VARIOUS REPRESENTATIONS OF THE GENERALIZED KOSTKA POLYNOMIALS

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Dedicated to George Andrews on the occasion of his sixtieth birthday

Abstract. The generalized Kostka polynomials $K_{\lambda R}(q)$ are labeled by a partition λ and a sequence of rectangles R. They are q-analogues of multiplicities of the finite-dimensional irreducible representation $W(\lambda)$ of $\mathfrak{gl}_n$ with highest weight λ in the tensor product $W(R_1)\otimes\cdots\otimes W(R_L)$. We review several representations of the generalized Kostka polynomials, such as the charge, path space, quasi-particle and bosonic representation. In addition we describe a bijection between Littlewood–Richardson tableaux and rigged configurations, and sketch a proof that it preserves the appropriate statistics. This proves in particular the equality of the quasi-particle and charge representation of the generalized Kostka polynomials.

1. Introduction

The Kostka number $K_{\lambda\mu}$ labeled by two partitions $\lambda=(\lambda_1,\dots,\lambda_n)$ and $\mu=(\mu_1,\dots,\mu_L)$ is the multiplicity of the finite-dimensional irreducible representation $W(\lambda)$ of $\mathfrak{gl}_n$ with highest weight λ in the tensor product $W(\mu_1)\otimes\cdots\otimes W(\mu_L)$, that is

$$K_{\lambda\mu}=[W(\lambda):W(\mu_1)\otimes\cdots\otimes W(\mu_L)].$$

This multiplicity is equal to the cardinality of the set of column-strict Young tableaux of shape λ and content μ. There is a natural q-deformation of the Kostka numbers given by the Kostka polynomials which are formally defined as the connection coefficients between the Schur and the Hall–Littlewood polynomials [24].

Date: December 1998.

1991 *Mathematics Subject Classification.* Primary 05A19, 05A15; Secondary 81R50, 82B23.

Key words and phrases. Generalized Kostka polynomials, charge, path space, Littlewood–Richardson tableaux, rigged configurations.

* Supported by the "Stichting Fundamenteel Onderzoek der Materie".

† Partially supported by NSF grant DMS-9800941.

There are several explicit expressions for the Kostka polynomials. Lascoux and Schützenberger [20] express them as the generating function of column-strict Young tableaux with a charge statistic. They show that the set of column-strict Young tableaux of fixed content has the structure of a graded poset with the covering relation given by the cocyclage. In their study of the XXX model using Bethe Ansatz techniques, Kirillov and Reshetikhin [16] obtained an expression for the Kostka polynomials in terms of rigged configurations. Rigged configurations index the solutions of the Bethe Ansatz equations; they are sequences of partitions obeying certain conditions together with quantum numbers or riggings labeling the parts of the partitions. This representation is of interest from the physics perspective as it reflects the quasi-particle content of the underlying statistical mechanical model [13, 14]. A third representation in terms of path spaces was given by Nakayashiki and Yamada [25], and in a slightly different formulation using the plactic monoid by Lascoux, Leclerc and Thibon [22]. Path spaces first occurred in the corner transfer matrix study of exactly solvable lattice models (see for example [1]) and are closely related to the crystal theory of Kashiwara [10]. The statistic in this case is given by the energy function on paths. We will refer to these three representations as charge, quasi-particle and path space representation, respectively.

Recently certain generalizations of the Kostka polynomials were introduced and studied [18, 28, 30, 31, 32, 33]. These generalized Kostka polynomials $K_{\lambda R}(q)$ are labeled by a partition λ and a sequence of rectangles $R = (R_1, \dots, R_L)$, that is, each $R_i = (\eta_i^{\mu_i})$ is a partition of rectangular shape. They are q-analogues of the multiplicity

$$K_{\lambda R}(1) = [W(\lambda) : W(R_1) \otimes \cdots \otimes W(R_L)], \tag{1}$$

and when all R_i are single rows (in which case $R_i = (\eta_i)$), the generalized Kostka polynomial reduces to the Kostka polynomial $K_{\lambda\eta}(q)$. The multiplicity $K_{\lambda R}(1)$ is equal to the cardinality of the set of Littlewood–Richardson tableaux [5].

Akin to the Kostka polynomials, the generalized Kostka polynomials have a charge, path space and quasi-particle representation. We will discuss these representations in Sections 2, 3 and 5, respectively. A fourth representation as the sum of q-supernomials over the A_n Weyl group is given in Section 4. q-Supernomials extend the q-binomial coefficients and correspond to the generating function of unrestricted paths. Because most formulas of the form of this fourth representation can be interpreted as (finitizations of) characters or branching functions of bosonic algebras, they are referred to as bosonic representations.

The generalized Kostka polynomials also arise as Poincaré polynomials of isotypic components of graded $GL(n)$-modules supported in the closure of a nilpotent conjugacy class [33]. This yields in particular another tableau formula for the generalized Kostka polynomials in terms of catabolizable tableaux [31, Proposition 19]. These connections are however beyond the scope of this paper.

The relations between the different representations of the generalized Kostka polynomials discussed here have been established in various papers. The equality between the bosonic and path space representation was shown in [28], the equality between the path space and the charge representation was established in [28, 32], and finally the equivalence of the charge and quasi-particle representation was recently proven in [17]. In [17] a bijection between the set of Littlewood–Richardson tableaux and rigged configurations is given, which is reviewed in Section 6. This bijection preserves the statistics. A sketch of the proof of this property is given in Section 7. The equality between the quasi-particle and the bosonic representation can be viewed as a Rogers–Ramanujan-type identity. We conclude in Section 8 with a discussion of some open problems related to the generalized Kostka polynomials.

