)\,|_{x^\alpha}=1$ replaces I.3 b). To work under this alternate characterization, we need only check that the polynomials we are constructing satisfy I.4. This presents no additional difficulty. For instance, in the case of $G_{(m,0)}$, we only need to show that the coefficient of x^m in E_m does not vanish. Now, the occurrence of x^m in $(x;q)_k$ where $0\le k\le m$ is only possible when $k=m$. However, the restriction that $j+k\le m$ forces j to be zero, thus the coefficient of $x^m\,y^0$ can be computed exactly to be

$$\frac{(-1)^m\,q^{\binom{m+1}{2}+\binom{m}{2}}\,t^m}{(q;q)_m} \tag{2.11}$$

which is clearly non-zero!

Macdonald gives an explicit formula for the polynomial P_λ when λ is a one part partition. Namely, (see eq. 4.9 p. 323 of [11]) he shows that

$$P_{(m)}(x,y;q,t) \;=\; \frac{(q;q)_m}{(t;q)_m}\sum_{|\mu|=m}\frac{(t;q)_\mu}{(q;q)_\mu}\;m_\mu\;\;. \tag{2.12}$$

Macdonald in [11] shows that the polynomial P_λ (up to a scalar factor) can be obtained by a Hecke algebra symmetrization of his polynomial E_α whenever α is a composition that rearranges to λ. Knop [5] and Sahi [14] show that E_α may be recovered from the top of homogeneous component of their polynomials. Denoting the top component of G_α by G^{top}, we deduce that, whenever α rearranges to λ, we must have

$$P_\lambda(x;q,t)\;\doteq\;\sum_{\omega\in S_n}t^{-l(\omega)}\,T_\omega\,G^{top}_\alpha(x;q,t)\;\;. \tag{2.13}$$

Here again, we have used the "$\doteq$" sign since due to different normalizations used in this paper from those adopted in [12], [14] and [5], equality holds true up to a scalar factor. This given, we may use our formula 2.1 to obtain explicit expressions for the Macdonald polynomials. We shall carry this out here in the two variable case when $\alpha=(m,0)$.

Taking the top component of the right hand side of 2.1 we get

$$G^{top}_{(m,0)}(x,y;q,t)\;=\;\frac{(-1)^m\,q^{\binom{m+1}{2}+\binom{m}{2}}t^m}{(t;q)_{m+1}}\;\sum_{j=0}^{m}\frac{q^j\;(t;q)_j\;(t;q)_{m+1-j}}{(q;q)_{m-j}\;(q;q)_j}\;x^{m-j}\;y^j\;\;.$$

Two applications of formula 2.3 transforms this to

$$G^{top}_{(m,0)}(x,y;q,t)\;=\;\frac{(-1)^m\,q^{\binom{m+1}{2}+\binom{m}{2}}\;t^m}{(q;q)_m}\sum_{j=0}^{m}\frac{q^j\;(t;q)_j\;(q^{-m};q)_j}{t^j\;(q;q)_j\;(q^{-m}/t;q)_j}\;x^{m-j}\;y^j\;\;. \tag{2.14}$$

The two variable case of the operator T_{s_i}, defined in I.7, may be written as

$$T_s \;=\; s \;+\; \frac{(1-t)}{x-y}\, x\,(1-s)$$

where s denotes the transposition which exchanges x and y. Setting

$$S \;=\; T_{id} + \frac{1}{t}\, T_s \;=\; 1 + \frac{1}{t}\, s + \frac{(1-t)}{t} \frac{x}{(x-y)}\,(1-s)$$

2.13 implies that

$$P_{(m)} \;\doteq\; S\;\, G^{top}_{(m,0)}(x,y;q,t)\;\; .$$

It is interesting to see how this result can be directly derived from our explicit formulas. This is the contents of our next result.

Theorem 2.1

$$P_{(m)} \;=\; \frac{(-1)^m\,(q;q)_m}{q^{\binom{m+1}{2}+\binom{m}{2}}\;t^{m-1}}\;\frac{(1-tq^m)}{(1-t^2q^m)}\;\;S\,G^{top}_{(m,0)}(x,y;q,t)\;\; . \qquad 2.15$$

To verify this identity we need to prove three auxiliary lemmas.

For convenience let us set

$$\overline{G}^{top}_{(m,0)}(x,y;q,t) \;=\; \frac{(-1)^m\,(q;q)_m}{q^{\binom{m+1}{2}+\binom{m}{2}}\;t^{m-1}}\;\frac{(1-tq^m)}{(1-t^2q^m)}\;\;G^{top}_{(m,0)}(x,y;q,t) \qquad 2.16$$

and note that from 2.14 and 2.16 we derive that

$$\overline{G}^{top}_{(m,0)}(x,y;q,t) \;=\; \sum_{j=0}^{m} c_j\; x^{m-j}y^j$$

with

$$c_j \;=\; \frac{(1-tq^m)}{(1-t^2q^m)}\;\frac{q^j\,(t;q)_j\;(q^{-m};q)_j}{t^{j-1}\,(q;q)_j\;(q^{-m}/t;q)_j}\;\; . \qquad 2.17$$

Lemma 2.1

$$S\,\overline{G}^{top}_{\alpha}(x,y;q,t) \;=\; \sum_{a=0}^{m}\left((\frac{1}{t}-1)\sum_{j=0}^{a-1}(\bar{c}_j - c_{m-j}) \;+\; \frac{c_a}{t} \;+\; c_{m-a}\right) x^a\,y^{m-a} \qquad 2.18$$

Proof First examine the action of S on an arbitrary monomial, $x^a\,y^b$. There are three separate cases to consider, $a<b$, $a>b$ and $a=b$.

Let $a>b$. Then

$$S\,x^a\,y^b = \Big(1+\frac{1}{t}\,s+\frac{(1-t)}{t}\,\frac{x}{(x-y)}\,(1-s)\Big)\,x^a\,y^b = \frac{(x^{a+1}\,y^b - x^b\,y^{a+1}) + t\,(x^{b+1}\,y^a - x^a\,y^{b+1})}{t\,(x-y)}$$

$$= \frac{1}{t}\sum_{r=0}^{a-b} x^{a-r}\,y^{r+b} \;-\; \sum_{r=1}^{a-b-1} x^{a-r}\,y^{r+b} \;\;=\;\; (\frac{1}{t}-1)\sum_{r=1}^{a-b-1} x^{a-r}\,y^{r+b} \;\;+\;\; \frac{x^ay^b + x^by^a}{t}$$

The two remaining cases are determined in a similar fashion to obtain:

$$S\,x^a\,y^b \;=\; \begin{cases} (1-\frac{1}{t})\,f(b,a) \;+\; x^a y^b \;+\; x^b y^a & a<b \\[2mm] (\frac{1}{t}-1)\,f(a,b) \;+\frac{1}{t}\,x^a y^b+\frac{1}{t}\,x^b y^a & a>b \\[2mm] (1+\frac{1}{t})\,x^a\,y^a & a=b \end{cases}$$

Where for convenience we have set

$$f(a,b) \;=\; x^{a-1}y^{b+1} \;+\; \cdots \;+\; x^{b+1}y^{a-1} \qquad \text{for} \qquad a>b\;. \tag{2.19}$$

Thus, applying S to 2.17 we get

$$\begin{aligned} S\,\overline{G}^{top}_{(m,0)} = \sum_{0\,\le\, j\,<\,m/2} c_j \left[(\tfrac{1}{t}-1)\,f(m-j,j) \;+\tfrac{1}{t}\,x^{m-j}y^j+\tfrac{1}{t}\,x^j y^{m-j}\right] \;+ \\ \sum_{m/2\,<\,j\,\le\,m} c_j \left[(1-\tfrac{1}{t})\,f(j,m-j) \;+\; x^{m-j}y^j+x^j y^{m-j}\right] \;+ \\ c_{m/2}\,(1+\tfrac{1}{t})\,x^{m/2}y^{m/2}\;\;. \end{aligned}$$

with the convention to set $c_{m/2}=0$ when m is not even.

