Rectangular Scott-type Permanents

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Abstract. — Let $x_1, x_2, \ldots, x_n$ be the zeroes of a polynomial $P(x)$ of degree n and $y_1, y_2, \ldots, y_m$ be the zeroes of another polynomial $Q(y)$ of degree m. Our object of study is the permanent $\operatorname{per}(1/(x_i - y_j))_{1\le i\le n,\,1\le j\le m}$, here named "Scott-type" permanent, the case of $P(x) = x^n - 1$ and $Q(y) = y^n + 1$ having been considered by R. F. Scott. We present an efficient approach to determining explicit evaluations of Scott-type permanents, based on generalizations of classical theorems by Cauchy and Borchardt, and of a recent theorem by Lascoux. This continues and extends the work initiated by the first author ("Généralisation de l'identité de Scott sur les permanents," to appear in Linear Algebra Appl.). Our approach enables us to provide numerous closed form evaluations of Scott-type permanents for special choices of the polynomials $P(x)$ and $Q(y)$, including generalizations of all the results from the above mentioned paper and of Scott's permanent itself. For example, we prove that if $P(x) = x^n - 1$ and $Q(y) = y^{2n} + y^n + 1$ then the corresponding Scott-type permanent is equal to $(-1)^{n+1} n!$.

Résumé. — Soient $x_1, x_2, \ldots, x_n$ les zéros d'un polynôme $P(x)$ de degré n et $y_1, y_2, \ldots, y_m$ les zéros d'un autre polynôme $Q(y)$ de degré m. Notre objet d'étude est le permanent $\operatorname{per}(1/(x_i - y_j))_{1\le i\le n,\,1\le j\le m}$, appelé ici permanent de type Scott. Le cas de $P(x) = x^n - 1$ et $Q(y) = y^n + 1$ a été considéré par R. F. Scott. Nous présentons une approche efficace pour déterminer les évaluations explicites des permanents de type Scott, basée sur des généralisations des théorèmes classiques de Cauchy et Borchardt, et d'un théorème récent de Lascoux. La présente étude prolonge le travail du premier auteur ("Généralisation de l'identité de Scott sur les permanents," à apparaître dans Linear Algebra Appl.). Notre approche nous permet de fournir de nombreuses évaluations explicites des permanents de type Scott pour des choix spéciaux des polynômes $P(x)$ et $Q(y)$, y compris des généalisations de tous les résultats de l'article mentionné ci-dessus et du permanent de Scott lui-même. Par exemple, nous prouvons que si $P(x) = x^n - 1$ et $Q(y) = y^{2n} + y^n + 1$ alors le permanent correspondant de type Scott est égal à $(-1)^{n+1} n!$.

1. Introduction

In 1881, Scott [16] stated, without proof, the following result:

Let $x_1, x_2, \ldots, x_n$ be the zeroes of $x^n - 1$ and $y_1, y_2, \ldots, y_n$ be the zeroes of $y^n + 1$. Let A be the $n \times n$ matrix $(1/(x_i - y_j))_{1\le i,j\le n}$. Then

$$\operatorname{per}(A) = \begin{cases} (-1)^{\frac{n-1}{2}} \dfrac{n(1\cdot 3\cdot 5\cdots(n-2))^2}{2^n}, & \textit{if } n \textit{ is odd,} \\ 0, & \textit{if } n \textit{ is even.} \end{cases}$$

In 1978, in his monograph *Permanents*, Minc [13, p. 155] included this result in a list of conjectures on permanents. Since then, several proofs have been given [3, 8, 14, 17], one of which by Minc himself. All of these proofs are heavily based on the fact that the zeroes of the polynomials $x^n - 1$ and $y^n + 1$ can be written in simple explicit terms. Thus, neither of these proofs extends to the more general problem which is the subject of this paper:

> *Given a polynomial $P(x)$ of degree n and a polynomial $Q(y)$ of degree m, let $x_1, x_2, \dots, x_n$ be the zeroes of $P(x)$ and $y_1, y_2, \dots, y_m$ be the zeroes of $Q(y)$. Evaluate the permanent of the $n \times m$ matrix $(1/(x_i - y_j))_{1 \le i \le n,\, 1 \le j \le m}$.*

As usual, the permanent $\operatorname{per}(A)$ of an $n \times m$ matrix A is defined as the sum of all possible products of n coefficients of A chosen such that no two of the coefficients are taken from the same row nor from the same column (see [13, Ch. 1, (1.1)]). Given the assumptions of the above problem, we call the permanent of the matrix $(1/(x_i - y_j))_{1 \le i \le n,\, 1 \le j \le m}$ a *Scott-type permanent*, and denote it by $\operatorname{PER}(P(x), Q(y))$.

In [6], the first author presented a new approach to this type of problem in the case $n = m$, i.e., in the case that both polynomials have the same degree. This approach does not rely at all on explicit analytic forms of zeroes of polynomials. Instead, it makes essential use of recent symmetric functions techniques, in particular of a theorem due to Lascoux [11], which the latter author established in his étude on the square ice model of statistical mechanics.

In the present paper, we are going to extend this approach to arbitrary n and m. This requires extensions of classical theorems of Cauchy and Borchardt (see Theorems (Cauchy+) and (Borchardt+) in Section 2), and an extension of Lascoux's theorem (see Theorem (Lascoux+)). As a result (see Theorem 1), we are able to express any Scott-type permanent as the quotient of a determinant which features complete homogeneous and elementary symmetric functions in the zeroes of the two polynomials, divided by the resultant of the two polynomials. In particular, it follows immediately that any Scott-type permanent is rational in the coefficients of the polynomials $P(x)$ and $Q(y)$.

In Section 5 we apply this result to obtain explicit evaluations of Scott-type permanents in numerous special cases. Amongst others, we provide generalizations of all the results from [6], thus also covering Scott's permanent itself. For the proofs of the results in Section 5, we make use of two particular specializations of our main theorem, Theorem 1, which we derive in Section 3 (see Theorems 3 and 4), and of four determinant evaluations, which we state and establish separately in Section 4.

Finally, we also comment briefly on an alternative approach to the evaluation of Scott-type determinants, due to the first author [5]. It allows to express Scott-type permanents in terms of weighted sums over involutions (see Theorem 2 in Section 2). By combining this result with some of the evaluations in Section 5, we obtain interesting summation theorems, which are presented in Section 6.

2. The general theory

In [6], the main ingredients are theorems by Cauchy, Borchardt, Lascoux, and a lemma on the resultant. Since we intend to extend the approach of [6] to the case of *rectangular* Scott-type permanents (corresponding to polynomials of, possibly, different degrees), we have to first provide the appropriate extensions of these theorems.

Given positive integers m and n and two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$ of variables, we use the following notations:

$$R(X,Y) := \prod_{i=1}^{n}\prod_{j=1}^{m}(x_i - y_j) \quad \text{and} \quad \Delta(X) := \prod_{i<j}(x_i - x_j).$$

$$\left(\frac{1}{x_i - y_j}\right) := \begin{pmatrix} \frac{1}{x_1 - y_1} & \frac{1}{x_1 - y_2} & \cdots & \frac{1}{x_1 - y_m} \\ \frac{1}{x_2 - y_1} & \frac{1}{x_2 - y_2} & \cdots & \frac{1}{x_2 - y_m} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{x_n - y_1} & \frac{1}{x_n - y_2} & \cdots & \frac{1}{x_n - y_m} \end{pmatrix}.$$

We first state a variation on Cauchy's evaluation of his double alternant (*cf.* [4; 15, vol. 1, pp. 342–345]).