2. Charge representation

The multiplicity $K_{\lambda R}(1)$ as defined in (1) is equal to the cardinality of the set of Littlewood–Richardson tableaux. There are several ways to define LR tableaux. Here we define the set $\mathrm{CLR}(\lambda; R)$ where "C" indicates a column labeling. Later we will also need the set of row LR tableaux denoted by $\mathrm{RLR}(\lambda; R)$. For a given sequence of rectangles $R = (R_1, \ldots, R_L)$ define the standard tableaux Z_i ($1 \le i \le L$) of shape $R_i = (\eta_i^{\mu_i})$ by inserting the numbers

$$(c-1)\mu_i + \sum_{j=1}^{i-1} |R_i| < k \le c\mu_i + \sum_{j=1}^{i-1} |R_i|$$

into the c-th column of R_i. For example, for $R = ((2,2),(3,3))$ we have, using the English convention for tableaux,

$$Z_1 = \begin{matrix} 1 & 3 \\ 2 & 4 \end{matrix} \quad \text{and} \quad Z_2 = \begin{matrix} 5 & 7 & 9 \\ 6 & 8 & 10 \end{matrix}.$$

This means that Z_i is a standard tableau over the alphabet $B_i = \{|R_1| + \cdots + |R_{i-1}| + 1 < \cdots < |R_1| + \cdots + |R_i|\}$. For a tableau T denote by $T|_B$ the restriction of T to the alphabet B. The row-reading word of a skew tableau T is given by $\mathrm{word}(T) = \cdots w_2 w_1$ where w_i is the word of the i-th row of T. Denote by $P(w)$ the Schensted P-tableau of the word w and define $P(T) := P(\mathrm{word}(T))$. Finally denote the set of all standard tableaux of shape λ by $\mathrm{ST}(\lambda)$. Then the set $\mathrm{CLR}(\lambda; R)$ is defined as

$$\mathrm{CLR}(\lambda; R) = \{T \in \mathrm{ST}(\lambda) | P(T|_{B_i}) = Z_i\}.$$

Example 1. Let $\lambda = (3, 2)$ and $R = ((1), (2, 2))$. Then

$$T = \begin{matrix} 1 & 2 & 4 \\ 3 & 5 & \end{matrix}$$

is in $\mathrm{CLR}(\lambda; R)$ since $P(T|_{B_1}) = 1$ and $P(T|_{B_2}) = \begin{matrix} 2 & 4 \\ 3 & 5 \end{matrix}$. On the other hand $T = \begin{matrix} 1 & 3 & 5 \\ 2 & 4 & \end{matrix}$ is not an LR tableau with respect to R since $P(T|_{B_2}) = \begin{matrix} 2 & 3 & 5 \\ 4 & & \end{matrix}$.

It was shown in [28, Section 6] and [30] that the set $\mathrm{CLR}(R) = \cup_\lambda \mathrm{CLR}(\lambda; R)$ has the structure of a graded poset with covering relation given by the R-cocyclage and grading function given by the generalized charge, denoted c_R. The generalized Kostka polynomial is the generating function of LR tableaux with the charge statistic [28, 30]

$$K_{\lambda R}(q) = \sum_{T \in \mathrm{CLR}(\lambda; R)} q^{c_R(T)}. \tag{2}$$

This extends the charge representation of the Kostka polynomial $K_{\lambda\eta}(q)$ of Lascoux and Schützenberger [20, 21].

The charge of an LR tableau can be given explicitly. To this end we need to introduce bijections on the set of LR tableaux which correspond to a permutation of the rectangles in R. Define by $s_p R$ the sequence of rectangles obtained from R by exchanging R_p and R_{p+1}. Furthermore, denote by $\mathrm{ev} : \mathrm{ST}(\lambda) \to \mathrm{ST}(\lambda)$ Schützenberger's evacuation involution [29]. This involution restricts to a bijection $\mathrm{ev} : \mathrm{CLR}(\lambda; R) \to \mathrm{CLR}(\lambda; R^{\mathrm{ev}})$ where $R^{\mathrm{ev}} = (R_L, \ldots, R_1)$.

Definition-Proposition 1. For $1 \le p \le L - 1$ there are unique bijections $\sigma_p : \mathrm{CLR}(\lambda; R) \to \mathrm{CLR}(\lambda; s_p R)$ satisfying the following properties:

1. If $p < L - 1$ then σ_p commutes with restriction to the initial interval $B_1 \cup \cdots \cup B_{L-1}$ where B_i are as in the definition of $\mathrm{CLR}(\lambda; R)$.

2. For $p = L-1$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{CLR}(\lambda; R) & \xrightarrow{\sigma_p} & \mathrm{CLR}(\lambda; s_p R) \\ {\scriptstyle \mathrm{ev}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\ \mathrm{CLR}(\lambda; R^{\mathrm{ev}}) & \xrightarrow[\sigma_1]{} & \mathrm{CLR}(\lambda; s_1(R^{\mathrm{ev}})) \end{array}$$

Proof. One may reduce to the case $p = L-1$ using 1, then to the case $p = 1$ using 2, and then to the case $p = 1$ and $L = 2$ using 1 again. In this case the sets of tableaux are all empty or all singletons by [30, Prop. 33] and the result holds trivially. ☐

Then the generalized charge on $\mathrm{CLR}(R)$ is given by [30, 28]

$$c_R(T) = \frac{1}{L!} \sum_{\sigma \in S_L} \sum_{i=1}^{L-1} (L-i) d_{i,\sigma R}(\sigma T). \tag{3}$$

Here $\sigma = \sigma_{p_1} \circ \cdots \circ \sigma_{p_k}$ is an element of the symmetric group, $d_{i,R}(T) = d_{R_i,R_{i+1}}(\mathrm{word}(T)|_{B_i \cup B_{i+1}})$ where B_i is the alphabet corresponding to R_i in the definition of $\mathrm{CLR}(\lambda; R)$ and $d_{R_1,R_2}(w)$ is the number of cells of the shape $P(w)$ that lie in columns strictly to the right of the $\max(\eta_1, \eta_2)$-th column.