Splitting the two sums and making the change of variables $j\to m-j$ in the first portion of the second sum, we can regroup the resulting terms and obtain

$$\begin{aligned} S\,\overline{G}^{top}_{(m,0)} = \sum_{0\,\le\, j\,<\,m/2} (c_j-c_{m-j})\,(\tfrac{1}{t}-1)\,f(m-j,j) \;+ \\ \sum_{0\,\le\, j\,<\,m/2} (\tfrac{1}{t}\,c_j+c_{m-j})\,x^{m-j}y^j \;\;+\; \sum_{m/2\,<\,j\,\le\,m} (c_j+\tfrac{1}{t}\,c_{m-j})\,x^j y^{m-j} \\ +\;\; c_{m/2}\,(1+\tfrac{1}{t})\,x^{m/2}y^{m/2}\;\;. \end{aligned} \tag{2.20}$$

Note now that 2.19 for $a=m-j$ and $b=j$ gives

$$f(m-j,j) = \; x^{m-j-1}y^{j+1}+\cdots+x^{j+1}y^{m-j-1} \;\;=\; \sum_{a=0}^{m} x^{m-a}y^a \;\;\chi\big(j+1\le a\le m-j-1\big)\;\;.$$

Substituting this in the first sum on the right-hand side of 2.20 and, for a moment, calling the result S_1 we get

$$\begin{aligned} S_1 \;\;&=\;\; \sum_{a=0}^{m} x^{m-a}y^a \sum_{0\le j<\,m/2} (c_j-c_{m-j})(\tfrac{1}{t}-1)\;\;\chi\big(j+1\le a\le m-j-1\big) \\ &=\;\; \sum_{0\le a\le\, m/2} x^{m-a}y^a \sum_{0\le j\le a-1} (c_j-c_{m-j})(\tfrac{1}{t}-1) \\ &\qquad\qquad +\;\; \sum_{m/2\,<a\le m} x^{m-a}y^a \sum_{0\le j\le m-a-1} (c_j-c_{m-j})(\tfrac{1}{t}-1)\;\;. \end{aligned}$$

Substituting this in 2.20 and grouping terms we may write

$$S\,\overline{G}_\alpha^{top} = \sum_{0\le a\le m/2} x^{m-a}y^a\Big(\sum_{0\le j\le a-1}(c_j - c_{m-j})(\tfrac{1}{t}-1) \;+\; \tfrac{1}{t}\,c_a + c_{m-a}\Big)$$
$$+\sum_{m/2<a\le m} x^{m-a}y^a\Big(\sum_{0\le j\le m-a-1}(c_j - c_{m-j})(\tfrac{1}{t}-1) \;+\; c_a + \tfrac{1}{t}\,c_{m-a}\Big)\ ,$$

and this gives 2.18 since for $a>\frac{m}{2}$ we do have

$$\sum_{0\le j\le m-a-1}(c_j - c_{m-j})(\tfrac{1}{t}-1) \;+\; c_a + \tfrac{1}{t}\,c_{m-a} = \sum_{0\le j\le a-1}(c_j - c_{m-j})(\tfrac{1}{t}-1) \;+\; \tfrac{1}{t}\,c_a + c_{m-a}$$

This completes the proof of lemma 2.1.

Lemma 2.2

$$c_j - c_{m-j} = \frac{(1-tq^m)\,(t;q)_j\,(q^{-m};q)_j\,(1-q^{-m+2j})}{t^j\,(1-t^2q^m)\,(q;q)_j\,(q^{-m}/t;q)_{j+1}} \qquad 2.21$$

Proof The definition in 2.17 directly implies

$$c_j - c_{m-j} \;=\; \frac{(1-tq^m)}{(1-t^2q^m)}\left(\frac{q^j\,(t;q)_j\,(q^{-m};q)_j}{t^{j-1}\,(q;q)_j\,(q^{-m}/t;q)_j} \;-\; \frac{q^{m-j}\,(t;q)_{m-j}\,(q^{-m};q)_{m-j}}{t^{m-j-1}\,(q;q)_{m-j}\,(q^{-m}/t;q)_{m-j}}\right)\ . \qquad 2.22$$

Application of the q-shifted factorial property 2.3 and

$$(aq^{-n};q)_{n-k} \;=\; \frac{(q/a;q)_n}{(q/a;q)_k}\left(\frac{a}{q}\right)^{n-k} q^{\binom{k}{2}-\binom{n}{2}} \qquad 2.23$$

allow the expression in 2.22 to take the form

$$c_j - c_{m-j} \;=$$

$$= \frac{1-tq^m}{1-t^2q^m}\left(\frac{q^j\,(t;q)_j\,(q^{-m};q)_j}{t^{j-1}\,(q;q)_j\,(q^{-m}/t;q)_j} \;-\; \frac{q^{-\binom{m}{2}}\,(t;q)_m\,(q;q)_m\,(qt;q)_j\,(q^{-m};q)_j}{t^{m+j-1}\,(q;q)_m\,(q^{-m}/t;q)_m(q;q)_j\,(q^{-m+1}/t;q)_j}\right)$$

$$= \frac{(1-tq^mq^j)\ (t;q)_j\,(q^{-m};q)_j}{(1-t^2q^m)\,t^{j-1}\,(q;q)_j\,(q^{-m}/t;q)_{j+1}}\left((1-q^{-m+j}/t) \;-\; \frac{(t;q)_m\,(1-tq^j)\,(1-q^{-m}/t)\,(-1)^m}{q^{\binom{m}{2}+j}\,t^m\,(q^{-m}/t;q)_m\,(1-t)}\right)\ .$$

Using the transformations

$$(t;q)_m\,(1-q^{-m}/t) \;=\; \frac{-q^{-m}\,(t;q)_{m+1}}{t}$$

$$(q^{-m}/t;q)_m\,(1-t) \;=\; (-1)^m\,\frac{(t;q)_{m+1}}{t^m\,q^{\binom{m+1}{2}}}$$

we obtain

$$c_j - c_{m-j} \;=\; \frac{q^j\,(1-tq^m)\,(t;q)_j\,(q^{-m};q)_j}{t^{j-1}\,(1-t^2q^m)\,(q;q)_j\,(q^{-m}/t;q)_{j+1}}\left((1-q^{-m+j}/t) \;+\; \frac{q^{-j}\,(1-tq^j)}{t}\right)$$

$$=\; \frac{(1-tq^m)\,(t;q)_j\,(q^{-m};q)_j\,(1-q^{-m+2j})}{t^j\,(1-t^2q^m)\,(q;q)_j\,(q^{-m}/t;q)_{j+1}}\;.$$

This completes the proof of Lemma 2.2.

Now, in the two variable case, the Macdonald polynomial given in 2.12 reduces to

$$P_{(m)}(x,y;q,t) \;=\; \frac{(q;q)_m}{(t;q)_m}\sum_{a=0}^{m}\frac{(t;q)_{m-a}\,(t;q)_a}{(q;q)_{m-a}\,(q;q)_a}\;x^{m-a}y^a$$

2.24

$$=\; \sum_{a=0}^{m}\frac{q^a\,(t;q)_a\,(q^{-m};q)_a}{t^a\,(q^{1-m}/t;q)_a\,(q;q)_a}\;x^{m-a}\,y^a\;.$$

Denoting the coefficient of $x^{m-a}\,y^a$ by m_a, that is

$$m_a \;=\; \frac{q^a\,(t;q)_a\,(q^{-m};q)_a}{t^a\,(q^{1-m}/t;q)_a\,(q;q)_a}\;, \tag{2.25}$$

we see that Lemma 2.1 immediately allows the identity in 2.15 to be expressed in the following manner:

$$\sum_{a=0}^{m} m_a\,x^{m-a}\,y^a \;=\; \sum_{a=0}^{m}\left((\frac{1}{t}-1)\sum_{j=0}^{a-1}(c_j-c_{m-j}) \;+\; \frac{c_a}{t} \;+\; c_{m-a}\right)\;x^{m-a}\,y^a\;.$$