Theorem (Cauchy+). — *For $m \geq n$, let*

$$C(X,Y) := \begin{pmatrix} \frac{1}{x_1 - y_1} & \frac{1}{x_1 - y_2} & \cdots & \frac{1}{x_1 - y_m} \\ \frac{1}{x_2 - y_1} & \frac{1}{x_2 - y_2} & \cdots & \frac{1}{x_2 - y_m} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{x_n - y_1} & \frac{1}{x_n - y_2} & \cdots & \frac{1}{x_n - y_m} \\ 1 & 1 & \cdots & 1 \\ y_1 & y_2 & \cdots & y_m \\ y_1^2 & y_2^2 & \cdots & y_m^2 \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{m-n-1} & y_2^{m-n-1} & \cdots & y_m^{m-n-1} \end{pmatrix}.$$

Then

$$\det(C(X,Y)) = (-1)^{n(n-1)/2}\frac{\Delta(X)\Delta(Y)}{R(X,Y)}.$$

Proof. — If $n = m$, this is exactly Cauchy's theorem. The general case can be either established directly, or, it may be observed that the "general" case is in fact *implied* by Cauchy's theorem. To see this, consider the above identity with $n = m$. Given $k < m$, expand both sides as power series in $1/x_{k+1}, \ldots, 1/x_m$, and compare coefficients of $1/x_{k+1}x_{k+2}^2 \cdots x_m^{m-k}$ on both sides. □

Next we state the required extension of Borchardt's theorem [2; 15, vol. 2, pp. 173–175]. It can be established by reading through the proof of Borchardt's theorem given in [1, Proof of Cor. 5.1], ignoring however the restriction $m = n$ (see also [5]).

THEOREM (BORCHARDT+). — *For $m \geq n$, let*

$$B(X,Y) := \begin{pmatrix} \dfrac{1}{(x_1-y_1)^2} & \dfrac{1}{(x_1-y_2)^2} & \cdots & \dfrac{1}{(x_1-y_m)^2} \\ \dfrac{1}{(x_2-y_1)^2} & \dfrac{1}{(x_2-y_2)^2} & \cdots & \dfrac{1}{(x_2-y_m)^2} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{1}{(x_n-y_1)^2} & \dfrac{1}{(x_n-y_2)^2} & \cdots & \dfrac{1}{(x_n-y_m)^2} \\ 1 & 1 & \cdots & 1 \\ y_1 & y_2 & \cdots & y_m \\ y_1^2 & y_2^2 & \cdots & y_m^2 \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{m-n-1} & y_2^{m-n-1} & \cdots & y_m^{m-n-1} \end{pmatrix}.$$

Then

$$\det(B(X,Y)) = \det(C(X,Y)) \times \operatorname{per}\left(\frac{1}{x_i - y_j}\right).$$

Our next goal is to derive the required extension of (a special case of) Lascoux's theorem [11, Theorem q]. Let $Z = \{z_1, z_2, \ldots, z_n\}$ be another set of variables, of equal cardinality as X. For $1 \leq i \leq n$ we define the divided difference ∂_i by

$$\partial_i : f \mapsto \frac{f - f^{\sigma_i}}{x_i - z_i},$$

where σ_i is the transposition which interchanges x_i and z_i. It is easy to see that

$$\partial_i \frac{1}{z_i - y} = \frac{1}{(x_i - y)(z_i - y)}.$$

Since the operator ∂_i acts only on one row of the matrix $C(Z,Y)$ (to be precise, the i-th row), it follows that

$$\partial_1\partial_2\cdots\partial_n(\det(C(Z,Y))|_{Z=X} = \det(B(X,Y)). \tag{1}$$

Now, to generalize Lascoux's theorem to our case, one simply reads through the proof of Theorem q in [11], on introducing slight modifications if necessary. The result is:

Theorem (Lascoux+). — *Let $H(X)$ be the $n\times(m+n-1)$ matrix defined by*

$$H(X) := \big(h_{j-i}(X)\big)_{1\le i\le n, 1\le j\le m+n-1},$$

where $h_s(X)$ denotes the complete homogeneous symmetric function of degree s in the variables X (cf. [12, Ch. 1]), and $E(Y)$ be the $(m+n-1)\times n$ matrix defined by

$$E(Y) := \Big((j-2k+2)(-1)^{m-j+k-1}e_{m-j+k-1}(Y)\Big)_{1\le j\le m+n-1, 1\le k\le n},$$

where $e_s(Y)$ denotes the elementary symmetric function of degree s in the variables Y (cf. [12, Ch. 1]). Then

$$\partial_1\partial_2\cdots\partial_n(\Delta(Z)R(X,Y))|_{Z=X} = \Delta(X)\det\big(H(X)E(Y)\big).$$

Now we are in the position to state our main theorem, which will enable us, in Section 5, to evaluate numerous Scott-type permanents in closed form. The theorem implies immediately that any Scott-type permanent $\mathrm{PER}(P(x), Q(y))$ is a rational function in the coefficients of the two polynomials $P(x)$ and $Q(y)$.

Theorem 1. — *Let m and n be arbitrary positive integers, and let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$ be two sets of variables. Then*

$$\operatorname{per}\left(\frac{1}{x_i - y_j}\right) = \frac{\det\big(H(X)E(Y)\big)}{R(X,Y)}.$$

Proof. — First let $m \ge n$. By combining Theorems (Cauchy+) and (Borchardt+), and Equation (1), we obtain

$$(-1)^{n(n-1)/2}\frac{R(X,Y)}{\Delta(X)\Delta(Y)}\partial_1\partial_2\cdots\partial_n(\det(C(Z,Y))|_{Z=X}$$

for the permanent. Now we apply Theorem (Cauchy+) again in order to replace $C(Z,Y)$ by the corresponding product form guaranteed by the theorem. After having used that the divided differences ∂_i commute with $\Delta(Y)/R(X,Y)R(Z,Y)$ (because the latter expression is symmetric in x_i and z_i), Theorem (Lascoux+) applies and yields the desired result.

If $m < n$, the permanent clearly vanishes. According to Theorem (Lascoux+), it suffices to establish that

$$U := \partial_1 \partial_2 \cdots \partial_n (\Delta(Z) R(X,Y)) = 0.$$

To begin with, we rewrite the Vandermonde determinant evaluation as $\Delta(Z)R(X,Y) = (-1)^{n(n-1)/2} \det(z_i^{j-1} R(x_i, Y))$. Writing $Q(x_i) := R(x_i, Y) = a_m x_i^m + \cdots + a_1 x_i + a_0$, we obtain for U the expression

$$U = \det(\partial_i z_i^{j-1} R(x_i, Y)) = \det\big(z_i^{j-1} Q(x_i) - x_i^{j-1} Q(z_i)\big) \Big/ \prod (x_i - z_i).$$

Since for $m < n$, we have

$$\sum_{j=0}^{m} a_m \big(z_i^j Q(x_i) - x_i^j Q(z_i)\big) = 0,$$

the $m+1$ elements $z_i^j Q(x_i) - x_i^j Q(z_i)$, $0 \le j \le m$, are linearly dependent. Hence, $U = 0$. ∎

In [5], the first author obtained another expression for the permanent, in form of a certain weighted sum over involutions. To state and explain this formula, for $s \in X$ define

$$L(s; X, Y) := \sum_{x \neq s} \frac{1}{x - s} + \sum_{y \in Y} \frac{1}{s - y}.$$

Let us denote by $\mathcal{I}(n)$ the set of involutions on $\{1, 2, \ldots, n\}$. Given an involution $\sigma \in \mathcal{I}(n)$, we define the weight $\Psi(\sigma)$ of σ by

$$\Psi(\sigma; X, Y) := \prod_{(ij) \in \sigma} \frac{1}{(x_i - x_j)^2} \prod_{(k) \in \sigma} L(x_k; X, Y)$$

where the first product is over all transpositions (ij) in the disjoint cycle decomposition of σ, and where the second product is over all fixed points k of σ. Then the result from [5] is the following.