3. Path space representation

It was shown in [28, 32] that the generalized Kostka polynomials have a path space representation. Denote by $\mathcal{B}_\lambda$ the set of column-strict Young tableaux of shape λ over the alphabet $\{1, 2, \ldots, n\}$. Let us restrict here to partitions with at most $n-1$ parts. The partition λ can be identified with a weight $\Lambda = \Lambda_{\lambda_1^t} + \Lambda_{\lambda_2^t} + \cdots$ where λ^t denotes the transpose of λ and Λ_i are the fundamental weights of A_{n-1}. Then the set $\mathcal{B}_\lambda$ is the crystal base of the irreducible integral highest weight module of highest weight Λ over the quantized universal enveloping algebra $U_q(A_{n-1})$ [11]. The action of the raising and lowering operators e_i and f_i on $\mathcal{B}_\lambda$ for $i \in I = \{1, 2, \ldots, n\}$ are also given in [11] which are compatible with the tensor product structure. There is an inclusion $U_q(A_{n-1}) \subset U'_q(A_{n-1}^{(1)})$ where $U'_q(A_{n-1}^{(1)})$ is the universal enveloping algebra of the derived algebra $A'^{(1)}_{n-1}$ [9]. The weight lattice of $A'^{(1)}_{n-1}$ is the $\mathbb{Z}$-span of the fundamental weights $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}\}$. In ref. [12] the actions of e_i and f_i for $i \in J = \{0, 1, \ldots, n\}$ were defined, and it was shown that for rectangular shapes λ the crystal $\mathcal{B}_\lambda$ is perfect.

Write $\phi_i(b)$ (resp. $\epsilon_i(b)$) for the maximum index m such that $f_i^m(b) \neq 0$ (resp. $e_i^m(b) \neq 0$). Given crystals $\mathcal{B}_1$ and $\mathcal{B}_2$ define the following crystal structure on the tensor product $\mathcal{B}_2 \otimes \mathcal{B}_1$ (which differs from the literature, but is consistent with the Robinson–Schensted–Knuth correspondence). For $b_2 \otimes b_1 \in \mathcal{B}_2 \otimes \mathcal{B}_1$ one defines

$$\phi_i(b_2 \otimes b_1) = \phi_i(b_2) + \max(0, \phi_i(b_1) - \epsilon_i(b_2))$$
$$\epsilon_i(b_2 \otimes b_1) = \epsilon_i(b_1) + \max(0, -\phi_i(b_1) + \epsilon_i(b_2)).$$

When $\phi_i(b_2 \otimes b_1) > 0$ (resp. $\epsilon_i(b_2 \otimes b_1) > 0$) one defines

$$f_i(b_2 \otimes b_1) = \begin{cases} b_2 \otimes f_i(b_1) & \text{if } \phi_i(b_1) > \epsilon_i(b_2) \\ f_i(b_2) \otimes b_1 & \text{if } \phi_i(b_1) \leq \epsilon_i(b_2) \end{cases}$$

and respectively

$$e_i(b_2 \otimes b_1) = \begin{cases} b_2 \otimes e_i(b_1) & \text{if } \phi_i(b_1) \geq \epsilon_i(b_2) \\ e_i(b_2) \otimes b_1 & \text{if } \phi_i(b_1) < \epsilon_i(b_2). \end{cases}$$

For a partition λ and a sequence of rectangles R define the set of unrestricted paths

$$\mathcal{P}_{\lambda R} = \{b_L \otimes \cdots \otimes b_1 | b_i \in \mathcal{B}_{R_i} \text{ and } \sum_{i=1}^{L} \mathrm{content}(b_i) = \lambda\}.$$

Say that $b \in \mathcal{P}_{\lambda R}$ is classically restricted if $e_i(b) = 0$ for all $i \in I$. The set of classically restricted paths is defined as

$$\mathcal{P}^{\mathrm{r}}_{\lambda R} = \{b \in \mathcal{P}_{\lambda R} | b \text{ is classically restricted}\}.$$

There exists an energy function on the set of paths. The definitions here follow [25]. Suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ are perfect crystals of finite-dimensional $U_q(A_{n-1})$-modules; then $\mathcal{B}_2 \otimes \mathcal{B}_1$ is connected. There is an isomorphism $\mathcal{B}_2 \otimes \mathcal{B}_1 \cong \mathcal{B}_1 \otimes \mathcal{B}_2$. This is called the local isomorphism. Let the image of $b_2 \otimes b_1 \in \mathcal{B}_2 \otimes \mathcal{B}_1$ under this isomorphism be denoted $b_1' \otimes b_2'$. Then there is a unique (up to global additive constant) map $H : \mathcal{B}_2 \otimes \mathcal{B}_1 \to \mathbb{Z}$ such that

$$H(e_i(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} -1 & \text{if } i = 0,\ e_0(b_2 \otimes b_1) = e_0 b_2 \otimes b_1 \\ & \text{and } e_0(b_1' \otimes b_2') = e_0 b_1' \otimes b_2' \\ 1 & \text{if } i = 0,\ e_0(b_2 \otimes b_1) = b_2 \otimes e_0 b_1 \\ & \text{and } e_0(b_1' \otimes b_2') = b_1' \otimes e_0 b_2' \\ 0 & \text{otherwise.} \end{cases}$$

This map is called the local energy function.