Thus to prove Theorem 2.1 we need only verify that

$$m_a \;-\; \frac{c_a}{t} \;-\; c_{m-a} \;=\; (\frac{1}{t}-1)\sum_{j=0}^{a-1}(c_j-c_{m-j})\;. \tag{2.26}$$

Lemma 2.3 Setting $R_a = m_a \,-\, c_a/t \,-\, c_{m-a}$ *we have*

$$R_a \;=\; \frac{(1-tq^m)}{(1-t^2q^m)}\;\frac{(t;q)_a(q^{-m};q)_a}{t^a(q^{-m}/t;q)_a(q;q)_{a-1}}$$

Proof Definitions 2.17 and 2.25 directly give

$$\begin{aligned} R_a \;=\; &\frac{q^a\,(q^{-m};q)_a\,(t;q)_a}{t^a\,(q^{1-m}/t;q)_a\,(q;q)_a} \\ &-\;\frac{q^a\,(1-tq^m)\,(q^{-m};q)_a\,(t;q)_a}{t^a\,(q^{-m}/t;q)_a\,(q;q)_a\,(1-t^2q^m)} \\ &-\;\frac{t\,q^{m-a}\,(1-tq^m)\,(q^{-m};q)_{m-a}\,(t;q)_{m-a}}{t^{m-a}\,(q^{-m}/t;q)_{m-a}\,(q;q)_{m-a}\,(1-t^2q^m)}\;. \end{aligned} \tag{2.27}$$

Transforming the third term by means of formula 2.3 and 2.23, we get

$$\frac{t\,q^{m-a}\,(1-tq^m)\,(q^{-m};q)_{m-a}\,(t;q)_a}{t^{m-a}\,(q^{-m}/t;q)_{m-a}\,(q;q)_{m-a}\,(1-t^2q^m)} \quad = \quad \frac{(q^{-m};q)_a\,(tq;q)_a(1-tq^m)\,(t;q)_m}{t^{m+a}\,(q^{-m};q)_m\,(q^{1-m}/t;q)_a\,(q;q)_a\,(1-t^2q^m)} \ .$$

This allows the expression in the right-hand side of 2.27 to be reduced to the form:

$$\frac{(1-tq^m)}{(1-t^2q^m)}\ \frac{(t;q)_a\,(q^{-m};q)_a}{t^a\,(q^{-m}/t;q)_a\,(q;q)_{a-1}}\ \times$$

$$\left(\frac{(1-t^2q^m)\,(1-q^{-m}/t)\,q^a}{(1-tq^m)\,(1-q^{-m+a}/t)\,(1-q^a)} - \frac{q^a}{(1-q^a)} - \frac{t\,(1-tq^a)\,(1-q^{-m}/t)}{q^{-m}\,(1-tq^m)\,(1-q^a)\,(1-q^{-m+a}/t)}\right) \ .$$

The expression in the parentheses simplifies to 1 proving Lemma 2.3.

Proof of Theorem 2.1 Combining Lemma 2.2 and Lemma 2.3 reduces 2.26 to the form

$$\frac{(1-tq^m)\,(t;q)_a\,(q^{-m};q)_a}{(1-t^2q^m)\,t^a\,(q;q)_{a-1}\,(q^{-m}/t;q)_a} \quad = \quad \frac{(1/t-1)\,(1-tq^m)}{(1-t^2q^m)} \sum_{j=0}^{a-1} \frac{(t;q)_j\,(q^{-m};q)_j\,(1-q^{-m+2j})}{t^j\,(q;q)_j\,(q^{-m}/t;q)_{j+1}}$$

Thus we are left to verify that

$$\frac{(t;q)_a\,(q^{-m};q)_a}{t^a\,(q;q)_{a-1}\,(q^{-m}/t;q)_a} \quad = \quad \frac{(1-t)}{t} \sum_{j=0}^{a-1} \frac{(t;q)_j\,(q^{-m};q)_j\,(1-q^{-m+2j})}{t^j\,(q;q)_j\,(q^{-m}/t;q)_{j+1}} \ .$$

However this is immediate since for all $j>0$ we have

$$\frac{(1-t)}{t}\ \frac{(t;q)_j\,(q^{-m};q)_j\,(1-q^{-m+2j})}{t^j\,(q;q)_j\,(q^{-m}/t;q)_{j+1}} \quad = \quad \frac{(t;q)_{j+1}\,(q^{-m};q)_{j+1}}{t^{j+1}\,(q;q)_j\,(q^{-m}/t;q)_{j+1}} \ - \ \frac{(t;q)_j\,(q^{-m};q)_j}{t^j\,(q;q)_{j-1}\,(q^{-m}/t;q)_j} \ ,$$

and moreover this relation remains true even for $j=0$ provided we set $(q;q)_{-1}=\infty$. This completes our proof.

3. The matrix of the Knop operator Ξ_1 and a ${}_6\Phi_5$ summation formula.

Our original proof of Theorem I.1 was based on the characterization of the polynomials G_α as eigenfunctions of the Knop-operators Ξ_i. This approach required the construction of explicit formulas for the entries of the matrix expressing the action of Ξ_1 on the basis $\{(x;q)_k(y;q)_l\}_{k,l}$. The computations and identities that result from this proof turn out to be quite interesting in themselves. In particular they reveal an intimate connection between the Knop-Sahi polynomials and some of the deeper identities of the theory of basic hypergeometric series. In this section we shall give an outline of this alternate proof focussing on the salient features and omitting some of the more laborious details. The reader is referred to [13] for the complete treatment.

Our point of departure is the following simple observation.

Proposition 3.1 *If a polynomial $P(x,y;q,t)$ is of degree m and satisfies $\Xi_1\, P(x,y;q,t) = q^m t\, P(x,y;q,t)$ then it is necessarily a multiple of $G_{(m,0)}(x,y;q,t)$.*

Proof Formula I.2 gives that $\overline{\alpha}_1 = q^m t$ if and only if $m = \alpha_1 \geq \alpha_2$. Thus from Property I.4 we derive that $\Xi_1\, G_\alpha = q^m t\, G_\alpha$ if and only if $\alpha = (m,i)$ for some $0 \leq i \leq m$. Thus the elements, $\{G_{(m,0)}, G_{(m,1)}, \ldots, G_{(m,m)}\}$, form a basis for the $q^m t$-eigenspace of Ξ_1. In particular, this gives that our polynomial $P(x,y;q,t)$ must have the expansion

$$P(x,y;q,t) \;=\; \sum_{i=0}^{m} d_i\, G_{(m,i)}(x,y;q,t) \;. \tag{3.1}$$

Note now that the term $x^m y^m$ must occur in $G_{(m,m)}$ and at the same time it cannot occur anywhere else in 3.1 since all the other polynomials (including P) have degree strictly less than $2m$. This forces $d_m = 0$. Similar reasoning recursively applied yields $d_i = 0$ for all $i \geq 1$. Thus $P(x,y;q,t) = d_0\, G_{(m,0)}(x,y;q,t)$ as asserted.