Theorem 2. — *Let m and n be arbitrary positive integers, and let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$ be two sets of variables. Then*

$$\operatorname{per}\left(\frac{1}{x_i - y_j}\right) = \sum_{\sigma \in \mathcal{I}(n)} \Psi(\sigma; X, Y).$$

Example 1. — Let $n = 1$, $X = \{x\}$, $Y = \{y_1, y_2, \ldots, y_m\}$. Then

$$\begin{aligned} H(X) &:= \big(h_{j-1}(X)\big)_{i=1,1\le j\le m} = (h_0(x), h_1(x), \ldots, h_{m-1}(x)), \\ E(Y) &:= \Big(j(-1)^{m-j} e_{m-j}(Y)\Big)_{1\le j\le m, k=1} \\ &= (1(-1)^{m-1}e_{m-1}(Y), 2(-1)^{m-2}e_{m-2}(Y), \ldots, me_0(Y))^t. \end{aligned}$$

We have

$$H(X)E(Y) = \sum_{j=0}^{m-1} (j+1)(-1)^{m-j-1} e_{m-j-1}(Y)x^j.$$

Therefore,

$$\begin{aligned} \operatorname{per}\left(\frac{1}{x-y}\right) &= \frac{1}{x-y_1} + \frac{1}{x-y_2} + \cdots + \frac{1}{x-y_m} \quad \text{[Definition, Th. 2]} \\ &= \frac{\sum_{j=0}^{m-1}(j+1)(-1)^{m-j-1}e_{m-j-1}(Y)x^j}{(x-y_1)(x-y_2)\cdots(x-y_m)}. \quad \text{[Th. 1]} \end{aligned}$$

Example 2. — For $n = 4$ and $m = 3$, Theorem 1 yields the following identity:

$$\det\left(\begin{pmatrix} h_0 & h_1 & h_2 & h_3 & h_4 & h_5 \\ 0 & h_0 & h_1 & h_2 & h_3 & h_4 \\ 0 & 0 & h_0 & h_1 & h_2 & h_3 \\ 0 & 0 & 0 & h_0 & h_1 & h_2 \end{pmatrix} \times \begin{pmatrix} e_2 & e_3 & 0 & 0 \\ -2e_1 & 0 & 2e_3 & 0 \\ 3e_0 & -e_1 & -e_2 & 3e_3 \\ 0 & 2e_0 & 0 & -2e_2 \\ 0 & 0 & e_0 & e_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = 0.$$

For $n = 2$ and $m = 1$, Theorem 2 yields

$$\frac{1}{(x_1-x_2)^2} + \left(\frac{1}{x_2-x_1} + \frac{1}{x_1-y}\right)\left(\frac{1}{x_1-x_2} + \frac{1}{x_2-y}\right) = 0.$$

3. The case of $P(x) = x^n - 1$ and of $P(x) = x^{n-1} + \cdots + x + 1$

In this section we specialize Theorem 1 to the case that the x_i's are the zeroes of the polynomial $P(x) = x^n - 1$ or of $P(x) + x^{n-1} + \cdots + x + 1$, and the y_i's are the zeroes of an arbitrary other polynomial. (This covers, for example, the case of Scott's identity). For the remainder of this section, we fix m and n, $m \ge n$.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of zeroes of $x^n - 1$, and let $Y = \{y_1, y_2, \ldots, y_m\}$ be the set of zeroes of $Q(x) = a_m y^m + a_{m-1}y^{m-1} + \cdots + a_1 y^1 + a_0$, with $a_m = 1$. We write

$$\operatorname{PER}(P,Q) := \operatorname{per}\left(\frac{1}{x_i - y_j}\right) = \frac{\det(H(X)E(Y))}{R(X,Y)}. \tag{2}$$

Since

$$\sum h_i(X)t^i = \frac{1}{\prod_i(1-tx_i)} = \frac{1}{(1-t^n)} = 1 + t^n + t^{2n} + \cdots,$$

we have

$$h_k(X) = \begin{cases} 1, & \text{if } k = 0 \pmod{n}, \\ 0, & \text{if } k \neq 0 \pmod{n}. \end{cases}$$

We denote by $\mathbf{I}_k$ the $k \times k$ identity matrix, and by $\mathbf{0}_{l,c}$ the $l \times c$ matrix with all entries equal to 0. For all r, we write $r\%n$ for the number between 1 and n that satisfies $r \pmod{n} = r\%n \pmod{n}$. Then we have

$$H(X) = \left(\mathbf{I}_n \,\middle|\, \mathbf{I}_n \,\middle|\, \cdots \,\middle|\, \mathbf{I}_n \,\middle|\, \begin{matrix} \mathbf{I}_{m'} \\ \mathbf{0}_{n-m',m'} \end{matrix} \right),$$

with $m' = (m+n-1)\%n$.

Furthermore, let $\operatorname{diag}_n^i(c_1, c_2, \ldots, c_n)$ denote the $n \times n$ "diagonal" matrix, in which the (broken) diagonal starts in the i-th row,

$$\begin{pmatrix} \mathbf{0}_{i-1,n-i+1} & \operatorname{diag}_{i-1}(c_{n-i+2}, c_{n-i+3}, \ldots, c_n) \\ \operatorname{diag}_{n-i+1}(c_1, c_2, \ldots, c_{n-i+1}) & \mathbf{0}_{n-i+1,i-1} \end{pmatrix}.$$

According to the definition of $E(Y)$, a simple calculation yields that

$$H(X)E(Y) = \sum_{r=0}^{m} \operatorname{diag}_n^{r\%n}(ra_r, (r-1)a_r, \ldots, (r-n+1)a_r).$$

Example 3. — For $n = 3$ and $m = 4$, let $P(x) = x^3 - 1$ and $Q(y) = y^4 + a_3y^3 + a_2y^2 + a_1y + a_0$. Then $H(X)E(Y)$ is the sum of the following matrices:

$$\begin{pmatrix} 0 & -a_0 & 0 \\ 0 & 0 & -2a_0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 2a_2 & 0 & 0 \\ 0 & a_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2a_3 & 0 \\ 0 & 0 & a_3 \\ 3a_3 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In the theorem below, we summarize our findings.

Theorem 3. — *Let $P(x) = x^n - 1$ and $Q(y) = a_my^m + \cdots + a_1y^1 + a_0$, a_m not necessarily 1. Writing*

$$\operatorname{Fes}(Q) = \det\left(\sum_{r=0}^{m} \operatorname{diag}_n^{r\%n}(ra_r, (r-1)a_r, \ldots, (r-n+1)a_r)\right), \tag{3}$$

we have

$$\mathrm{PER}(P,Q) = \frac{\mathrm{Fes}(Q)}{\mathrm{Res}(P,Q)},$$

where Res *is the classical resultant of two polynomials.*

Proof. — We have already seen that the theorem is true for $a_m = 1$. On the other hand, there hold $\mathrm{Fes}(\lambda Q) = \lambda^n \mathrm{Fes}(Q)$ and $\mathrm{Res}(P,\lambda Q) = \lambda^n \mathrm{Res}(P,Q)$, as is easily verified. □

Now let us consider the case that $X = \{x_1, x_2, \ldots, x_{n-1}\}$ is the set of zeroes of $P(x) = x^{n-1} + \cdots + x + 1$. We perform an analysis very similar to the one before, using the fact that we have

$$\sum h_i(X)t^i = \frac{1}{\prod_i (1 - tx_i)} = \frac{1-t}{\prod_i (1 - t^n)} = 1 - t + t^n - t^{n+1} + - \cdots.$$

In order to state the result, we introduce the following notation: We write $\widetilde{\mathrm{diag}}^i_{n-1}(c_1, c_2, \ldots, c_{n-1})$ for the $(n-1) \times (n-1)$ matrix

$$\begin{pmatrix} \mathbf{0}_{i-1,n-i} & \mathbf{0}_{i-1,1} & \mathrm{diag}_{i-2}(c_{n-i+2}, c_{n-i+3}, \ldots, c_{n-1}) \\ \mathrm{diag}_{n-i}(c_1, c_2, \ldots, c_{n-i}) & \mathbf{0}_{n-i,1} & \mathbf{0}_{n-i+1,i-2} \end{pmatrix}$$

if $i > 1$, respectively $\mathrm{diag}_{n-1}(c_1, c_2, \ldots, c_{n-1})$ if $i = 1$. This is again a matrix with a (possibly broken) diagonal, in which the diagonal "jumps over" one row and column in the case that it is broken. (Note the slight discrepancy in dimension between the diagonal and the zero matrices.)