Let R be a sequence of rectangles. Given $b = b_L \otimes \cdots \otimes b_1$ with $b_i \in \mathcal{B}_{R_i}$, denote by $b_j^{(i+1)}$ the $(i+1)$-th tensor factor in the image of b

under the composition of local isomorphisms that switch $\mathcal{B}_{R_j}$ with $\mathcal{B}_{R_k}$ as k goes from $j-1$ down to $i+1$. Then define the energy function

$$E(b) = \sum_{1 \leq i < j \leq L} H(b_j^{(i+1)} \otimes b_i).$$

It was shown in [32] that the generalized Kostka polynomial can be expressed as

$$K_{\lambda R}(q) = \sum_{b \in \mathcal{P}^{\mathrm{r}}_{\lambda R}} q^{E(b)}. \tag{4}$$

This representation extends the path space representation of the Kostka polynomials of Nakayashiki and Yamada [25]. A slightly different energy function, more in the spirit of (3), was given in [28, 32] extending the representation of [22] in the Kostka case.

4. Bosonic representation

Similar to (4) of the previous section one may define the generating function of unrestricted paths as

$$S_{\lambda R}(q) = \sum_{b \in \mathcal{P}_{\lambda R}} q^{E(b)}. \tag{5}$$

For R a sequence of single rows or single columns these polynomials were studied in [3, 4, 7, 15]. For an arbitrary sequence of rectangles the polynomials (5) were introduced in [28] where they were called supernomials. The supernomials can be expressed in terms of the generalized Kostka polynomials and the Kostka numbers [28, Theorem 7.2]

$$S_{\lambda R}(q) = \sum_{\eta} K_{\eta\lambda} K_{\eta R}(q),$$

where the sum is over all partitions η of $|\lambda|$. The inverse of this relation yields the bosonic representation of the generalized Kostka polynomials [28, Corollary 7.3]

$$K_{\lambda R}(q) = \sum_{\tau \in S_n} \epsilon(\tau) S_{(\lambda_1+\tau_1-1,\dots,\lambda_n+\tau_n-n)R}(q).$$

5. Quasi-particle representation

This section follows [17, Section 2.2].

Recall that $R = (R_1, \dots, R_L)$ such that R_j has μ_j rows and η_j columns for $1 \leq j \leq L$. For a partition λ denote by λ^t its transpose and set $R^t = (R_1^t, \dots, R_L^t)$. A $(\lambda^t; R^t)$-configuration is a sequence of partitions $\nu = (\nu^{(1)}, \nu^{(2)}, \dots)$ with the size constraints

$$(6) \qquad |\nu^{(k)}| = \sum_{j>k} \lambda_j^t - \sum_{a=1}^{L} \mu_a \max(\eta_a - k, 0)$$

for $k \geq 0$ where by convention $\nu^{(0)}$ is the empty partition. For a partition ρ, define $m_n(\rho)$ as the number of parts equal to n and $Q_n(\rho) = \rho_1^t + \rho_2^t + \cdots + \rho_n^t$, the size of the first n columns of ρ. Let $\xi^{(k)}(R)$ be the partition whose parts are the heights of the rectangles in R of width k. The vacancy numbers for the $(\lambda^t; R^t)$-configuration ν are the numbers (indexed by $k \geq 1$ and $n \geq 0$) defined by

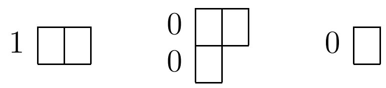
$$(7) \quad P_n^{(k)}(\nu) = Q_n(\nu^{(k-1)}) - 2Q_n(\nu^{(k)}) + Q_n(\nu^{(k+1)}) + Q_n(\xi^{(k)}(R)).$$

In particular $P_0^{(k)}(\nu) = 0$ for all $k \geq 1$. The $(\lambda^t; R^t)$-configuration ν is admissible if $P_n^{(k)}(\nu) \geq 0$ for all $k, n \geq 1$, and the set of admissible $(\lambda^t; R^t)$-configurations is denoted by $\mathrm{C}(\lambda^t; R^t)$. Set

$$\mathrm{cc}(\nu) = \sum_{k,n \geq 1} \alpha_n^{(k)} (\alpha_n^{(k)} - \alpha_n^{(k+1)})$$

where $\alpha_n^{(k)}$ is the size of the n-th column in $\nu^{(k)}$.

Example 2. Let $\lambda = (4, 3, 1)$ and $R = ((2), (2, 2), (1, 1))$. Then $\nu = ((2), (2, 1), (1))$ is a $(\lambda^t; R^t)$-configuration. We have $\xi^{(1)}(R) = (2)$ and $\xi^{(2)}(R) = (2, 1)$. The configuration ν may be represented as



where the vacancy numbers are indicated to the left of each part.

Define the q-binomial as

$$\begin{bmatrix} m+n \\ m \end{bmatrix} = \frac{(q)_{m+n}}{(q)_m (q)_n}$$

for $m, n \in \mathbb{Z}_{\geq 0}$ and zero otherwise where $(q)_m = (1-q)(1-q^2)\cdots(1-q^m)$. Then the quasi-particle expression of the generalized Kostka polynomials can be stated as follows.

Theorem 2 (Theorem 2.10, [17]). *For λ a partition and R a sequence of rectangles*

$$(8) \qquad K_{\lambda R}(q) = \sum_{\nu \in \mathrm{C}(\lambda^t; R^t)} q^{\mathrm{cc}(\nu)} \prod_{k,n \geq 1} \begin{bmatrix} P_n^{(k)}(\nu) + m_n(\nu^{(k)}) \\ m_n(\nu^{(k)}) \end{bmatrix}.$$

Expression (8) can be reformulated as the generating function over rigged configurations. To this end we need to define certain labelings of the rows of the partitions in a configuration. For this purpose one should view a partition as a multiset of positive integers. A rigged partition is by definition a finite multiset of pairs (n, x) where n is a positive integer and x is a nonnegative integer. The pairs (n, x) are referred to as strings; n is referred to as the length or size of the string and x as the label or quantum number of the string. A rigged partition is said to be a rigging of the partition ρ if the multiset consisting of the sizes of the strings, is the partition ρ. So a rigging of ρ is a labeling of the parts of ρ by nonnegative integers, where one identifies labelings that differ only by permuting labels among equal-sized parts of ρ.

