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# COMPUTING POWERS OF TWO GENERALIZATIONS OF THE LOGARITHM

Wadim Zudilin[‡] (Moscow)

Abstract. We prove multiple-series representations for positive integer powers of the series

$$L(z;\alpha)=\sum_{n=1}^{\infty}\frac{z^n}{n+\alpha},\ \ |z|<1,\ \alpha\geqslant 0,\quad \text{and}\quad \ell_q(z)=\sum_{n=1}^{\infty}\frac{z^nq^n}{1-q^n},\ \ |z|\leqslant 1,\ |q|<1.$$

The results generalize a known formula for powers of the series for the ordinary logarithm $-\log(1-z)=L(z;0)$.

The series for the ordinary logarithm,

$$\mathrm{Li}_1(z)=-\log(1-z)=\sum_{n=1}^{\infty}\frac{z^n}{n},\qquad |z|<1, \tag{1}$$

admits the following formula

$$\mathrm{Li}_1(z)^l=l!\sum_{n_l>n_{l-1}>\dots>n_1\geqslant 1}\frac{z^{n_l}}{n_1n_2\cdots n_l}. \tag{2}$$

It plays an important rôle in evaluating irrationality and transcendence measures for values of the logarithm at rational points (cf. [Ne]). On the other hand, information on the arithmetic nature of values of the following two generalizations of the series (1) is available only in particular cases at this moment. The first generalization, the $\alpha$-shift of the logarithm for $\alpha\geqslant 0$, is defined by the series

$$L(z)=L(z;\alpha)=\sum_{n=1}^{\infty}\frac{z^n}{n+\alpha},\qquad |z|<1.$$

The second generalization, known as a $q$-extension of the logarithm, is given by the formula

$$\ell_q(z)=\sum_{n=1}^{\infty}\frac{z^nq^n}{1-q^n}=\sum_{n=1}^{\infty}\frac{z^n}{p^n-1},\qquad |z|\leqslant 1,\quad |q|<1,\quad p=q^{-1}.$$

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Then one has

$$\lim_{\substack{q\to1\\|q|<1}}(1-q)\ell_q(z)=\mathrm{Li}_1(z)=L(z;0) \qquad \text{for} \quad |z|<1.$$

The aim of this note is to provide generalizations of the formula (2) for both $L(z;\alpha)$ (Theorem 1) and $\ell_q(z)$ (Theorem 2), which might become useful in the arithmetic study of values of the series.

**Theorem 1.** *For each $l=0,1,2,\dots$, the following identity holds:*

$$L(z)^l=l!L_l(z), \qquad l=0,1,2,\dots,$$

*where*

$$L_l(z)=\sum_{n_l>\dots>n_2>n_1\geqslant1}\frac{z^{n_l}}{(n_1+\alpha)\cdots(n_l+l\alpha)}, \quad l=1,2,\dots, \qquad L_0(z)=1.$$

*Proof.* For given $l$, we have

$$\begin{aligned}
z^{1-l\alpha}\frac{\mathrm{d}}{\mathrm{d}z}\big(z^{l\alpha}L_l(z)\big)&=z^{1-l\alpha}\frac{\mathrm{d}}{\mathrm{d}z}\sum_{n_l>\dots>n_1\geqslant1}\frac{z^{n_l+l\alpha}}{(n_1+\alpha)\cdots(n_{l-1}+(l-1)\alpha)(n_l+l\alpha)}\\
&=\sum_{n_{l-1}>\dots>n_1\geqslant1}\frac{1}{(n_1+\alpha)\cdots(n_{l-1}+(l-1)\alpha)}\sum_{n_l=n_{l-1}+1}^{\infty}z^{n_l}\\
&=\frac{z}{1-z}\sum_{n_{l-1}>\dots>n_1\geqslant1}\frac{z^{n_{l-1}}}{(n_1+\alpha)\cdots(n_{l-1}+(l-1)\alpha)}\\
&=\frac{z}{1-z}L_{l-1}(z).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
z^{1-l\alpha}\frac{\mathrm{d}}{\mathrm{d}z}\big(z^{l\alpha}L_l(z)\big)&=z^{1-l\alpha}\left(l\alpha z^{l\alpha-1}L_l(z)+z^{l\alpha}\frac{\mathrm{d}}{\mathrm{d}z}L_l(z)\right)\\
&=l\alpha L_l(z)+z\frac{\mathrm{d}}{\mathrm{d}z}L_l(z).
\end{aligned}$$

Therefore,

$$z\frac{\mathrm{d}}{\mathrm{d}z}L_l(z)=-l\alpha L_l(z)+\frac{z}{1-z}L_{l-1}(z), \qquad l=1,2,\dots.$$