Note that since the polynomial $K_m(x,y;q,t)$ given in 2.1 is clearly of degree m and, as we have seen in section 1, it statisfies the normalization $K_m((\overline{m,0});q,t) = 1$, Proposition 3.1 reduces the proof of Theorem I.1 to showing that

$$\Xi_1 K_m(x,y;q,t) = q^m t\, K_m(x,y;q,t) \;. \tag{3.2}$$

To this end let us set

$$K_m(x,y;q,t) = \sum_{k,l} z_k\, w_l\, C_m^{(k,l)} \quad \text{and} \quad \Xi_1\, z_k\, w_l = \sum_{a,b} z_a\, w_b\, M_{(a,b;k,l)}$$

where for simplicity we let $z_k = (x;q)_k$ and $w_l = (y;q)_l$. Substituting in 3.2 and equating coefficients of $z_a\, w_b$ we derive that 3.2 holds if and only if we have

$$\sum_{k,l} M_{(a,b;k,l)}\, C_m^{(k,l)} \;=\; q^m t\, C_m^{(a,b)} \;. \tag{3.3}$$

Now it develops that the coefficients $M_{(a,b;k,l)}$ may be given the following explicit expressions:

Theorem 3.1 *For $a+b \leq k+l \leq m$ we have*

$$M_{(a,b;k,l)} = \begin{cases} \begin{cases} t\, q^a & \text{if } a \geq b \\ q^a & \text{if } a < b \end{cases} & \text{for } \ k=a \text{ and } \ l=b \\ \frac{(t-1)\, q^{l^2-la-lb+ab-l+a+b}\, (q;q)_{k-b}\, (q;q)_{k-a}}{(q;q)_{k+l-a-b}\, (q;q)_{a-l}\, (q;q)_{b-l}} & \text{for } k>a \text{ and } \ l \leq b \ \text{ when } \ k>l \\ \frac{(1-t)\, q^{k^2+k-ka-kb+ab}\, (q;q)_{l-b-1}\, (q;q)_{l-a-1}}{(q;q)_{k+l-a-b}\, (q;q)_{a-k-1}\, (q;q)_{b-k-1}} & \text{for } l>a \text{ and } \ k<b \ \text{ when } \ k<l \end{cases}$$

and $M_{(a,b;k,l)} = 0$ *otherwise.*

In view of 3.3 and the preceeding observations, the proof of Theorem I.1 by this approach reduces to the following identity:

Proposition 3.2

$$q^m t\, C_m^{(a,b)} \;=\; C_m^{(a,b)} \times \left\{ \begin{matrix} q^a\, t & \text{if } a \geq b \\ q^a & \text{if } a < b \end{matrix} \right\} + \sum_{\substack{k>l \\ k>a\ \&\ l\leq b \\ a+b\leq l+k\leq m}} m^{(1)}_{k,l}\, C_m^{(k,l)} + \sum_{\substack{k<l \\ l>a\ \&\ k<b \\ a+b\leq l+k\leq m}} m^{(2)}_{k,l}\, C_m^{(k,l)} \qquad 3.4$$

where

$$a) \quad m^{(1)}_{k,l} = \frac{(t-1)\,(q;q)_{k-b}\,(q;q)_{k-a}\, q^{l^2-al-bl-l+ab+a+b}}{(q;q)_{b-l}\,(q;q)_{a-l}\,(q;q)_{l+k-a-b}}\,,$$

$$3.5$$

$$b) \quad m^{(2)}_{k,l} = \frac{(1-t)\,(q;q)_{l-a-1}\,(q;q)_{l-b-1}\, q^{k^2+k-ak+ab-bk}}{(q;q)_{b-k-1}\,(q;q)_{a-k-1}\,(q;q)_{l+k-a-b}}\,.$$

The expressions for the entries $M_{(a,b;k.l)}$ of the matrix of the operator Ξ_1 given by Theorem 3.1 are an immediate consequences of the following

Proposition 3.3

$$\Xi_1\, z_k\, w_l = \begin{cases} t\, q^k\, z_k\, w_l + (t-1) \sum_{\substack{l\leq a\leq k-1 \\ l\leq b\leq k+l-a}} \frac{q^{l^2-la-bl+ab-l+a+b} (q;q)_{k-b}\,(q;q)_{k-a}}{(q;q)_{k+l-a-b}\,(q;q)_{a-l}\,(q;q)_{b-l}}\, z_a\, w_b & \text{if } k \geq l \\ q^k\, z_k\, w_l + (1-t) \sum_{\substack{k+1\leq a\leq l-1 \\ k+1\leq b\leq l+k-a}} \frac{q^{k^2+k-ak-kb+ab} (q;q)_{l-b-1}\,(q;q)_{l-a-1}}{(q;q)_{k+l-a-b}\,(q;q)_{a-k-1}\,(q;q)_{b-k-1}}\, z_a\, w_b & \text{if } k < l \end{cases} \qquad 3.6$$

The proof of this proposition depends primarily on the following cute identity.

Theorem 3.2

$$\frac{(x;q)_n - (y;q)_n}{x-y} = \sum_{0\leq a+b\leq n-1} -q^{ab+a+b}\, \frac{(q;q)_{n-b-1}\,(q;q)_{n-a-1}}{(q;q)_{n-a-b-1}\,(q;q)_a\,(q;q)_b}\, (x;q)_a\,(y;q)_b$$

Proof Express both sides in powers of x and y.

$$\sum_{0<k\leq n} \frac{(q^{-n};q)_k\, q^{n\,k}}{(q;q)_k} \left[\frac{x^k-y^k}{x-y}\right] = \sum_{0\leq a+b\leq n-1} \Big(\frac{-q^{ab+a+b}\,(q;q)_{n-b-1}\,(q;q)_{n-a-1}}{(q;q)_{n-a-b-1}\,(q;q)_a\,(q;q)_b} \times$$

$$\times \sum_{i=0}^{a} \frac{(q^{-a};q)_i\,(x\,q^a)^i}{(q;q)_i} \sum_{j=0}^{b} \frac{(q^{-b};q)_j\,(y\,q^b)^j}{(q;q)_j} \Big)$$

Expand the left hand side.

$$\sum_{k<n} \frac{(q^{-n};q)_k\, q^{n\,k}}{(q;q)_k} \sum_{0\leq l\leq k-1} x^l\, y^{k-1-l} \;=\; \sum_{0\leq a+b\leq n-1} \Big(\frac{-q^{ab+a+b}\,(q;q)_{n-b-1}\,(q;q)_{n-a-1}}{(q;q)_{n-a-b-1}\,(q;q)_a\,(q;q)_b} \times$$

$$\times \sum_{i=0}^{a} \sum_{j=0}^{b} \frac{q^{a\,i+b\,j}\,(q^{-a};q)_i\,(q^{-b};q)_j\, x^i\, y^j}{(q;q)_i\,(q;q)_j} \Big)$$

Take the coefficient of $x^r y^s$ on both sides, where $r, s \geq 0$.

$$\frac{(q^{-n};q)_{1+r+s}\, q^{n\,(r+s+1)}}{(q;q)_{r+s+1}} = \sum_{\substack{0\leq a+b\leq n-1\\ r\leq a\\ s\leq b}} -q^{ab+a+b+a\,r+b\,s}\, \frac{(q;q)_{n-b-1}\,(q;q)_{n-a-1}(q^{-a};q)_r\,(q^{-b};q)_s}{(q;q)_{n-a-b-1}\,(q;q)_a\,(q;q)_b\,(q;q)_r\,(q;q)_s}$$

Transforming the factors $(q;q)_{n-a-1}$ and $(q;q)_{n-a-b-1}$ using 2.3, and $(q^{-a};q)_r$ and $(q^{-b};q)_s$ using 2.10, reduces this equality to a form in which the factors depending on n can be removed from the summand, and the dependence on a and b will appear only in the indices of the q-shifted factorials.

$$\frac{q^{n(r+s+1)}(q;q)_r\,(q;q)_s\,(q^{-n};q)_{1+r+s}}{(-1)^{r+s}\,q^{\binom{r}{2}+\binom{s}{2}}\,(q;q)_{n-1}\,(q;q)_{r+s+1}} = \sum_{\substack{r\leq a\leq n-1\\ s\leq b\leq n-1-a}} \frac{-q^{a+b}\,(q^{-n+1};q)_{a+b}}{(q^{-n+1};q)_a\,(q^{-n+1};q)_b\,(q;q)_{a-r}\,(q;q)_{b-s}}$$