A rigging J of the $(\lambda^t; R^t)$-configuration ν is a sequence of riggings of the partitions $\nu^{(k)}$ such that every label x of a part of $\nu^{(k)}$ of size n, satisfies the inequalities

$$0 \leq x \leq P_n^{(k)}(\nu). \tag{9}$$

The pair (ν, J) is called a rigged configuration. The set of riggings of admissible $(\lambda^t; R^t)$-configurations is denoted by $\mathrm{RC}(\lambda^t; R^t)$. Let $(\nu, J)^{(k)}$ be the k-th rigged partition of (ν, J). A string $(n, x) \in (\nu, J)^{(k)}$ is said to be singular if $x = P_n^{(k)}(\nu)$, that is, its label takes on the maximum value.

Observe that the definition of the set $\mathrm{RC}(\lambda^t; R^t)$ is completely insensitive to the order of the rectangles in the sequence R. However the notation involving the sequence R is useful when discussing the bijection $\overline{\phi}_R : \mathrm{CLR}(\lambda; R) \to \mathrm{RC}(\lambda^t; R^t)$ between LR tableaux and rigged configurations as defined in the next section, since the ordering on R is essential in the definition of $\mathrm{CLR}(\lambda; R)$.

The set of rigged configurations is endowed with a natural statistic cc [18, (3.2)] defined by

$$\mathrm{cc}(\nu, J) = \mathrm{cc}(\nu) + \sum_{k,n \geq 1} |J_n^{(k)}|$$

for $(\nu, J) \in \mathrm{RC}(\lambda^t; R^t)$. Here $|\rho|$ is the size of the partition ρ and $J_n^{(k)}$ denotes the partition inside the rectangle of height $m_n(\nu^{(k)})$ and width $P_n^{(k)}(\nu)$ given by the labels of the parts of $\nu^{(k)}$ of size n. Since the q-binomial $\left[{P+m \atop m}\right]$ is the generating function of partitions with at most m parts each not exceeding P, Theorem 2 is equivalent to the following theorem.

Theorem 2′ (Theorem 2.12, [17]). *For λ a partition and R a sequence of rectangles*

$$K_{\lambda R}(q) = \sum_{(\nu,J)\in \mathrm{RC}(\lambda^t;R^t)} q^{\mathrm{cc}(\nu,J)}. \tag{10}$$

A detailed proof of this Theorem is given in [17]. In the following section we describe the bijection $\phi_R : \mathrm{CLR}(\lambda;R) \to \mathrm{RC}(\lambda^t;R^t)$ of [17] and sketch the proof in section 7 that this bijection preserves the statistics, that is $c_R(T) = \mathrm{cc}(\phi_R(T))$.

6. The bijection between LR tableaux and rigged configurations

In this section we define the bijection $\overline{\phi}_R : \mathrm{CLR}(\lambda;R) \to \mathrm{RC}(\lambda^t;R^t)$ between LR tableaux and rigged configurations. The bijection which preserves the statistics is

$$\phi_R = \mathrm{comp} \circ \overline{\phi}_R$$

where comp : $\mathrm{RC}(\lambda;R) \to \mathrm{RC}(\lambda;R)$ complements the rigging labels. That is, for $(\nu,J) \in \mathrm{RC}(\lambda;R)$ a string $(n,x) \in (\nu,J)^{(k)}$ is mapped to $(n, P_n^{(k)}(\nu) - x)$.

The bijection $\overline{\phi}_R$ is defined recursively based on the following two operations on sequences of rectangles $R = (R_1, \dots, R_L)$:

I. Let $R^\wedge$ be the sequence of rectangles obtained from R by splitting off the last column of R_L; formally, $R_j^\wedge = R_j$ for $1 \le j \le L-1$, $R_L^\wedge = ((\eta_L - 1)^{\mu_L})$ and $R_{L+1}^\wedge = (1^{\mu_L})$.

II. If the last rectangle of R is a single column, let $\overline{R}$ be given by removing one cell from the column R_L; $\overline{R}_j = R_j$ for $1 \le j \le L-1$ and $\overline{R}_L = (1^{\mu_L - 1})$.

Remark 1. Given any sequence of rectangles, there is a unique sequence of transformations of the form $R \to R^\wedge$ or $R \to \overline{R}$ resulting in the empty sequence, where $R \to R^\wedge$ is only used when the last rectangle of R has more than one column.

For both transformations on sequences of rectangles, there are natural (injective) maps on the corresponding sets of LR tableaux and rigged configurations. The analogue of transformation I on LR tableaux is the inclusion

$$\imath^\wedge : \mathrm{CLR}(\lambda;R) \to \mathrm{CLR}(\lambda;R^\wedge).$$

When the last rectangle of R is a single column define

$$\mathrm{CLR}(\lambda^-;\overline{R}) = \bigcup_{\rho \lessdot \lambda} \mathrm{CLR}(\rho;\overline{R})$$

where $\rho \lessdot \lambda$ means that $\rho \subset \lambda$ and λ/ρ is a single cell. Define the injective map

$$\begin{aligned} \mathrm{CLR}(\lambda;R) &\to \mathrm{CLR}(\lambda^-;\overline{R}) \\ T &\mapsto T^- \end{aligned}$$

where T^- is the LR tableau obtained by removing the maximum letter from T. This corresponds to transform II.