If we define

$$\tilde L_l(z)=\frac{1}{l!}L(z)^l=\frac{1}{l!}\left(\sum_{n=1}^{\infty}\frac{z^n}{n+\alpha}\right)^l,$$

then

$$z\frac{\mathrm{d}}{\mathrm{d}z}\tilde{L}_l(z) = \frac{1}{l!}z\frac{\mathrm{d}}{\mathrm{d}z}L(z)^l = \frac{1}{(l-1)!}L(z)^{l-1}\cdot z\frac{\mathrm{d}}{\mathrm{d}z}L(z)$$
$$= \frac{1}{(l-1)!}L(z)^{l-1}\left(-\alpha L(z)+\frac{z}{1-z}\right) = -\alpha l\tilde{L}_l(z)+\frac{z}{1-z}\tilde{L}_{l-1}(z).$$

Since $L_0(z)=\tilde{L}_0(z)=1$ and $L_l(z)=(z/(1+\alpha))^l/l!+O(z^{l+1})$, $\tilde{L}_l(z)=(z/(1+\alpha))^l/l!+O(z^{l+1})$, we obtain the desired result. $\square$

**Theorem 2.** *For each $l=1,2,\dots,$*

$$\ell_q(z)^l = \sum_{n_l>n_{l-1}>\dots>n_1\geqslant 1}\frac{z^{n_l}q^{n_l}\Phi_l(q^{n_1},q^{n_2-n_1},\dots,q^{n_l-n_{l-1}})}{(1-q^{n_1})(1-q^{n_2})\cdots(1-q^{n_l})}$$
$$= \sum_{n_l>n_{l-1}>\dots>n_1\geqslant 1}\frac{z^{n_l}\Phi_l(p^{n_1},p^{n_2-n_1},\dots,p^{n_l-n_{l-1}})}{(p^{n_1}-1)(p^{n_2}-1)\cdots(p^{n_l}-1)}, \tag{3}$$

*where $\Phi_l(x_1,\dots,x_l)$ is the polynomial*

$$\Phi_l(x_1,\dots,x_l) = (x_1^{l-1}+x_1^{l-2}+\dots+x_1+1)(x_2^{l-2}+\dots+x_2+1)\cdots(x_{l-1}+1)$$
$$= \prod_{j=1}^{l}\frac{x_j^{l+1-j}-1}{x_j-1}. \tag{4}$$

*In particular, one has*

$$\ell_{1/p}(z)^2 = \sum_{n_2>n_1\geqslant 1}\frac{z^{n_2}(p^{n_1}+1)}{(p^{n_1}-1)(p^{n_2}-1)},$$
$$\ell_{1/p}(z)^3 = \frac{1}{2}\sum_{n_3>n_2>n_1\geqslant 1}\frac{z^{n_3}(p^{2n_1}+p^{n_1}+1)(p^{n_2-n_1}+1)}{(p^{n_1}-1)(p^{n_2}-1)(p^{n_3}-1)}.$$

The proof requires two auxiliary identities.

**Lemma 1.** *For $l=2,3,\dots,$ the equality*

$$\frac{1}{(x_1-1)(x_1x_2-1)\cdots(x_1x_2\cdots x_{l-1}-1)\cdot(x-1)}$$
$$= \sum_{j=1}^{l}\frac{x_1\dots x_{j-1}}{\prod_{k=1}^{j-1}(x_1\cdots x_k-1)\cdot\prod_{k=j-1}^{l-1}(x_1\cdots x_kx-1)}$$

*holds identically in the variables $x_1,\dots,x_{l-1}$ and $x$.* (Empty products should be replaced by 1.)

The proof exploits a simple inductive argument, and therefore is omitted.

**Lemma 2.** *For each $l=1,2,\dots,$ the following identity holds:*

$$\begin{aligned}&\frac{1}{(x_1-1)(x_2-1)\cdots(x_l-1)}\\ &\qquad=\frac{1}{l!}\sum_{\sigma\in\mathfrak{S}_l}\frac{\Phi_l(x_{\sigma(1)},\dots,x_{\sigma(l)})}{(x_{\sigma(1)}-1)(x_{\sigma(1)}x_{\sigma(2)}-1)\cdots(x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(l)}-1)},\end{aligned}\tag{5}$$

*where*

$$\Phi_l(x_1,\dots,x_l)=\sum_{\substack{k_i=0\\ i=1,\dots,l}}^{l-i}x_1^{k_1}x_2^{k_2}\cdots x_l^{k_l}\tag{6}$$

*is the polynomial defined in* (4) *and $\mathfrak{S}_l$ denotes the set of all permutations of* $\{1,2,\dots,$ $l\}$.