We now make the change of variables $a \to a+r$ and $b \to b+s$, obtaining

$$\frac{q^{n(r+s+1)}(q;q)_r\,(q;q)_s\,(q^{-n};q)_{1+r+s}}{(-1)^{r+s}\,q^{\binom{r}{2}+\binom{s}{2}}\,(q;q)_{n-1}\,(q;q)_{r+s+1}} = \sum_{\substack{r\leq a+r\leq n-1\\ s\leq b+s\leq n-1-a-r}} \frac{-q^{a+r+s+b}\,(q^{-n+1};q)_{a+b+r+s}}{(q^{-n+1};q)_{r+a}\,(q^{-n+1};q)_{s+b}\,(q;q)_a\,(q;q)_b}$$

The q-shifted factorial property,

$$(a;q)_{k+n} = (a;q)_k\,(aq^k;q)_n \tag{3.7}$$

modifies the summands to consist of factors indexed by only one of the variables. This allows one variable to be considered fixed while summing over the other.

$$\begin{aligned}\frac{q^{n(r+s+1)}\,(q^{-n+1};q)_r\,(q^{-n+1};q)_s\,(q;q)_r\,(q;q)_s\,(q^{-n};q)_{1+r+s}}{(-1)^{r+s+1}\,q^{r+s+\binom{r}{2}+\binom{s}{2}}\,(q;q)_{n-1}\,(q;q)_{r+s+1}\,(q^{-n+1};q)_{r+s}} &= \\ = \sum_{a=0}^{n-1-r} q^a\,\frac{(q^{-n+1+r+s};q)_a}{(q^{-n+1+r};q)_a\,(q;q)_a} \sum_{b=0}^{n-1-a-s-r} q^b\,\frac{(q^{-n+1+r+s+a};q)_b}{(q^{-n+1+s};q)_b\,(q;q)_b}\end{aligned} \tag{3.8}$$

This places us in a position to use the summation identity

$$ {}_2\phi_1(0,q^{-n};c;q,q) \;=\; \frac{(-1)^n\,c^n\,q^{\binom{n}{2}}}{(c;q)_n}\;, \tag{3.9}$$

which is the particular case of 2.6 obtained by making the replacement

$$\frac{(c/a;q)_n}{(c;q)_n}\,a^n \;=\; (-1)^n\,c^n\,q^{\binom{n}{2}}\,\frac{(a\,q^{-n+1}/c;q)_n}{(c;q)_n}\;,$$

an letting $a \to 0$.

Applying 3.9 with $n = n-1-r-s-a$ and $c = q^{-n+1+s}$, the double sum in 3.8 reduces to a single sum and 3.8 becomes:

$$\frac{(-1)^{r+s+1}\, q^{n(r+s+1)}\, (q^{-n+1};q)_r\, (q^{-n+1};q)_s\, (q;q)_r\, (q;q)_s\, (q^{-n};q)_{1+r+s}}{q^{r+s+\binom{r}{2}+\binom{s}{2}}\, (q;q)_{n-1}\, (q;q)_{r+s+1}\, (q^{-n+1};q)_{r+s}} \quad = \tag{3.10}$$

$$= \sum_{a=0}^{n-1-r} q^a\, \frac{(q^{-n+1+r+s};q)_a}{(q^{-n+1+r};q)_a\, (q;q)_a} \frac{(-1)^{n-1-r-s-a}\, (q^{-n+1+s})^{n-1-r-s-a}\, q^{\binom{n-1-r-s-a}{2}}}{(q^{-n+1+s};q)_{n-1-r-s-a}}$$

Applying 2.3 to $(q^{-n+1+s};q)_{n-1-r-s-a}$ transforms 3.10 to

$$\frac{(-1)^n\, q^{n(r+s+1)}\, (q^{-n+1+s};q)_{n-1-r-s}\, (q^{-n+1};q)_r\, (q^{-n+1};q)_s\, (q;q)_r\, (q;q)_s\, (q^{-n};q)_{1+r+s}}{q^{r+s+\binom{r}{2}+\binom{s}{2}}\, (q;q)_{n-1}\, (q;q)_{r+s+1}\, (q^{-n+1};q)_{r+s}}$$

$$= \sum_{1\le a\le n-1-r} q^a\, \frac{(q^{-n+1+r+s};q)_a\, (q^{1+r};q)_a}{(q^{-n+1+r};q)_a\, (q;q)_a}\ . \tag{3.11}$$

Let $c = q^{1+r-n}$, $n = n-1-r-s$ and $a = q^{1+r}$ to permit the application of identity 2.6. The sum is thus eliminated and 3.11 reduces to

$$\frac{(-1)^n\, q^{n^2+n\,r-r-r^2-\binom{n}{2}}\, (q^{-n+1+s};q)_{n-1-r-s}\, (q^{-n+1};q)_r\, (q^{-n+1};q)_s\, (q;q)_r\, (q;q)_s\, (q^{-n};q)_{1+r+s}}{(q;q)_{n-1}\, (q;q)_{r+s+1}\, (q^{-n+1};q)_{r+s}}$$

$$= \frac{(q^{-n};q)_{n-1-r-s}\, (q^{1+r})^{n-1-r-s}}{(q^{-n+1+r};q)_{n-1-r-s}}$$

This equality can be easily be seen to hold, proving the theorem.

Proof of Proposition 3.3 Note that we may express the action of Ξ_1 in terms of T_{s_1} and ϕ in the form

$$\Xi_1 \;=\; t\,T_{s_1}^{-1}\,(\phi\,T_{s_1} + 1/y - 1/y\,\phi\,T_{s_1})\,T_{s_1}^{-1}\ .$$

This can be seen by combining 1.6 with $i = 1$ with the definition I.15 for $n = i = 2$ and $x_2 = y$. Thus, application of Ξ_1 to $z_k w_l$ immediately produces

$$\Xi_1\, z_k\, w_l \;=\; t\,T_{s_1}^{-1}\,\phi\, z_k\, w_l + t\,T_{s_1}^{-1}\, 1/y\, T_{s_1}^{-1}\, z_k\, w_l - t\,T_{s_1}^{-1}\, 1/y\,\phi\, z_k\, w_l\ .$$

The definition of ϕ and 1.5 b) yields

$$\Xi_1\,(x;q)_k\,(y;q)_l \;=\; \frac{1}{x}(x;q)_k\,(y;q)_l + \frac{x-1}{x}\frac{(t\,x-y)}{(x-y)}\,(qx;q)_k\,(y;q)_l + \frac{(y-1)(1-t)}{(x-y)}\,(qy;q)_k\,(x;q)_l\ .$$

Manipulation gives

$$\Xi_1\,(x;q)_k\,(y;q)_l \;=\; q^k\,(x;q)_k\,(y;q)_l + \frac{(1-t)}{x-y}[(x;q)_{k+1}\,(y;q)_l - (y;q)_{k+1}\,(x;q)_l]\ .$$

This reduces to two cases;

$$\Xi_1\, z_k\, w_l \;=\; \begin{cases} q^k\, z_k\, w_l + \frac{(1-t)}{(x-y)}[(x\,q^l;q)_{k-l+1} - (y\,q^l;q)_{k-l+1}] z_l\, w_l & \text{if } k \ge l \\ q^k\, z_k\, w_l + \frac{(1-t)}{(x-y)}[(y\,q^{k+1};q)_{l-k-1} - (x\,q^{k+1};q)_{l-k-1}] z_{k+1}\, w_{k+1} & \text{if } k < l \end{cases}$$

This given, two applications of Theorem 3.2 yield that for $k \ge l$ we have

$$\Xi_1\, z_k w_l \;=\; q^k\, z_k\, w_l \,+\, (t-1) \sum_{\substack{0\le a\le k-l \\ 0\le b\le k-l-a}} \frac{q^{ab+a+l+b}(q;q)_{k-b-l}\,(q;q)_{k-l-a}}{(q;q)_{k-l-a-b}\,(q;q)_k\,(q;q)_l}\, z_{l+a}\, w_{l+b}\,,$$

and for $k < l$,

$$\Xi_1\, z_k w_l \;=\; q^k\, z_k\, w_l \,-\, (t-1) \sum_{\substack{0\le a\le l-k-2 \\ 0\le b\le l-k-2-a}} \frac{q^{ab+b+k+a+1}(q;q)_{l-k-b-2}\,(q;q)_{l-k-a-2}}{(q;q)_{l-k-a-b-2}\,(q;q)_a\,(q;q)_b}\, z_{k+1+a}\, w_{k+1+b}\,.$$