The analogue of transform I for rigged configurations is given by the map

$$\jmath^\wedge : \mathrm{RC}(\lambda^t;R^t) \to \mathrm{RC}(\lambda^t;(R^\wedge)^t)$$

by declaring that $\jmath^\wedge(\nu,J)$ is obtained from $(\nu,J) \in \mathrm{RC}(\lambda^t;R^t)$ by adding a singular string of length μ_L to each of the first $\eta_L - 1$ rigged partitions. Note that $\jmath^\wedge$ is the identity map if R_L is a single column. It is shown in [17, Lemma 3.9] that $\jmath^\wedge$ is a well-defined injection that preserves the vacancy numbers of the underlying configurations. An example of the map $\jmath^\wedge$ is the first transformation of Table 1 in Appendix A.

Suppose the last rectangle of R is a single column. Define the set

$$\mathrm{RC}(\lambda^{-t};\overline{R}^t) = \bigcup_{\rho \lessdot \lambda} \mathrm{RC}(\rho^t;\overline{R}^t).$$

The key algorithm on rigged configurations is given by the map

$$\overline{\delta} : \mathrm{RC}(\lambda^t;R^t) \to \mathrm{RC}(\lambda^{-t};\overline{R}^t),$$

defined as follows. Let $(\nu,J) \in \mathrm{RC}(\lambda^t;R^t)$. Define $\overline{\ell}^{(0)} = \mu_L$. By induction select the singular string in $(\nu,J)^{(k)}$ whose length $\overline{\ell}^{(k)}$ is minimal such that $\overline{\ell}^{(k-1)} \le \overline{\ell}^{(k)}$. Let $\overline{\mathrm{rk}}(\nu,J)$ denote the smallest k for which no such string exists, and set $\overline{\ell}^{(k)} = \infty$ for $k \ge \overline{\mathrm{rk}}(\nu,J)$. Then $\overline{\delta}(\nu,J)$ is obtained from (ν,J) by shortening each of the selected singular strings by one, changing their labels so that they remain singular, and leaving the other strings unchanged. It is shown in [17, Proposition 3.12] that the map $\overline{\delta}$ is a well-defined injection such that $\overline{\delta}(\nu,J) \in \mathrm{RC}(\rho^t;\overline{R}^t)$ where ρ is obtained from λ by removing the corner cell in the column of index $\overline{\mathrm{rk}}(\nu,J)$.

Example 3. We continue Example 2 and consider the rigged configuration (ν, J)

$$1\ \begin{array}{|c|c|}\hline & \\ \hline\end{array}\ 1 \qquad \begin{array}{r|c|c|l} \cline{2-3} 0 & & & 0 \\ \cline{2-3} 0 & & \multicolumn{1}{|l}{0} & \\ \cline{2-2}\end{array} \qquad 0\ \begin{array}{|c|}\hline \\ \hline\end{array}\ 0$$

where the vacancy numbers are indicated to the left and the riggings to the right of each part. By definition $\overline{\ell}^{(0)} = \mu_2 = 2$. Now we need to pick a singular string in $\nu^{(1)}$ of length $\overline{\ell}^{(1)}$ minimal such that $\overline{\ell}^{(1)} \geq 2$. The part of length two in $\nu^{(1)}$ is singular; hence $\overline{\ell}^{(1)} = 2$. Next select a singular string in $\nu^{(2)}$ of length $\overline{\ell}^{(2)}$ minimal such that $\overline{\ell}^{(2)} \geq \overline{\ell}^{(1)}$. The part of length two in $\nu^{(2)}$ is singular; hence $\overline{\ell}^{(2)} = 2$. There is no singular string of length greater or equal to two in $\nu^{(3)}$. This implies $\overline{\mathrm{rk}}(\nu, J) = 3$. The selected strings are indicated by $*$ in the following diagram

$$1\ \begin{array}{|c|c|}\hline & * \\ \hline\end{array}\ 1 \qquad \begin{array}{r|c|c|l} \cline{2-3} 0 & & * & 0 \\ \cline{2-3} 0 & & \multicolumn{1}{|l}{0} & \\ \cline{2-2}\end{array} \qquad 0\ \begin{array}{|c|}\hline \\ \hline\end{array}\ 0\ .$$

Now $\overline{\delta}(\nu, J)$ is obtained by shortening the selected strings keeping them singular so that $\overline{\delta}(\nu, J)$ is represented by

$$1\ \begin{array}{|c|}\hline \\ \hline\end{array}\ 1 \qquad \begin{array}{r|c|l} \cline{2-2} 0 & & 0 \\ \cline{2-2} 0 & & 0 \\ \cline{2-2}\end{array} \qquad 0\ \begin{array}{|c|}\hline \\ \hline\end{array}\ 0$$

which is indeed a $(\overline{\lambda}^t; \overline{R}^t)$-configuration with $\overline{\lambda} = (4, 2, 1)$ and $\overline{R} = ((2), (2, 2), (1))$.

The bijection $\overline{\phi}_R : \mathrm{CLR}(\lambda; R) \to \mathrm{RC}(\lambda^t; R^t)$ is defined inductively based on Remark 1.

Definition-Proposition 3. There is a unique family of bijections $\overline{\phi}_R : \mathrm{CLR}(\lambda; R) \to \mathrm{RC}(\lambda^t; R^t)$ indexed by R, such that:

1. If the last rectangle of R is a single column, then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{CLR}(\lambda; R) \arrow[r, "-"] \arrow[d, "\overline{\phi}_R"'] & \mathrm{CLR}(\lambda^-; \overline{R}) \arrow[d, "\overline{\phi}_{\overline{R}}"] \\
\mathrm{RC}(\lambda^t; R^t) \arrow[r, "\overline{\delta}"'] & \mathrm{RC}(\lambda^{-t}; \overline{R}^t).
\end{tikzcd}$$

2. The following diagram commutes:

$$\begin{tikzcd} \mathrm{CLR}(\lambda;R) \arrow[r, "\imath^\wedge"] \arrow[d, "\overline{\phi}_R"'] & \mathrm{CLR}(\lambda;R^\wedge) \arrow[d, "\overline{\phi}_{R^\wedge}"] \\ \mathrm{RC}(\lambda^t;R^t) \arrow[r, "\jmath^\wedge"'] & \mathrm{RC}(\lambda^t;R^{\wedge t}). \end{tikzcd}$$

The proof of this Definition-Proposition is given in [17]. An example illustrating this bijection is given in Appendix A.