Theorem 4. — *Let* $P(x) = x^{n-1} + \cdots + x + 1$ *and* $Q(y) = a_m y^m + \cdots + a_1 y^1 + a_0$, a_m *not necessarily* 1. *Writing*

$$\begin{aligned}\widetilde{\mathrm{Fes}}(Q) = \det\Bigg(&\sum_{r=0}^{m} \widetilde{\mathrm{diag}}^{r\%n}_{n-1}(ra_r, (r-1)a_r, \ldots, (r-n+2)a_r) \\ &- \sum_{r=0}^{m} \widetilde{\mathrm{diag}}^{(r-1)\%n}_{n-1}(ra_r, (r-1)a_r, \ldots, (r-n+2)a_r)\Bigg),\end{aligned} \tag{4}$$

we have

$$\mathrm{PER}(P,Q) = \frac{\widetilde{\mathrm{Fes}}(Q)}{\mathrm{Res}(P,Q)},$$

where, again, Res *is the classical resultant of two polynomials.*

According to Theorems 3 and 4, for accomplishing the evaluation of the permanent, it is necessary to evaluate the numerator $\mathrm{Fes}(Q)$, respectively $\widetilde{\mathrm{Fes}}(Q)$, and the denominator $\mathrm{Res}(P,Q)$. For the evaluation of the resultant $\operatorname{Res}(P, Q)$, we make use of the following lemma, which, for example, appears explicitly in [7]. (In fact, it follows from a special case of Proposition 6 in the next section.)

LEMMA 5. — *Let d be the greatest common divisor of m and n, and let A and C be two nonzero constants. Then the resultant of the two polynomials $Ax^m - B$ and $Cx^n - D$ is given by*

$$\operatorname{Res}(Ax^m - B, Cx^n - D) = (-1)^m \big(A^{n/d} D^{m/d} - B^{n/d} C^{m/d}\big)^d.$$

What concerns the evaluation of $\operatorname{Fes}(Q)$, respectively $\widetilde{\operatorname{Fes}}(Q)$, we refer the reader to the next section for the determinant evaluations that we are going to use. In combination with Lemma 5, these will allow us to evaluate Scott-type permanents in numerous special cases, see Section 5.

4. Determinant evaluations

PROPOSITION 6. — *Let n and r be positive integers, $r \le n$, and $x_1, x_2, \dots, x_n$, $y_1, y_2, \dots, y_n$ be indeterminates. Then, with $d = \gcd(r, n)$, we have*

$$\det \begin{pmatrix} x_1 & 0 & \dots & 0 & y_{n-r+1} & 0 & & \\ 0 & x_2 & 0 & & 0 & y_{n-r+2} & 0 & \\ & & \ddots & & & & \ddots & 0 \\ 0 & & & & & & 0 & y_n \\ y_1 & 0 & & & & & & \\ 0 & y_2 & 0 & & & & & \\ & 0 & \ddots & 0 & & & \ddots & 0 \\ & & 0 & y_{n-r} & 0 & & 0 & x_n \end{pmatrix}$$
$$= \prod_{i=1}^{d} \left(\prod_{j=1}^{n/d} x_{i+(j-1)d} - (-1)^{n/d} \prod_{j=1}^{n/d} y_{i+(j-1)d} \right). \tag{5}$$

(I.e., in the matrix there are only nonzero entries along two diagonals, one of which is a broken diagonal.)

Proof. — Let $A = (A_{ij})_{1 \le i,j \le n}$ be the matrix of which the determinant is taken in (5). By definition, we have

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \prod_{i=1}^{n} A_{i\sigma(i)}, \tag{6}$$

where S_n denotes the symmetric group of order n. Since the matrix A is a sparse matrix, only those permutations σ contribute to the sum which have the property $\sigma(i)=i$ or $\sigma(i)=i+r \bmod n$ for all i. Thus, the decomposition into disjoint cycles of such a permutation consists only of cycles of length 1, and of cycles of length n/d of the form $(i,i+r,\dots,i+(n/d-1)r)$ (where, again, all integers have to be taken modulo n). Using these observations and the fact that the sign of any cycle of length n/d is $(-1)^{n/d-1}$ in (6) yields (5) immediately. $\square$

THEOREM 7. — *Let a,b,c,d,e be indeterminates. For any positive integer n and integers i,j let $n(i,j)$ denote 1 plus (the representative between 0 and $n-1$ of) the residue class of $i-j$ mod n. Then*

$$\det_{1\le i,j\le n}\Big((n(i,j)+c)(n(i,j)a+b)+d-(j-1)(n(i,j)a+e)\Big)$$
$$=\det\begin{pmatrix} \begin{matrix}(c+1)(a+b)\\ +d\end{matrix} & \begin{matrix}(c+n)(na+b)\\ +d-(na+e)\end{matrix} & \dots & \begin{matrix}(c+2)(2a+b)\\ +d-(n-1)(2a+e)\end{matrix}\\ \begin{matrix}(c+2)(2a+b)\\ +d\end{matrix} & \begin{matrix}(c+1)(a+b)\\ +d-(a+e)\end{matrix} & \dots & \begin{matrix}(c+3)(3a+b)\\ +d-(n-1)(3a+e)\end{matrix}\\ \vdots & \vdots & \dots & \vdots\\ \begin{matrix}(c+n)(na+b)\\ +d\end{matrix} & \begin{matrix}(c+n-1)\\ \times((n-1)a+b)+d\\ -((n-1)a+e)\end{matrix} & \dots & \begin{matrix}(c+1)(a+b)\\ +d-(n-1)(a+e)\end{matrix}\end{pmatrix}$$
$$=(-n)^{n-1}U_n(a,b,c,d,e)\prod_{i=3}^{n}(ia+b+ca), \tag{7}$$

where $U_n(a,b,c,d,e)$ is the polynomial

$$U_n(a,b,c,d,e)=\frac{(n+1)(n+2)}{3}a^2+\frac{(n+1)(2n+7)}{6}ab+\frac{(n+1)}{2}b^2$$
$$+\frac{(n+1)(2n+7)}{6}a^2c+\frac{(3n+5)}{2}abc+b^2c+\frac{(n+1)}{2}a^2c^2+abc^2+\frac{(n+3)}{2}ad$$
$$+bd+acd-\frac{(n-1)(2n+5)}{6}ae-\frac{(n-1)}{2}be-\frac{(n-1)}{2}ace.$$

In the case that $n=2$, the product in (7) has to be read as 1, and in the case that $n=1$, the product has to be interpreted as $1/(2a+b+ca)$.

Proof. — In the cases $n=1$ and $n=2$, the claim can be verified directly. For $n\ge 3$, we use the "identification of factors" method as explained in [10, Sec. 2.4] or [9, Sec. 2].

We proceed in several steps. An outline is as follows. In the first step we show that $\prod_{i=3}^{n}(ia+b+ca)$ is a factor of the determinant as a polynomial in a,b,c,d,e. In the second step we prove that $U_n(a,b,c,d,e)$ is a factor of the determinant. Then, in the third step, we determine the maximal degree of the determinant as a polynomial in a, and also in b, c, d, and in e. It turns out that the maximal degree is n as a polynomial in a, the same being true as a polynomial in b and as a polynomial in c, while it is

1 as a polynomial in d, the same being true as a polynomial in e. On the other hand, the degree in a, and also in b and in c, of the product on the right-hand side of (7), which by the first two steps divides the determinant, is exactly n. It is exactly 1 in d and also in e. Therefore we are forced to conclude that the determinant equals

$$C(n)U_n(a,b,c,d,e)\prod_{i=3}^{n}(ia+b+ca), \tag{8}$$

where $C(n)$ is a constant independent of a,b,c,d,e. Finally, in the fourth step, we determine the constant $C(n)$, which turns out to equal $(-n)^{n-1}$. Clearly, this would finish the proof of theorem.

Step 1. For $i=3,\dots,n$ the term $(ia+b+ca)$ is a factor of the determinant. We claim that, if $b=-ia-ca$, we have

$$\begin{aligned}-\big(\text{row }(n-i)\big)+3\big(\text{row }(n-i+1)\big)&\\ -3\big(\text{row }(n-i+2)\big)+\big(\text{row }(n-i+3)\big)&=0\end{aligned}$$

as long as $n>i$. (Here, $(\text{row }i)$ denotes the i-th row of the matrix underlying the determinant in (7).) In the case that $n=i$, we claim that we have

$$-3\big(\text{row }1\big)+3\big(\text{row }2\big)-\big(\text{row }3\big)+\big(\text{row }n\big)=0$$

as long as $n>3$, and that we have $-\big(\text{row }1\big)+\big(\text{row }2\big)=0$ if $n=3$. All these claims are easily verified by an obvious case-by-case analysis.