A change of variables allows the action to be described in terms of basis elements indexed only by a and b. For $k \ge l$, make the change of variables, $a \to a-l$ and $b \to b-l$, and for $k < l$, let $a \to a-k-1$ and $b \to b-k-1$.

$$\Xi_1\, z_k w_l \;=\; \begin{cases} q^k\, z_k\, w_l \,+\, (t-1)\sum_{\substack{l\le a\le k \\ l\le b\le k+l-a}} \frac{q^{ab-al-lb+l^2+a-l+b}\,(q;q)_{k-b}\,(q;q)_{k-a}}{(q;q)_{k+l-a-b}\,(q;q)_{a-l}\,(q;q)_{b-l}}\, z_a\, w_b & \text{for } k \ge l \\ \\ q^k\, z_k\, w_l \,-\, (t-1)\sum_{\substack{k+1\le a\le l-1 \\ k+1\le b\le l+k-a}} \frac{q^{ab-ak-kb+k^2+k}\,(q;q)_{l-b-1}\,(q;q)_{l-a-1}}{(q;q)_{k+l-a-b}\,(q;q)_{a-k-1}\,(q;q)_{b-k-1}}\, z_a\, w_b & \text{for } k < l \end{cases}$$

Subtraction of the $k = a$ term from the sum in the first case yields the equalities in 3.6 and completes the proof of the proposition.

Proof of Proposition 3.2 Only the case $a \ge b$ will be given here. The case of $a < b$ can be seen in a similar manner. The variable changes $k \to a+k$, $l \to b-l$ in the first sum of 3.4, and $k \to b-l$, $l \to a+k$ in the second sum reduces the problem to showing

$$(q^m\, t - q^a\, t)\, C_m^{(a,b)} = \sum_{\substack{a+k>b-l \\ k>0\ \&\ l\ge 0 \\ l\le k \\ k-l+a+b\le m}} m^{(1)}_{(k+a,b-l)}\, C_m^{(k+a,b-l)} \;+\; \sum_{\substack{a+k>b-l \\ k>0\ \&\ l>0 \\ l\le k \\ k-l+a+b\le m}} m^{(2)}_{(b-l,k+a)}\, C_m^{(b-l,k+a)}\,.$$

Denote the right hand side by R_m and substitute our explicit expressions for $C_m^{(k,l)}$, $m^{(1)}_{k,l}$, $m^{(2)}_{k,l}$ given by 2.1 and 3.5 a) and b). Because we are only considering the case $a \ge b$ the condition $a+k > b-l$ is redundant and can therefore be eliminated. This gives

$$R_m = \sum_{\substack{l\le k \\ k>0\ \&\ l\ge 0 \\ k-l+a+b\le m}} \frac{(-1)^m\,(t-1)\,q^{\binom{m+1}{2}+l^2-lk-lb+ba+bk+a+b}\,(t;q)_{m-a-k}\,(t;q)_{m-b+l+1}\,(q;q)_{a-b+k}\,(q;q)_k}{t^{l-a-k-b}\,(t;q)_{m+1}\,(q;q)_{a+k}\,(q;q)_{b-l}\,(q;q)_{m-a-k-b+l}\,(q;q)_{a-b+l}\,(q;q)_l\,(q;q)_{k-l}}$$

$$+\sum_{\substack{l\le k\\ k>0\ \&\ l>0\\ k-l+a+b\le m}} \frac{(-1)^m\,(1-t)\,q^{\binom{m+1}{2}+l^2-lk-lb+bk+ba+k+a}\,(t;q)_{m-a-k+1}\,(t;q)_{m-b+l}\,(q;q)_{a-b+k-1}\,(q;q)_{k-1}}{t^{l-a-k-b}\,(t;q)_{m+1}(q;q)_{a+k}\,(q;q)_{b-l}\,(q;q)_{m-a-k-b+l}\,(q;q)_{a-b+l-1}\,(q;q)_{l-1}\,(q;q)_{k-l}}$$

Transformation of terms using properties 2.3 and 3.7 allows the summands to be expressed with q-shifted factorials that are either indexed by k and independent of l or indexed by l with a possible dependence on k, but no further dependence on l. This converts $R_m$to the form

$$\sum_{\substack{l\le k\\ k>0\ \&\ l\ge 0\\ k-l+a+b\le m}} \left(\frac{(-1)^m\,t^{a+b}\,q^{\binom{m+1}{2}+b(a+1)+a+k}}{t^l(t;q)_{m+1}\,(q;q)_a\,(q;q)_b\,(q;q)_{m-a-b}}\right)\times$$
$$\frac{q^l\,(t-1)(t;q)_{m-a}\,(t;q)_{m-b+1}(q^{a+b-m};q)_k\,(q^{a-b+1};q)_k\,(tq^{m-b+1};q)_l\,(q^{-b};q)_l\,(q^{-k};q)_l}{(q^{1-m+a}/t;q)_k\,(q^{a+1};q)_k\,(q^{-a-b+m-k+1};q)_l\,(q^{a-b+1};q)_l\,(q;q)_l}\ +$$
$$+\sum_{\substack{l\le k\\ k>0\ \&\ l>0\\ k-l+a+b\le m}} \left(\frac{(-1)^m\,t^{a+b}\,q^{\binom{m+1}{2}+b(a+1)+a+k}}{t^l(t;q)_{m+1}\,(q;q)_a\,(q;q)_b\,(q;q)_{m-a-b}}\right)\times$$
$$\left.\frac{(1-t)(t;q)_{m-a+1}\,(t;q)_{m-b}\,(tq^{m-b};q)_l\,(q;q)_{k-1}\,(q^{a-b};q)_k\,(q^{-b};q)_l\,(q^{a+b-m};q)_k\,(q^{-k},q)_l}{(q;q)_k\,(q^{-m+a}/t;q)_k\,(q^{a+1};q)_k\,(q;q)_{l-1}\,(q^{1-k-a-b+m};q)_l\,(q^{a-b};q)_l}\right)$$

We can see now that $C_m^{(a,b)} = \frac{(-1)^m\,t^{a+b}\,q^{\binom{m+1}{2}+b(a+1)}\,(t;q)_{m-b+1}\,(t;q)_{m-a}}{(t;q)_{m+1}\,(q;q)_a\,(q;q)_b\,(q;q)_{m-a-b}}$ can be factored out of R_m reducing the problem to verifying the equality:

$$t\,q^m - t\,q^a \;=\; \sum_{\substack{l\le k\\ k>0\ \&\ l\ge 0\\ k-l+a+b\le m}} \frac{q^{k+l+a}\,(t-1)\,(q^{a-b+1};q)_k\,(q^{a+b-m};q)_k\,(tq^{m-b+1};q)_l\,(q^{-b};q)_l\,(q^{-k};q)_l}{t^l\,(q^{1-m+a}/t;q)_k\,(q^{a+1};q)_k\,(q^{-a-b+m-k+1};q)_l\,(q^{a-b+1};q)_l\,(q;q)_l}$$
$$+\sum_{\substack{l\le k\\ k>0\ \&\ l>0\\ k-l+a+b\le m}} \frac{q^m\,(1-t)\,(q^{a+b-m};q)_k\,(q^{a-b+1};q)_{k-1}\,(tq^{m-b+1};q)_{l-1}\,(q^{-b};q)_l\,(q^{-k+1};q)_{l-1}}{t^{l-1}\,(q^{1-m+a}/t;q)_{k-1}\,(q^{a+1};q)_k\,(q^{-a-b+m-k+1};q)_l\,(q^{a-b+1};q)_{l-1}\,(q;q)_{l-1}}\,.$$