7. Sketch of the proof of theorem $2'$

For the proof of Theorem $2'$ it remains to show that the bijection ϕ_R preserves the statistics.

Lemma 4. *Let $T \in \mathrm{CLR}(\lambda;R)$. Then $c_R(T) = \mathrm{cc}(\phi_R(T))$.*

The proof of this lemma is given in full length in [17]. Here we only sketch the main ideas.

There are further important maps on the sets of LR tableaux and rigged configurations. The maps which play a central rôle in the proof are the transposition maps on LR tableaux and rigged configurations and a statistic preserving embedding on LR tableaux. Let us briefly review their definitions and some of their properties.

Denote by $\mathrm{tr} : \mathrm{ST}(\lambda) \to \mathrm{ST}(\lambda^t)$ the ordinary transposition of standard tableaux. Analogous to the definition of $\mathrm{CLR}(\lambda;R)$ let us define the set

$$\mathrm{RLR}(\lambda;R) = \{T \in \mathrm{ST}(\lambda) | P(T|_{B_i}) = Z'_i\}$$

where Z'_i is the standard tableau of shape $R_i = (\eta_i^{\mu_i})$ obtained by inserting the numbers

$$(r-1)\eta_i + \sum_{j=1}^{i-1} |R_i| < k \le r\eta_i + \sum_{j=1}^{i-1} |R_i|$$

into the r-th row of R_i. There is a bijection $\gamma_R : \mathrm{CLR}(\lambda;R) \to \mathrm{RLR}(\lambda;R)$ given by relabeling as follows. Suppose the letter j occurs in Z_i in cell s. Then, to obtain $\gamma_R(T)$ from $T \in \mathrm{CLR}(\lambda;R)$, replace the letter j in T by the letter occurring in cell s of Z'_i for all letters j. The transpose map tr restricts to a bijection $\mathrm{tr} : \mathrm{CLR}(\lambda;R) \to \mathrm{RLR}(\lambda^t;R^t)$. Then the LR-transpose

$$\mathrm{tr}_{\mathrm{LR}} : \mathrm{CLR}(\lambda;R) \to \mathrm{CLR}(\lambda^t;R^t)$$

is defined as $\mathrm{tr}_{\mathrm{LR}} := \mathrm{tr} \circ \gamma_R$.

An analogous RC-transpose bijection exists for the set of rigged configurations denoted by $\mathrm{tr}_{\mathrm{RC}} : \mathrm{RC}(\lambda^t; R^t) \to \mathrm{RC}(\lambda; R)$, which was described in [18, Section 9]. Let $(\nu, J) \in \mathrm{RC}(\lambda^t; R^t)$ and let ν have the associated matrix m with entries m_{ij} as in [18, (9.2)]

$$m_{ij} = \alpha_j^{(i-1)} - \alpha_j^{(i)}$$

for $i, j \geq 1$, where $\alpha_j^{(i)}$ is the size of the j-th column of the partition $\nu^{(i)}$, recalling that $\nu^{(0)}$ is defined to be the empty partition. The configuration ν^t in $(\nu^t, J^t) = \mathrm{tr}_{\mathrm{RC}}(\nu, J)$ is defined by its associated matrix m^t given by

$$m_{ij}^t = -m_{ji} + \chi((i,j) \in \lambda) - \sum_{a=1}^{L} \chi((i,j) \in R_a)$$

for all $i, j \geq 1$. Here $(i, j) \in \lambda$ means that the cell (i, j) is in the Ferrers diagram of the partition λ with i specifying the row and j the column, and $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. Recall that the rigging J is determined by partitions $J_n^{(k)}$ inside the rectangle of height $m_n(\nu^{(k)})$ and width $P_n^{(k)}(\nu)$ given by the labels of the parts of $\nu^{(k)}$ of size n. The partition $J_k^{t(n)}$ corresponding to $(\nu^t, J^t) = \mathrm{tr}_{\mathrm{RC}}(\nu, J)$ is defined as the transpose of the complementary partition to $J_n^{(k)}$ in the rectangle of height $m_n(\nu^{(k)})$ and width $P_n^{(k)}(\nu)$.

It is shown in [17, Theorem 7.1] that the diagram

$$\begin{array}{ccc} \mathrm{CLR}(\lambda; R) & \xrightarrow{\mathrm{tr}_{\mathrm{LR}}} & \mathrm{CLR}(\lambda^t; R^t) \\ {\scriptstyle \phi_R}\big\downarrow & & \big\downarrow{\scriptstyle \phi_{R^t}} \\ \mathrm{RC}(\lambda^t; R^t) & \xrightarrow[\mathrm{tr}_{\mathrm{RC}}]{} & \mathrm{RC}(\lambda; R) \end{array} \tag{11}$$

commutes.