Step 2. The polynomial $U_n(a,b,c,d,e)$ is a factor of the determinant. We claim that if d is chosen so that $U_n(a,b,c,d,e)$ vanishes, we have

$$\sum_{j=1}^{n}((j+1)a+b+ca)\big(\text{column }j\big)=0.$$

Again, it is a routine task to verify this identity.

Step 3. The determinant is a polynomial in a (in b, respectively in c) of maximal degree n, and a polynomial in d (respectively in e) of maximal degree 1. The first claim follows from the fact that each term in the defining expansion of the determinant has degree n in a (as well as in b, respectively in c). To establish the second claim, we simply subtract the first row of the determinant from all other rows, with the effect that only the entries in the first row contain d and e after these transformations. Since the right-hand side of (7), which by Steps 1 and 2 divides the determinant as a polynomial in a,b,c,d,e, also has degree n in a, in b, and in c, and degree 1 in d, and in e, the determinant and the right-hand side of (7) differ only by a multiplicative constant.

Step 4. The evaluation of the multiplicative constant. By the preceding steps we know that the determinant equals (8). In particular, if we set $a = c = d = e = 0$ and $b = 1$, we have

$$\det_{1\le i,j\le n} (n(i,j)) = C(n)(n+1)/2. \tag{9}$$

The matrix on the left-hand side of (9) is a circulant matrix with entries $1, 2, \dots, n$. Hence, its determinant equals

$$\prod_{\omega\,:\,\text{zero of } x^n-1} (1 + 2\omega + 3\omega^2 + \cdots + n\omega^{n-1}).$$

The sum is easily evaluated by observing that it is the derivative of a geometric series. It turns out to be equal to $-n/(1-\omega)$. The resulting product simplifies by the observation

$$\prod_{\omega\,:\,\text{zero of } x^n-1,\ \omega\ne 1} (1-\omega) = (1 + x + \cdots + x^{n-1})\big|_{x=1} = n. \tag{10}$$

Thus, the determinant in (9) equals $(-n)^{n-1}(n+1)/2$. Therefore $C(n)$ is equal to $(-n)^{n-1}$.

This finishes the proof of (7) and thus of the theorem. $\square$

Theorem 8. — *Let a be an indeterminate. For any positive integer n and integers i, j let $s(i,j)$ denote (the representative between 0 and $n-1$ of) the residue class of $i - j + 1$ mod n. Then*

$$\det_{1\le i,j\le n-1}\left(\begin{Bmatrix} (n-m-1) + j(1+a-n) & \text{if } i = j-2 \pmod n \\ (n-1)(m-1) + j(1-a) & \text{if } i = j-3 \pmod n \\ (n-m-3-2s(i,j)) + j & \text{otherwise} \end{Bmatrix}\right)$$
$$= (-1)^{n-1}\frac{1}{n}\prod_{i=2}^{n}(nm - ia). \tag{11}$$

Proof. — The matrix underlying this determinant is a matrix whose elements have a uniform definition, except for two (broken) diagonals, the one with $i = j-2 \pmod n$, and the one with $i = j-3 \pmod n$.

To begin with, we reorder the rows so that the next-to-last row becomes the first row, the last row becomes the second row, and then follow the remaining rows in their original order.

Now we add all the rows to the (new) last row. In the resulting matrix, we change the sign of the last row and, subsequently, move it up so that it becomes the third row. As a result of these manipulations, we obtain

$$(-1)^{n-1} \det_{1\le i,j\le n-1} \left(\begin{cases} (n-1)(m-1)+j(1-a) & \text{if } i=j \\ (n-m-1)+j(a-n+1) & \text{if } i=j+1 \\ -m-n+1-2i+3j & \text{if } i<j \\ -m+n+1-2i+3j & \text{if } i>j+1 \end{cases} \right). \quad (12)$$

Next we apply further row operations. We subtract the second row from the first, the third from the second, …, the $(n-1)$-st row from the $(n-2)$-nd row, in that order. Subsequently, we repeat the same kind of operations, but stop before the last row, i.e., we subtract the second row from the first, the third from the second, …, the $(n-2)$-nd row from the $(n-3)$-rd row, in that order. As a result, the above determinant is converted into the determinant of the following matrix

$$\left(\begin{cases} nm-ja & \text{if } i=j-2 \\ -2nm+2n-j(3a-n) & \text{if } i=j-1 \text{ and } i<n-2 \\ nm-2n-j(3a-2n) & \text{if } i=j \text{ and } i<n-2 \\ j(a-n) & \text{if } i=j+1 \text{ and } i<n-2 \\ 2 & \text{if } i=n-2 \text{ and } j\le n-4 \\ (n-3)(a-n)+2 & \text{if } i=n-2 \text{ and } j=n-3 \\ nm-2n+2-(n-2)(2a-n) & \text{if } i=n-2 \text{ and } j=n-2 \\ n-nm+1+(n-1)(a-1) & \text{if } i=n-2 \text{ and } j=n-1 \\ -m-n+3+3j & \text{if } i=n-1 \text{ and } j\le n-3 \\ n-m-1+(n-2)(a-n+1) & \text{if } i=n-1 \text{ and } j=n-2 \\ (n-1)(m-a) & \text{if } i=n-1 \text{ and } j=n-1 \\ 0 & \text{otherwise} \end{cases} \right). \quad (13)$$

This is a matrix with four "special" diagonals (the diagonals with $i=j-2$, $j-1$, j, $j+1$, respectively) and two "special" rows (the last two rows). All other entries are zero.

Now we factor $(nm-3a)$ out of the first row. Subsequently, we substract $(a-n)$ times the (new) first row from the second. Now one is able to factor $(nm-4a)$ out of the (new) second row. Next, we substract $2(a-n)$ times the (new) second row from the third row. Etc. We stop this procedure in the $(n-3)$-rd row. Thus, the determinant of the matrix in (13) is equal to $\prod_{i=3}^{n-1}(nm-ia)$ times the determinant of the matrix

$$\begin{pmatrix} 1 & -2 & 1 & 0 & \dots & & & \\ 0 & 1 & -2 & 1 & 0 & \dots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & & 0 \\ & & & 0 & 1 & -2 & & 1 \\ a_{n-2,1} & a_{n-2,2} & \dots & \dots & \dots & \dots & & a_{n-2,n-1} \\ a_{n-1,1} & a_{n-1,2} & \dots & \dots & \dots & \dots & & a_{n-1,n-1} \end{pmatrix}, \tag{14}$$

where the entries in the last two rows are still the same as in (13).

We now perform the final set of transformations. We add column 1 through column $n-2$ to column $n-1$, and then we add

$$\sum_{j=1}^{n-3}(n-j-1)\big(\text{column } j\big)$$

to column $n-2$. The effect is that a block matrix is obtained of the form

$$\begin{pmatrix} U & 0 \\ * & M \end{pmatrix},$$

where U is an $(n-3)\times(n-3)$ upper-triangular matrix with 1s on the diagonal, and where M is the 2×2 matrix

$$\begin{pmatrix} nm-n-2a+2 & n-2 \\ \frac{1}{2}(n-2)(n+m+2a-nm-3) & \frac{5}{2}n-\frac{n^2}{2}+m-a-3 \end{pmatrix}.$$

Clearly, the determinant of U is 1, while the determinant of M is $(m-a)(nm-2a)$. Putting everything together, we have completed the proof of (11). ∎

Corollary 9. — *Let a be an indeterminate. For any positive integer n and integers i,j let $s(i,j)$ denote (the representative between 0 and $n-1$ of) the residue class of $i-j+1$ mod n. Then*

$$\det_{1\le i,j\le n-1}\left(\begin{Bmatrix} -2s(i,j)-a+j-1 & \text{if } i\ne j-2 \pmod n \\ (n-1)(n+a-j-1) & \text{if } i=j-2 \pmod n \end{Bmatrix}\right) = n^{n-2}\prod_{i=0}^{n-2}(i+a). \tag{15}$$