Recall that we are working only with the case $a\ge b$. This condition assures that $(q^{a-b+1};q)_k$ does not vanish, thus the only zero in the denominators of the summands may come from the factor $(q^{-a-b+m-k+1};q)_l$. Observe now that since both sums are over $l\le k$ we may apply the following transformation

$$\frac{(q^{a+b-m};q)_k}{(q^{-a-b+m-k+1};q)_l} = (-1)^l\,q^{-l(1-a-b+m)+kl-\binom{l}{2}}\,(q^{a+b-m};q)_{k-l}\,. \qquad 3.12$$

which removes all denominators zeros and assures that the summands will vanish only if there is at least one zero in the numerator. Refer again to the previous equation. Because this term vanishes for $k-l+a+b > m$, the sumands will vanish if $k-l+a+b > m$. Thus the restriction $k-l+a+b \le m$ be eliminated from both sums and we are left with showing the equality

$$t\,q^m - t\,q^a \;=\; \sum_{\substack{1\le k<\infty\\0\le l<\infty\\l\le k}} \frac{q^{k+l+a}\,(t-1)\,(q^{a-b+1};q)_k\,(q^{a+b-m};q)_k\,(tq^{m-b+1};q)_l\,(q^{-b};q)_l\,(q^{-k};q)_l}{t^l\,(q^{1-m+a}/t;q)_k\,(q^{a+1};q)_k\,(q^{-a-b+m-k+1};q)_l\,(q^{a-b+1};q)_l\,(q;q)_l}$$

3.13

$$\sum_{\substack{1\le k<\infty\\1\le l<\infty\\l\le k}} \frac{q^m\,(1-t)\,(q^{a+b-m};q)_k\,(q^{a-b+1};q)_{k-1}\,(tq^{m-b+1};q)_{l-1}\,(q^{-b};q)_l\,(q^{-k+1};q)_{l-1}}{t^{l-1}\,(q^{1-m+a}/t;q)_{k-1}\,(q^{a+1};q)_k\,(q^{-a-b+m-k+1};q)_l\,(q^{a-b+1};q)_{l-1}\,(q;q)_{l-1}}\;.$$

Adding a $(k,l)=(0,0)$ term to the first sum converts it to a sum over $0\le k\le\infty$, and the change of variables, $k\to k+1$, $l\to l+1$, in the second allows the sums to be combined. The right hand side of 3.13 can thus be transformed as follows:

$$-q^a\,(t-1) + \sum_{\substack{0\le k<\infty\\0\le l<\infty\\l\le k}} \frac{q^{k+l+a}\,(t-1)\,(q^{a-b+1};q)_k\,(q^{a+b-m};q)_k\,(tq^{m-b+1};q)_l\,(q^{-b};q)_l\,(q^{-k};q)_l}{t^l\,(q^{1-m+a}/t;q)_k\,(q^{a+1};q)_k\,(q^{-a-b+m-k+1};q)_l\,(q^{a-b+1};q)_l\,(q;q)_l}$$

$$+\sum_{\substack{0\le k<\infty\\0\le l<\infty\\l\le k}} \frac{q^m\,(1-t)\,(q^{a+b-m};q)_{k+1}\,(q^{a-b+1};q)_k\,(tq^{m-b+1};q)_l\,(q^{-b};q)_{l+1}\,(q^{-k};q)_l}{t^l\,(q^{1-m+a}/t;q)_k\,(q^{a+1};q)_{k+1}\,(q^{-a-b+m-k};q)_{l+1}\,(q^{a-b+1};q)_l\,(q;q)_l}$$

$$= -q^a\,(t-1) + \sum_{\substack{0\le k<\infty\\0\le l<\infty\\l\le k}} \Bigg(\frac{(t-1)\,(q^{a-b+1};q)_k\,(q^{a+b-m};q)_k\,(tq^{m-b+1};q)_l\,(q^{-b};q)_l\,(q^{-k};q)_l}{t^l\,(q^{1-m+a}/t;q)_k\,(q^{a+1};q)_k\,(q^{-a-b+m-k+1};q)_l\,(q^{a-b+1};q)_l\,(q;q)_l} \times q^{k+b+a}\,\frac{(1-q^{a+k+1-b+l})}{(1-q^{a+k+1})}\Bigg)$$

The proof of proposition 3.2 is now a matter of validating the equality

$$1-tq^{m-a} \;=\; \sum_{\substack{0\le k<\infty\\0\le l<\infty\\l\le k}} \Bigg(\frac{q^{k+b}\,(1-t)}{t^l} \times\frac{(q^{a-b+1};q)_k\,(q^{a+b-m};q)_k\,(tq^{m-b+1};q)_l\,(q^{-b};q)_l\,(q^{-k};q)_l\,(1-q^{a+k+1-b+l})}{(q^{1-m+a}/t;q)_k\,(q^{a+1};q)_k\,(q^{-a-b+m-k+1};q)_l\,(q^{a-b+1};q)_l\,(q;q)_l\,(1-q^{a+k+1})}\Bigg)\;. \qquad 3.14$$

Let the right hand side be denoted, F_m. Note that the simple equality

$$\frac{(1-q^{a+k+1-b+l})}{(1-q^{a+k+1})} = \frac{(1-q^{a+k-b+1})\,(q^{a-b+k+2};q)_l}{(1-q^{a+k+1})\,(q^{a-b+k+1};q)_l}$$

allows F_m to be expressed entirely in terms of q-shifted factorials and factors of q and t, yielding

$$F_m = \sum_{\substack{0\le k<\infty\\0\le l<\infty\\l\le k}} \frac{q^{k+b}\,(1-t)\,(q^{a-b+1};q)_{k+1}\,(q^{a+b-m};q)_k\,(tq^{m-b+1};q)_l\,(q^{-b};q)_l\,(q^{-k};q)_l\,(q^{a-b+k+2};q)_l}{t^l\,(q^{1-m+a}/t;q)_k\,(q^{a+1};q)_{k+1}\,(q^{-a-b+m-k+1};q)_l\,(q^{a-b+1};q)_l\,(q;q)_l(q^{a-b+k+1};q)_l}$$

Next, the change of variables $j+l=k$ gives

$$\begin{aligned}F_m &= \sum_{\substack{0\le j<\infty\\0\le l<\infty}} \Bigg(\frac{q^{j+l+b}\,(1-t)}{t^l}\\ &\times \frac{(q^{a-b+1};q)_{j+l+1}\,(q^{a+b-m};q)_{j+l}\,(tq^{m-b+1};q)_l\,(q^{-b};q)_l\,(q^{-j-l};q)_l\,(q^{a-b+j+l+2};q)_l}{(q^{1-m+a}/t;q)_{j+l}\,(q^{a+1};q)_{j+l+1}\,(q^{-a-b+m-j-l+1};q)_l\,(q^{a-b+1};q)_l\,(q;q)_l(q^{a-b+j+l+1};q)_l}\Bigg)\end{aligned}$$

We now need to transform the summand into one that has only factors that are indexed by j and entirely independent of l, and factors indexed by l that have no other dependence on l. To this end, we apply property 3.7 to the factors $(q^{a-b+1};q)_{j+1+l}$, $(q^{a+b-m};q)_{j+l}$, $(q^{1-m+a}/t;q)_{j+l}$, and $(q^{a+1};q)_{j+1+l}$, obtaining

$$\begin{aligned}F_m &= \sum_{\substack{0\le j<\infty\\0\le l<\infty}} \Bigg(\frac{q^{j+b}\,(1-t)\,(q^{a-b+1};q)_{j+1}\,(q^{a+b-m};q)_j}{(q^{1-m+a}/t;q)_j\,(q^{a+1};q)_{j+1}}\\ &\times \frac{q^{l(a+b-m)}\,(q^{a-b+2+j};q)_l\,(q^{a+b-m+j};q)_l\,(tq^{m-b+1};q)_l\,(q^{-b};q)_l\,(q^{j+1};q)_l\,(q^{a-b+j+2};q)_{2l}(q^{a-b+j+1};q)_l}{t^l\,(q^{1-m+a+j}/t;q)_l\,(q^{a+j+2};q)_l\,(q^{a-b+1};q)_l\,(q;q)_l\,(q^{a+b-m+j};q)_l\,(q^{a-b+j+1};q)_{2l}\,(q^{a-b+j+2};q)_l}\Bigg)\end{aligned}$$

