On a septuple product identity

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For polynomials f and g in the variable n with integer coefficients let us define

$$\Omega(f) = \sum_{n=-\infty}^{\infty} (-1)^n q^{f(n)}$$

and

$$\Omega(f, g) = \sum_{n=-\infty}^{\infty} (-1)^n q^{f(n)} x^{g(n)}.$$

In [1] Farkas and Kra state and prove the following identity

Theorem 1

$$\begin{aligned}
& \prod_{n=1}^{\infty} (1 - q^{2n})^2 (1 - xq^{2n-2})(1 - x^{-1}q^{2n})(1 - x^2 q^{4n-2}) \\
\times \quad & (1 - x^{-2}q^{4n-2})(1 - x^2 q^{4n-4})(1 - x^{-2}q^{4n}) \\
= \quad & \Omega(5n^2 + n)(\Omega(5n^2 + 3n, 5n + 3) + \Omega(5n^2 - 3n, 5n)) \\
- \quad & \Omega(5n^2 + 3n)(\Omega(5n^2 + n, 5n + 2) + \Omega(5n^2 - n, 5n + 1)).
\end{aligned}$$

□

Their proof exploits identities involving theta functions. Foata and Han [2] gave a more elementary deduction of (1) from Jacobi's triple product formula and Watson's quintuple product formula. Here we give a still more elementary derivation just from Jacobi's triple product formula.

Let P denote the product on the of the identity in Theorem 1. First observe that we can rewrite P as

$$\prod_{n=1}^{\infty} (1 - q^{2n})^2 (1 - xq^{2n-2})(1 - x^{-1}q^{2n})(1 - x^2 q^{2n-2})(1 - x^{-2}q^{2n}). \tag{1}$$

We now use Jacobi's triple product formula in the form

$$\prod_{n=1}^{\infty}(1-q^{2n})(1-xq^{2n-2})(1-x^{-1}q^{2n}) = \sum_{k=-\infty}^{\infty}(-1)^k x^k q^{k^2-k}. \tag{2}$$

We obtain (2) by replacing q by q^2 and k by $-k$ in formula (1.1) of [2]. Replacing x by x^2 in (2) gives

$$\prod_{n=1}^{\infty}(1-q^{2n})(1-x^2q^{2n-2})(1-x^{-2}q^{2n}) = \sum_{k=-\infty}^{\infty}(-1)^k x^{2k} q^{k^2-k}. \tag{3}$$

Multiplying (2) by (3) and using (1) gives

$$P = \sum_{k,l=-\infty}^{\infty}(-1)^{k+l}x^{k+2l}q^{k^2-k+l^2-l}.$$

Thus

$$P = \sum_{r=-\infty}^{\infty} a_r(q)x^r$$

where

$$a_r(q) = \sum_{k+2l=r}(-1)^{k+l}q^{k^2-k+l^2-l} = \sum_{l=-\infty}^{\infty}(-1)^{r-l}q^{5l^2-(4r-1)l+r^2-r}.$$

We now evaluate $a_r(q)$ by dividing into cases according to the residue of r modulo 5. If $r = 5m$ then

$$5l^2-(4r-1)l+r^2-r = 5(l-2m)^2+(l-2m)+5m^2-3m$$

and so

$$a_{5m}(q) = (-1)^m q^{5m^2-3m}\sum_{n=-\infty}^{\infty}(-1)^n q^{5n^2+n} = (-1)^m q^{5m^2-3m}\Omega(5n^2+n).$$

If $r = 5m+1$ then

$$5l^2-(4r-1)l+r^2-r = 5(2m-l)^2+3(2m-l)+5m^2-m$$

and so

$$a_{5m+1}(q) = -(-1)^m q^{5m^2-m}\sum_{n=-\infty}^{\infty}(-1)^n q^{5n^2+3n} = -(-1)^m q^{5m^2-m}\Omega(5n^2+3n).$$

If $r = 5m+2$ then

$$5l^2-(4r-1)l+r^2-r = 5(l-2m-1)^2+3(l-2m-1)+5m^2+m$$

and so

$$a_{5m+2}(q) = -(-1)^m q^{5m^2+m} \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+3n} = -(-1)^m q^{5m^2+m} \Omega(5n^2+3n).$$

If $r = 5m+3$ then

$$5l^2 - (4r-1)l + r^2 - r = 5(2m+1-l)^2 + (2m+1-l) + 5m^2 + 3m$$

and so

$$a_{5m+3}(q) = (-1)^m q^{5m^2+3m} \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+n} = (-1)^m q^{5m^2+3m} \Omega(5n^2+n).$$

If $r = 5m+4$ then

$$5l^2 - (4r-1)l + r^2 - r = 5(l-2m-2)^2 + 5(l-2m-2) + 5m^2 + 5m + 2$$

and so

$$a_{5m+4}(q) = (-1)^m q^{5m^2+5m+2} \sum_{n=-\infty}^{\infty} (-1)^n q^{5n^2+5n}.$$

But in this sum the terms for $n = j$ and $n = -j-1$ cancel and so $a_{5m+4}(q) = 0$. It follows that

$$\begin{aligned} P &= \Omega(5n^2+n) \sum_{m=-\infty}^{\infty} (-1)^m (q^{5m^2-3m} x^{5m} + q^{5m^2+3m} x^{5m+3}) \\ &\quad -\Omega(5n^2+3n) \sum_{m=-\infty}^{\infty} (-1)^m (q^{5m^2-m} x^{5m+1} + q^{5m^2+m} x^{5m+2}) \\ &= \Omega(5n^2+n)(\Omega(5n^2-3n, 5n) + \Omega(5n^2+3n, 5n+3)) \\ &\quad -\Omega(5n^2+3n)(\Omega(5n^2-n, 5n+1) + \Omega(5n^2+n, 5n+2)) \end{aligned}$$

which establishes Theorem 1.

Remark After submitting this manuscript, the author received a copy of [3] which contains a proof of an assertion equivalent to Theorem 1, using a similar method.

References

[1] H. M. Farkas & I. Kra, 'On the quintuple product identity', *Proc. Amer. Math. Soc.* **27** (1999), 771–778.

[2] D. Foata & G.-H. Han, 'The triple, quintuple and septuple product identities revisited', *Séminaire Lotharingien de Combinatoire*, B42o (1999), 12 pp.

[3] M. D. Hirschhorn, 'An identity of Ramanujan, and applications', *Contemp. Math.*, to appear.