Proof. — In the determinant, we move the last row on top, replace the (now) last row by the sum of all the rows, factor (-1) out of the resulting row, and finally move it up so that it becomes the second row, retaining the order of all the other rows. These operations did not change the value of the determinant. However, the resulting determinant is exactly the determinant in (12) with $a = n$ and $m = a + n$. As we have shown in the proof of Theorem 8, the latter determinant differs from the determinant in (11) just by a sign of $(-1)^{n-1}$. Thus, we obtain the right-hand side of (15). $\square$

5. Closed form evaluations for Scott-type permanents

THEOREM 10. — *Let n, m and r be positive integers and $d = \gcd(n, r)$. Then*

$$\operatorname{PER}\left(x^n - 1, \sum_{\ell=0}^{m} a_\ell y^{\ell n} + \sum_{\ell=0}^{m} b_\ell y^{\ell n + r}\right)$$

$$= -\frac{d^n \prod_{i=1}^{d} \left(\left(\sum_{\ell=0}^{m} a_\ell \right)^{n/d} \left(\sum_{\ell=0}^{m} (i - n\ell - 1) a_\ell \Big/ \sum_{\ell=0}^{m} d a_\ell \right)_{n/d} - \left(- \sum_{\ell=0}^{m} b_\ell \right)^{n/d} \left(\sum_{\ell=0}^{m} (i - r - n\ell - 1) b_\ell \Big/ \sum_{\ell=0}^{m} d b_\ell \right)_{n/d} \right)}{\left(\left(\sum_{\ell=0}^{m} a_\ell \right)^{n/d} - \left(- \sum_{\ell=0}^{m} b_\ell \right)^{n/d} \right)^d},$$

where $(\alpha)_k$ is the standard notation for shifted factorials, $(\alpha)_k := \alpha(\alpha + 1) \cdots (\alpha + k - 1)$, $k \geq 1$, and $(\alpha)_0 := 1$.

Proof. — Let us first consider the case that $n \nmid r$. According to Theorem 3, we have to compute the quotient $\operatorname{Fes}(Q)/\operatorname{Res}(x^n - 1, Q)$, where $Q = \sum_{\ell=0}^{m} a_\ell y^{\ell n} + \sum_{\ell=0}^{m} b_\ell y^{\ell n + r}$. In order to compute $\operatorname{Fes}(Q)$, in (3) we replace $a_{\ell n}$ by a_ℓ and $a_{\ell n + r}$ by b_ℓ, $\ell = 0, 1, \ldots, m$, and set all other a_ℓ's equal to zero. In the resulting determinant we move the last row to the top, thus creating a sign of $(-1)^{n-1}$, and finally apply Proposition 6 with $x_j = \sum_{\ell=0}^{m} (n\ell - j + 1) a_\ell$ and $y_j = \sum_{\ell=0}^{m} (n\ell + r - j + 1) b_\ell$, $j = 1, 2, \ldots, n$. For the evaluation of $\operatorname{Res}(x^n - 1, Q)$ we note that

$$\operatorname{Res}\left(x^n - 1, \sum_{\ell=0}^{m} a_\ell y^{\ell n} + \sum_{\ell=0}^{m} b_\ell y^{\ell n + r}\right)$$

$$= \prod_{\omega\,:\,\text{zero of } x^n - 1} \left(\sum_{\ell=0}^{m} a_\ell + \omega^r \sum_{\ell=0}^{m} b_\ell \right) = \operatorname{Res}\left(x^n - 1, y^r \sum_{\ell=0}^{m} b_\ell + \sum_{\ell=0}^{m} a_\ell \right).$$

Now we can apply Lemma 5.

If on the other hand $n \mid r$, then $d = n$. It can be verified directly that the claimed formula remains valid in that case, too. $\square$

Remark. — Given a polynomial $Q(y)$, there is no unique way to write it in the form $\sum_{\ell=0}^{m} a_\ell y^{\ell n} + \sum_{\ell=0}^{m} b_\ell y^{\ell n+r}$. For example, we may write $Q(y) = y^n + a + b$ as $Q(y) = (y^n + ay^0) + b$, or as $Q(y) = (y^n + (a+b))$ (i.e., either with $a_1 = 1$, $a_0 = a$, $r = 0$, $b_0 = b$, or with $a_1 = 1$, $a_0 = a + b$, $b_\ell = 0$ for all ℓ). Regardless which choice we make, Theorem 10 yields

$$\mathrm{PER}(x^n - 1, y^n + a + b) = (-1)^{n+1} \frac{\prod_{i=1}^{n} (i - (n-i)(a+b))}{(a+b+1)^n}.$$

Corollary 11. — *Let n and m be positive integers. Then*

$$\mathrm{PER}\left(x^n - 1, \sum_{\ell=0}^{m} a_\ell y^{\ell n}\right) = -\left(-n \sum_{\ell=0}^{m} \ell a_\ell \Big/ \sum_{\ell=0}^{m} a_\ell\right)_n.$$

Corollary 12. — *We have*

$$\mathrm{PER}(x^n - 1, y^{mn} + \cdots + y^{2n} + y^n + 1) = -(-mn/2)_n.$$

Corollary 13. — *If m is even then*

$$\mathrm{PER}(x^n + 1, y^{mn} + \cdots + y^{2n} + y^n + 1) = (-mn/2)_n.$$

Proof. — We use the case $a_\ell = (-1)^\ell$ in Corollary 11, and the fact that

$$\mathrm{per}\left(\frac{1}{x_i \sqrt[n]{-1} - y_j}\right)_{1 \le i \le n,\, 1 \le j \le mn} = -\mathrm{per}\left(\frac{1}{x_i - y_j / \sqrt[n]{-1}}\right)_{1 \le i \le n,\, 1 \le j \le mn}.$$

$\square$

Corollary 14. — *We have*

$$\mathrm{PER}\left(x^n - 1, \sum_{\ell=0}^{m} \ell y^{\ell n}\right) = -(-n(2m+1)/3)_n.$$

Corollary 15. — *We have*

$$\mathrm{PER}\left(x^n - 1, \sum_{\ell=0}^{m} \ell y^{\ell^2 n}\right) = -(-nm(m+1)/2)_n.$$

Corollary 16. — *We have*

$$\mathrm{PER}(x^n-1, y^{mn}+ay^{rn}+b) = -(-(m+ra)n/(a+b+1))_n.$$

Corollary 17. — *We have*

$$\mathrm{PER}(x^n-1, y^{mn}+1) = -(-mn/2)_n.$$

For $m=1$ one recovers Scott's identity stated at the beginning of the introduction.

Corollary 18. — *If $m+ra=a+b+1\neq 0$, then*

$$\mathrm{PER}(x^n-1, y^{mn}+ay^{rn}+b) = (-1)^{n+1}n!.$$

Corollary 19. — *If $a\neq -2$ then*

$$\mathrm{PER}(x^n-1, y^{2n}+ay^{n}+1) = (-1)^{n+1}n!.$$

Corollary 20. — *If $m+ra=0$ and $a+b+1\neq 0$, then*

$$\mathrm{PER}(x^n-1, y^{mn}+ay^{rn}+b) = 0.$$

Corollary 21. — *If $b\neq 1$ then*

$$\mathrm{PER}(x^n-1, y^{2n}-2y^{n}+b) = 0.$$

Corollary 22. — *Let n and m be positive integers and $d=\gcd(n,m)$. Then*

$$\mathrm{PER}(x^n-1, y^m+b) = -\frac{d^n\prod_{i=1}^{d}\left(\left(\frac{i-m-1}{d}\right)_{n/d}-(-b)^{n/d}\left(\frac{i-1}{d}\right)_{n/d}\right)}{\left(1-(-b)^{n/d}\right)^d}.$$

Proof. — In Theorem 10, set $m=0$, $a_0=b$, $b_0=1$, and replace r by m, in this order. ∎