The additional equivalence

$$\frac{(q^{a-b+j+2};q)_{2l}}{(q^{a-b+j+1};q)_{2l}} = \frac{(q^{1/2(a-b+j+3)};q)_l\,(-q^{1/2(a-b+j+3)};q)_l}{(q^{1/2(a-b+j+1)};q)_l\,(-q^{1/2(a-b+j+1)};q)_l}$$

puts the sum into a form in which all the q-shifted factorials are indexed as desired, yielding our final expression

$$\begin{aligned}F_m &= \sum_{0\le j<\infty} \Bigg(\frac{q^{j+b}\,(1-t)\,(q^{a-b+1};q)_{j+1}\,(q^{a+b-m};q)_j}{(q^{1-m+a}/t;q)_j\,(q^{a+1};q)_{j+1}}\\ &\times \sum_{0\le l<\infty}\left(\frac{q^{a+b-m}}{t}\right)^l \frac{(q^{-b};q)_l(tq^{m-b+1};q)_l(q^{j+1};q)_l(q^{a-b+1+j};q)_l(q^{1/2(a-b+j+3)};q)_l(-q^{1/2(a-b+j+3)};q)_l}{(q^{1-m+a+j}/t;q)_l(q^{a+j+2};q)_l(q^{a-b+1};q)_l(q;q)_l(q^{1/2(a-b+j+1)};q)_l(-q^{1/2(a-b+j+1)};q)_l}\Bigg).\end{aligned}$$

Our efforts are now rewarded by a rather pleasing discovery that the inner sum may be evaluated by means of the basic summation formula

$$
{}_6\phi_5 \left[\begin{matrix} a,\, qa^{1/2},\, -qa^{1/2},\, b,\, c,\, q^{-n} \\ a^{1/2},\, -a^{1/2},\, aq/b,\, aq/c,\, aq^{n+1} \end{matrix} ; q, \frac{aq^{n+1}}{bc} \right] = \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n} .
$$

In fact, if we let $a = q^{a-b+j+1}$, $b = t\, q^{m-b+1}$, and $c = q^{j+1}$ the q-shifted factorials appearing in the inner sum precisely fit the pattern needed for an application of this remarkable identity. This reduces F_m to a single sum and permits the following series of reductions:

$$
\begin{aligned}
F_m &= \sum_{0 \le j < \infty} \left(\frac{q^{j+b}\,(1-t)\,(q^{a-b+1};q)_{j+1}\,(q^{a+b-m};q)_j}{(q^{1-m+a}/t;q)_j\,(q^{a+1};q)_{j+1}} \cdot \frac{(q^{a-b+2+j};q)_b\,(q^{-m+a}/t;q)_b}{(q^{1-m+a+j}/t;q)_b\,(q^{a-b+1};q)_b} \right) \\
&= \frac{q^b\,(1-t)\,(q^{-m+a}/t;q)_b}{(q^{a-b+1};q)_b} \sum_{0 \le j < \infty} \frac{q^j\,(q^{a-b+1};q)_{j+1}\,(q^{a+b-m};q)_j\,(q^{a-b+2+j};q)_b}{(q^{1-m+a}/t;q)_j\,(q^{a+1};q)_{j+1}\,(q^{1-m+a+j}/t;q)_b} \\
&= \frac{q^b\,(1-t)\,(q^{-m+a}/t;q)_b}{(q^{a-b+1};q)_b} \sum_{0 \le j < \infty} \frac{q^j\,(q^{a-b+1};q)_{b+1+j}\,(q^{a+b-m};q)_j}{(q^{1-m+a}/t;q)_{b+j}\,(q^{a+1};q)_{j+1}} \\
&= \frac{q^b\,(1-t)\,(q^{-m+a}/t;q)_b\,(q^{a-b+1};q)_{b+1}}{(q^{a-b+1};q)_b\,(q^{1-m+a}/t;q)_b} \sum_{0 \le j < \infty} \frac{q^j\,(q^{a-b+b+2};q)_j\,(q^{a+b-m};q)_j}{(q^{1-m+a+b}/t;q)_j\,(q^{a+1};q)_{j+1}} \\
&= \frac{q^b\,(1-t)\,(q^{-m+a}/t;q)_b\,(q^{a-b+1};q)_{b+1}}{(q^{a-b+1};q)_b\,(q^{1-m+a}/t;q)_b\,(1-q^{a+1})} \sum_{0 \le j < \infty} \frac{q^j\,(q^{a+b-m};q)_j}{(q^{1-m+a+b}/t;q)_j}
\end{aligned}
$$

Now this last sum can be easily evaluated using identity 2.6 with $a = q$, $q^{-n} = q^{-(m-a-b)}$, and $c = q^{-m+a+b+1}/t$.

$$
\begin{aligned}
F_m &= \frac{q^b\,(1-t)\,(q^{-m+a}/t;q)_b\,(q^{a-b+1};q)_{b+1}}{(q^{a-b+1};q)_b\,(q^{1-m+a}/t;q)_b\,(1-q^{a+1})} \cdot \frac{q^{m-a-b}\,(q^{a+b-m}/t;q)_{m-a-b}}{(q^{-m+a+b+1}/t;q)_{m-a-b}} \\
&= \frac{q^{m-a}\,(1-t)\,(1-q^{-m+a}/t)}{(1-1/t)} \\
&= 1 - tq^{m-a}
\end{aligned}
$$

This establishes 3.14 and completes the proof of Proposition 3.2.

REFERENCES

[1] A. Garsia and M. Haiman, *Some bigraded S_n-modules and the Macdonald q,t-Kostka coefficients*, Electronic Journal of Algebraic Combinatorics, Foata Festschrift, Paper R24, (web site http://ejc.math.gatech.edu:8080/Journal/journalhome.html).

[2] A. M. Garsia and G. Tessler, *Plethystic Formulas for the Macdonald q,t-Kostka coefficients*, Advances in Mathematics, (to appear).

[3] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge U. press, Cambridge (1990).

[4] A. Kirillov and M. Noumi, *Raising operators for Macdonald Polynomials*, (preprint).

[5] F. Knop, *Integrality of Two Variable Kostka Functions*, (preprint).

[6] F. Knop, *Symmetric and non-symmetric Quantum Capelli Polynomials*, (preprint).

[7] L. Lapointe and L. Vinet, *A short proof of the integrality of the Macdonald (q,t)-Kostka coefficients*, Centre de Recherches Mathematiques (preprint #2360).

[8] L. Lapointe and L. Vinet, *Exact operator solution of the Calogero-Sutherland model*, Centre de Recherches Mathematiques (preprint #2368).

[8] L. Lapointe and L. Vinet, *Exact operator solution of the Calogero-Sutherland model*, Centre de Recherches Mathematiques (preprint #2368).

[9] A. Lascoux and M. P. Shützenberger, *Symmetry and Flag manifolds*, Invariant Theory, Springer L.N. 996 (1983) 118-144.

[10] I. G. Macdonald, *A new class of symmetric functions*, Actes du 20^e Séminaire Lotharingien, Publ. I.R.M.A. Strasbourg, (1988) 131-171.

[11] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Second Edition, Clarendon Press, Oxford (1995).

[12] I. G. Macdonald, *Affine Hecke Algebras and orthogonal Polynomials*, Séminaire Bourbaki, (1995) # 797.

[13] J. Morse, *Symmetric Functions and Basic Hypergeometric Series*, UCSD Doctoral Thesis.

[14] S. Sahi, *Interpolation and integrality for Macdonald's Polynomials*, (preprint)