Let $\mathrm{rows}(R)$ be obtained from the sequence of rectangles R by slicing all the rectangles of R into single rows. In refs. [30, 28] an embedding

$$\theta_R : \mathrm{CLR}(\lambda; R) \to \mathrm{CLR}(\lambda; \mathrm{rows}(R))$$

was defined and it was shown that θ_R preserves the charge c_R of LR tableaux. This embedding stems from an analogous embedding on column-strict tableaux given by Lascoux and Schützenberger [19, 21]. For rigged configurations, it follows immediately from the definitions that there is an inclusion $\mathrm{RC}(\lambda^t; R^t) \subseteq \mathrm{RC}(\lambda^t; \mathrm{rows}(R)^t)$. It is shown in [17, Theorem 8.3] that the diagram

$$
\begin{tikzcd}
\mathrm{CLR}(\lambda; R) \arrow[r, "\theta_R"] \arrow[d, "\phi_R"'] & \mathrm{CLR}(\lambda; \mathrm{rows}(R)) \arrow[d, "\phi_{\mathrm{rows}(R)}"] \\
\mathrm{RC}(\lambda^t; R^t) \arrow[r, "\text{inclusion}"'] & \mathrm{RC}(\lambda^t; \mathrm{rows}(R)^t)
\end{tikzcd}
\tag{12}
$$

commutes.

Now the proof of Lemma 4 follows directly from (11) and (12). Since the embedding θ_R preserves the statistics one can reduce the proof of Lemma 4 to the case that all rectangles in R are single rows using (12). By (11) it may be assumed that R consists of single columns only. Finally applying (12) again, it is sufficient to establish Lemma 4 for R a sequence of single boxes only. In this case the lemma is verified explicitly [17].

8. Outlook

Let us take this opportunity to discuss some open problems regarding the generalized Kostka polynomials.

It was conjectured in [15, 18] that the generalized Kostka polynomials coincide with special cases of the spin generating functions over ribbon tableaux of Lascoux, Leclerc and Thibon [23]. A proof of this conjecture would add yet another representation of the generalized Kostka polynomials to the list. The spin generating functions over ribbon tableaux are defined for an arbitrary sequence of partitions R, not just a sequence of rectangles.

The algebraic structure underlying the generalized Kostka polynomials is determined by the affine Kac–Moody algebras of type A (compare with Section 3). The crystal theory can be formulated very generally for any affine Kac–Moody algebra. Recently Hatayama et al. [8] conjectured quasi-particle representations for all untwisted affine algebras. It would be interesting to find proofs of these conjectures.

In addition to the sets of unrestricted and classically restricted paths one can also define the set of level-restricted paths by imposing additional conditions using the raising operator e_0. Quasi-particle representations for the generating functions of level-restricted paths have been conjectured (see e.g. [8, 28]). However, proofs only exist in isolated cases (see e.g. [2, 6, 27]). A bosonic representation for level-restricted paths was recently proven in [26].

Finally, it would be interesting to find explicit expressions for the supernomial coefficients as introduced in Section 4. So far, explicit

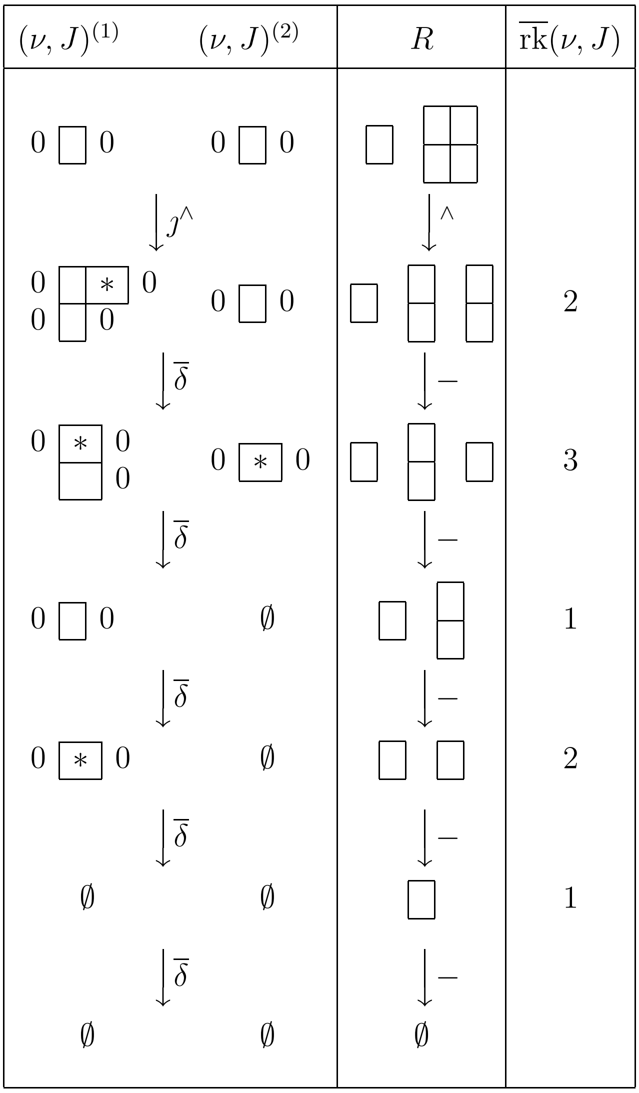


Table 1. Example of the bijection

formulas for the supernomials are only known for type A when R is either a sequence of single rows or single columns.

Appendix A. An example of the bijection

Here we give an example illustrating the bijection of Definition-Proposition 1. Set $\lambda = (3, 2)$ and $R = ((1), (2, 2))$. Then (ν, J) with $\nu = ((1), (1))$ and $J = ((0), (0))$ is a $(\lambda^t; R^t)$-configuration. In Table 1 the sequence of steps under $\jmath^\wedge$ and $\overline{\delta}$ is given. The vacancy numbers are indicated to the left of the corresponding part of the partition and the riggings are given on the right. The selected singular strings in each step are indicated by $*$. The LR tableau which corresponds to (ν, J) under the bijection $\overline{\phi}_R$ can be obtained as follows. First insert $|\lambda| = 5$ into column $\overline{\mathrm{rk}}(\nu, J) = 2$ of the shape λ. This yields

	5	

Next insert 4,3,2,1 into columns 3,1,2,1, respectively, where the second set of numbers are those of the last column in Table 1. This yields the LR tableau

1	2	4
3	5	

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