Corollary 23. — *If $\gcd(m,n)=1$, then*

$$\mathrm{PER}(x^n-1, y^m+b) = (-1)^{n+1}\frac{m(m-1)\cdots(m-n+1)}{1-(-b)^n}.$$

Corollary 24. — *If* $\gcd(m,n)=1$*, then*

$$\operatorname{PER}(x^{s(n-1)}+\cdots+x^{2s}+x^s+1,y^{s(m-1)}+\cdots+y^{2s}+y^s+1)$$
$$=\frac{\prod_{i=0}^{s-1}\left(\prod_{\ell=0}^{n-1}(i+\ell s)-\prod_{\ell=0}^{n-1}(i+\ell s-ms)\right)}{(mns)^s}.$$

Proof. — Consider the Scott-type permanent $\operatorname{PER}(x^{sn}-1,y^{sm}-q^{sm})$ (expressed in terms of its definition). When we multiply it by $(1-q)^s$ and then perform the limit $q\to 1$, then the permanent reduces to $\operatorname{PER}(x^{s(n-1)}+\cdots+x^s+1,y^{s(m-1)}+\cdots+y^s+1)$, as is straightforward to see. On the other hand, the permanent that we started with is the permanent in Corollary 22 with n replaced by sn, m replaced by sm, and $b=-q^{sm}$. Indeed, if we multiply the right-hand side from Corollary 22 (with these choices for the parameters) by $(1-q)^{sm}$, and then perform the limit $q\to 1$, we obtain exactly the claimed result. $\square$

Corollary 25. — *If* $\gcd(m,n)=1$*, then*

$$\operatorname{PER}(x^{n-1}+\cdots+x+1,y^{m-1}+\cdots+y+1)$$
$$=(-1)^{n+1}\frac{(m-1)\cdots(m-n+1)}{n}.$$

Corollary 26. — *If* $\gcd(m,n)=1$ *and n is odd, then*

$$\operatorname{PER}(x^n-1,y^m+1)=\frac{m(m-1)\cdots(m-n+1)}{2}.$$

Corollary 27. — *If n is odd, then*

$$\operatorname{PER}(x^n-1,y^{n+1}+1)=\frac{(n+1)!}{2}.$$

Corollary 28. — *Let n and r be positive integers (not necessarily $n>r$) and* $d=\gcd(n,r)$*. Then*

$$\operatorname{PER}(x^n-1,y^n+ay^r+b)$$
$$=-\frac{d^n\prod_{i=1}^{d}\left((b+1)^{\frac{n}{d}}\left(\frac{ib-b+i-n-1}{d(b+1)}\right)_{\frac{n}{d}}-(-a)^{\frac{n}{d}}\left(\frac{i-r-1}{d}\right)_{\frac{n}{d}}\right)}{\left((b+1)^{\frac{n}{d}}-(-a)^{\frac{n}{d}}\right)^d}.$$

Proof. — In Theorem 10, set $m = 1$, $a_0 = b$, $a_1 = 1$, $b_0 = a$, and $b_1 = 0$. ∎

COROLLARY 29. — *Let n and r be positive integers (not necessarily $n > r$) and $\gcd(n,r) = 1$. Then*

$$\mathrm{PER}(x^n - 1, y^n + ay^r + b) = (-1)^{n+1}\,\frac{\prod_{i=1}^{n}(i - (n-i)b)) - a^n(-r)_n}{(b+1)^n - (-a)^n}.$$

If $1 \le r \le n-1$ then $(-r)_n = 0$. Thus, one recovers the results in [6]. On the other hand, if we set $r = n+1$, we obtain, for example, the following result.

COROLLARY 30. — *We have*

$$\mathrm{PER}(x^n - 1, y^{n+1} + y^n - 1) = n^n - (-1)^n(n+1)!.$$

COROLLARY 31. — *We have*

$$\mathrm{PER}(x^n - 1, y^n + ny - 1) = 1.$$

THEOREM 32. — *Let n and m be positive integers and a be an arbitrary number. Then*

$$\mathrm{PER}\left(x^n - 1, \sum_{\ell=0}^{mn-1}(\ell + a)y^\ell\right) \\ = (-1)^{n-1}\frac{n(m-1)\,V_n(a,m)}{6\,(mn + 2a - 1)}\,(a + (m-1)n + 1)_{n-2}\,,$$

where $V_n(a,m)$ is the polynomial

$$V_n(a,m) = 1 - 6a + 6a^2 + n - 2an - 5mn + 10amn - mn^2 + 4m^2n^2.$$

Proof. — Let $Q = \sum_{\ell=0}^{mn-1}(\ell + a)y^\ell$. Using Theorem 3 again, we have to compute $\mathrm{Fes}(Q)/\mathrm{Res}(x^n - 1, Q)$. In order to compute $\mathrm{Fes}(Q)$, in (3) set $a_\ell = \ell + a$, $\ell = 0, 1, \ldots, mn-1$. In the resulting determinant we move the last row to the top, thus creating a sign of $(-1)^{n-1}$, and finally apply Theorem 7 with $a = 1$, $b = (m-1)n - 1$, $c = a - 1$, $d = n^2(m-1)(2m-1)/6 - an(m-1)/2$ and $e = a + n(m-1)/2 - 1$.

For the computation of $\mathrm{Res}(x^n-1,Q)$ we note that

$$\begin{aligned}\mathrm{Res}(x^n-1,Q) &= \prod_{\omega\,:\,\text{zero of } x^n-1}\left(\sum_{\ell=0}^{mn-1}(\ell+a)\omega^\ell\right)\\ &= \left(\binom{mn}{2}+mna\right)\prod_{\omega\,:\,\text{zero of } x^n-1,\ \omega\neq 1}\left(\sum_{\ell=1}^{mn}\ell\omega^{\ell-1}\right).\end{aligned}$$

The sum in the last line is the derivative of a geometric series, and is therefore easily evaluated. The result of the summation turns out to be $-mn/(1-\omega)$. The computation is completed by the observation (10), and some simplification. ∎

Corollary 33. — *Let n and m be positive integers, $n\geq 2$. Then*

$$\mathrm{PER}\left(x^n-1,\sum_{\ell=0}^{mn-1}\ell y^\ell\right)=(-1)^{n-1}\frac{(4mn-n-1)(mn-2)!}{6(mn-n-1)!}.$$

Corollary 34. — *Let n and m be positive integers, $n\geq 2$. Then*

$$\mathrm{PER}\left(x^n-1,\sum_{\ell=0}^{mn-1}(\ell+1)y^\ell\right)=(-1)^{n-1}\frac{(4mn-n+1)(mn-n)(mn-1)!}{6(mn-n+1)!}.$$

Corollary 35. — *Let n and m be positive integers, $n\geq 2$. Then*

$$\mathrm{PER}\left(x^n-1,\sum_{\ell=0}^{mn-1}(mn-\ell)y^\ell\right)=\frac{(m-1)\,(n+1)!}{6}.$$

Corollary 36. — *Let n and m be positive integers, $n\geq 2$. Then*

$$\mathrm{PER}\left(x^n-1,\sum_{\ell=0}^{mn-1}(mn-\ell-1)y^\ell\right)=\frac{(m-1)\,n!}{6}.$$

Remark. — It is also possible to move forward and derive formulas for $\mathrm{PER}\left(x^n-1,\sum_{\ell=0}^{mn/s-1}(\ell+a)y^{\ell s}\right)$, where s is some positive integer. This would require to find analogues of Theorem 7 in which $n(i,j)$ is replaced by $D\,n(i,j)$, where D is the inverse of $s/\gcd(n,s)$ modulo $n/\gcd(n,s)$. As calculations aided by the computer indicate, the resulting determinant evaluations have forms very similar to (7). That is, the result shows a product of linear factors in a and b as the one on the right-hand side of (7), and one irreducible polynomial of higher degree (such as $U_n(a,b,c,d,e)$ in (7)). However, as D increases, the degree of the irreducible polynomial also increases, whereas the amount of linear factors in a and b decreases, so that the results become more and more unwieldy. We therefore content ourselves with stating the result when s divides n.

THEOREM 37. — *Let n, m and s be positive integers so that $s \mid n$, and let a be an arbitrary number. Then*

$$\operatorname{PER}\left(x^n-1, \sum_{\ell=0}^{\frac{mn}{s}-1}(\ell+a)y^{\ell s}\right) = (-1)^{n-1}\frac{s^{n-2s}}{6^s\,(mn+2as-s)^s} \times \prod_{k=0}^{s-1}\left(\left(a+\tfrac{1}{s}(nm-n-k)+1\right)_{n/s-2} V_{n,s}(a,m,k)\right),$$

where $V_{n,s}(a,m,k)$ is the polynomial

$$\begin{aligned} V_{n,s}(a,m,k) &= 6k^2mn+6kmn^2-10km^2n^2+mn^3-5m^2n^3+4m^3n^3-6k^2s \\ &+12ak^2s-6kns+12akns+12kmns-24akmns-n^2s+2an^2s+6mn^2s \\ &-12amn^2s-5m^2n^2s+10am^2n^2s-2ks^2+12aks^2-12a^2ks^2-ns^2 \\ &+6ans^2-6a^2ns^2+mns^2-6amns^2+6a^2mns^2. \end{aligned}$$

Proof. — Let $Q=\sum_{\ell=0}^{mn-1}(\ell+a)y^{\ell s}$. Again, according to Theorem 3, we have to compute $\operatorname{Fes}(Q)/\operatorname{Res}(x^n-1,Q)$. In order to compute $\operatorname{Fes}(Q)$, in (3) set $a_{\ell s}=\ell+a$, $\ell=0,1,\ldots,mn/s-1$, and all other a_i's to zero. In the resulting determinant we move the last row to the top, thus creating a sign of $(-1)^{n-1}$. Since only every s-th a_i is nonzero, we are dealing with a determinant of a matrix in which a lot of entries are zero. If we permute rows and columns so that first come the rows and columns whose indices are congruent 1 mod s, then come the rows and columns whose indices are congruent 2 mod s, etc., then the matrix of which we want to compute the determinant assumes a block form, with $(n/s)\times(n/s)$ blocks on the diagonal, and zeroes otherwise. Therefore the determinant equals the product of the determinants of the s matrices of dimension $(n/s)\times(n/s)$ on the diagonal. Each of these determinants can be evaluated by means of Theorem 7. We leave it to the reader to fill in the details.

For the computation of $\operatorname{Res}(x^n-1,Q)$ we proceed as in the proof of Theorem 32. ∎

THEOREM 38. — *Let n and m be positive integers, $n \geq 2$, and a be* *an arbitrary number. Then*

$$\mathrm{PER}\left(x^{n-1}+\cdots+x+1,\ \sum_{\ell=0}^{mn-1}(\ell+a)y^{\ell}\right)\\ =(-1)^{n-1}\left(a+(m-1)n+1\right)_{n-1}.$$

Proof. — Let $P(x)=x^{n-1}+\cdots+x+1$ and $Q(y)=\sum_{\ell=0}^{mn-1}(\ell+a)y^{\ell}$. This time we apply Theorem 4. According to that theorem, we have to compute $\widetilde{\mathrm{Fes}}(Q)/\mathrm{Res}(P,Q)$. In order to compute $\widetilde{\mathrm{Fes}}(Q)$, in (4) we set $a_\ell=\ell+a$, $\ell=0,1,\dots,mn-1$. The resulting determinant is exactly m^{n-1} times the determinant in Corollary 9 with a replaced by $((m-1)n+a+1)$. For the computation of the resultant of P and Q we proceed as in the proof of Theorem 7. Simplification of the result yields the claimed expression. ∎

THEOREM 39. — *Let n and m be positive integers, $n\ge 2$, and a be an arbitrary number. Then*

$$\mathrm{PER}\left(x^{n-1}+\cdots+x+1,\ \sum_{\ell=0}^{mn-2}(\ell+a)y^{\ell}\right)\\ =(-1)^{n-1}\frac{(nm-n)_{n-1}\,(nm+a-1)^{n-1}}{(mn+a-1)^{n}-(a-1)^{n}}.$$

Proof. — Let $P(x)=x^{n-1}+\cdots+x+1$ and $Q(y)=\sum_{\ell=0}^{mn-1}(\ell+a)y^{\ell}$. Using Theorem 4 again, we have to compute $\widetilde{\mathrm{Fes}}(Q)/\mathrm{Res}(P,Q)$. In order to compute $\widetilde{\mathrm{Fes}}(Q)$, in (4) we set $a_\ell=\ell+a$, $\ell=0,1,\dots,mn-2$. The resulting determinant is exactly m^{n-1} times the determinant in Theorem 8 with m replaced by $mn+A-1$, a replaced by $(mn+A-1)/m$, and A replaced by a, in that order.

For the computation of $\mathrm{Res}(x^n+\cdots+x+1,Q)$ we proceed similarly as in the proof of Theorem 32. Using an observation from that proof, we note that

$$\begin{aligned}\mathrm{Res}(x^n+\cdots+x+1,Q)&=\prod_{\omega\,:\,\text{zero of }x^n-1,\ x\ne1}\left(\sum_{\ell=0}^{mn-2}(\ell+a)\omega^{\ell}\right)\\ &=\prod_{\omega\,:\,\text{zero of }x^n-1,\ x\ne1}\left(-\frac{mn}{1-\omega}-(mn+a-1)\omega^{mn-1}\right)\\ &=(-1)^{n-1}\prod_{\omega\,:\,\text{zero of }x^n-1,\ x\ne1}\frac{mn+a-1+\omega(1-a)}{\omega(1-\omega)}\\ &=\frac{(mn+a-1)^n-(a-1)^n}{n^2m}.\end{aligned}$$

The result follows now upon some simplification. ∎

6. Sums of involutions

As in [6, Sec. 4], we may obtain interesting summation theorems by combining special evaluations of Scott-type determinants (see Section 5) with Theorem 2. For example, if we combine Corollary 19 with Theorem 2, we obtain the following result.

PROPOSITION 40. — *Let $x_1, x_2, \ldots, x_n$ be the zeroes of $x^n - 1$. Then*

$$\sum_{\sigma \in \mathcal{I}(n)} \prod_{(ij) \in \sigma} \frac{1}{(x_i - x_j)^2} \prod_{(k) \in \sigma} \frac{n+1}{2x_k} = (-1)^{n+1} n!,$$

where the first product is over all transpositions (ij) of σ in its disjoint cycle decomposition, and where the second product is over all fixed points k of σ.

Similarly, if we combine Corollary 21 with Theorem 2, we obtain the result below.

PROPOSITION 41. — *Let $x_1, x_2, \ldots, x_n$ be the zeroes of $x^n - 1$. Then*

$$\sum_{\sigma \in \mathcal{I}(n)} \prod_{(ij) \in \sigma} \frac{1}{(x_i - x_j)^2} \prod_{(k) \in \sigma} \frac{n-1}{2x_k} = 0,$$

where the first product is over all transpositions (ij) of σ in its disjoint cycle decomposition, and where the second product is over all fixed points k of σ.

If we combine Corollary 31 with Theorem 2, then we obtain the following result.

PROPOSITION 42. — *Let $x_1, x_2, \ldots, x_n$ be the zeroes of $x^n - 1$. Then*

$$\sum_{\sigma \in \mathcal{I}(n)} \prod_{(ij) \in \sigma} \frac{1}{(x_i - x_j)^2} \prod_{(k) \in \sigma} \frac{2n + (n+1)x_k}{2x_k^2} = 1,$$

where the first product is over all transpositions (ij) of σ in its disjoint cycle decomposition, and where the second product is over all fixed points k of σ.

Finally, if we combine Corollary 27 with Theorem 2, we obtain the result below.

PROPOSITION 43. — *Let n be odd, and let $x_1, x_2, \ldots, x_n$ be the zeroes of $x^n - 1$. Then*

$$\sum_{\sigma \in \mathcal{I}(n)} \prod_{(ij) \in \sigma} \frac{1}{(x_i - x_j)^2} \prod_{(k) \in \sigma} \frac{1 - n + (3+n)x_k}{2(1+x_k)x_k} = \frac{(n+1)!}{2},$$

where the first product is over all transpositions (ij) of σ in its disjoint cycle decomposition, and where the second product is over all fixed points k of σ.

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