*Proof.* We apply induction on $l$. If $l=1$, then both sides of (5) are simply $1/(x_1-1)$. Suppose that $l>1$ and that we have proved identity (5) for $l$ replaced by $l-1$. Then applying Lemma 1 (with $x=x_l$) we obtain

$$\begin{aligned}&\frac{1}{(x_1-1)(x_2-1)\cdots(x_{l-1}-1)\cdot(x_l-1)}\\ &\quad=\frac{1}{(l-1)!}\sum_{\sigma\in\mathfrak{S}_{l-1}}\frac{\Phi_{l-1}(x_{\sigma(1)},\dots,x_{\sigma(l-1)})}{(x_{\sigma(1)}-1)(x_{\sigma(1)}x_{\sigma(2)}-1)\cdots(x_{\sigma(1)}\cdots x_{\sigma(l-1)}-1)\cdot(x_l-1)}\\ &\quad=\frac{1}{(l-1)!}\sum_{\sigma\in\mathfrak{S}_{l-1}}\sum_{j=1}^{l}\frac{\Phi_{l-1}(x_{\sigma(1)},\dots,x_{\sigma(l-1)})}{\prod_{k=1}^{j-1}(x_{\sigma(1)}\cdots x_{\sigma(k)}-1)}\\ &\qquad\qquad\times\frac{x_{\sigma(1)}\cdots x_{\sigma(j-1)}}{\prod_{k=j-1}^{l-1}(x_{\sigma(1)}\cdots x_{\sigma(j-1)}x_lx_{\sigma(j)}\cdots x_{\sigma(k)}-1)}\\ &\quad=\frac{1}{(l-1)!}\sum_{\tau\in\mathfrak{S}_l}\frac{\Phi_{l-1}(x_{\tau(1)},\dots,\widehat{x_{\tau(j)}},\dots,x_{\tau(l)})\cdot x_{\tau(1)}\cdots x_{\tau(j-1)}}{\prod_{k=1}^{l}(x_{\tau(1)}\cdots x_{\tau(k)}-1)},\end{aligned}\tag{7}$$

where in the latter sum $j$ abbreviates $\tau^{-1}(l)$, that is

$$\tau=\tau(j,\sigma)\in\mathfrak{S}_l\colon(1,2,\dots,l-1,l)\mapsto\big(\sigma(1),\dots,\sigma(j-1),l,\sigma(j),\dots,\sigma(l-1)\big),$$

and the notation $\widehat{x}$ means omitting the corresponding parameter. Since the product on the left-hand side of (7) is symmetric in $x_1,x_2,\dots,x_l$, we may replace $x_l$ by $x_i$ with $i\ne l$, to deduce an identity similar to but different from (7). If we now average the results over all $i = 1, \dots, l$, then we obtain

$$\begin{aligned}
&\frac{1}{(x_1-1)(x_2-1)\cdots(x_{l-1}-1)(x_l-1)} \\
&\quad = \frac{1}{(l-1)!}\cdot\frac{1}{l}\sum_{j=1}^{l}\sum_{\tau\in\mathfrak{S}_l}\frac{\Phi_{l-1}(x_{\tau(1)},\dots,\widehat{x_{\tau(j)}},\dots,x_{\tau(l)})\cdot x_{\tau(1)}\cdots x_{\tau(j-1)}}{\prod_{k=1}^{l}(x_{\tau(1)}\cdots x_{\tau(k)}-1)} \\
&\quad = \frac{1}{l!}\sum_{\tau\in\mathfrak{S}_l}\frac{\tilde{\Phi}_l(x_{\tau(1)},\dots,x_{\tau(l)})}{\prod_{k=1}^{l}(x_{\tau(1)}\cdots x_{\tau(k)}-1)},
\end{aligned}$$

where we set

$$\tilde{\Phi}_l(x_1,\dots,x_l) = \sum_{j=1}^{l}\Phi_{l-1}(x_1,\dots,\widehat{x_j},\dots,x_l)\cdot x_1x_2\cdots x_{j-1}. \tag{8}$$

It remains to verify that the polynomials $\tilde{\Phi}_l$ in (8) and $\Phi_l$ in (6) coincide:

$$\begin{aligned}
\tilde{\Phi}_l(x_1,\dots,x_l) &= \sum_{j=1}^{l}\sum_{\substack{k_i=0\\ i=1,\dots,l-1}}^{l-1-i} x_1^{k_1+1}\cdots x_{j-1}^{k_{j-1}+1}x_{j+1}^{k_j}\cdots x_l^{k_{l-1}} \\
&= \sum_{j=1}^{l}\sum_{\substack{k'_i=1\\ i=1,\dots,j-1}}^{l-i} x_1^{k'_1}\cdots x_{j-1}^{k'_{j-1}}\sum_{\substack{k'_i=0\\ i=j+1,\dots,l}}^{l-i} x_{j+1}^{k'_{j+1}}\cdots x_l^{k'_l} \\
&= \sum_{\substack{k'_i=0\\ i=1,\dots,l}}^{l-i} x_1^{k'_1}x_2^{k'_2}\cdots x_l^{k'_l} = \Phi_l(x_1,\dots,x_l),
\end{aligned}$$

where we use the bijection

$$(j,k_1,\dots,k_{l-1}) \mapsto (k'_1,k'_2,\dots,k'_l) = (k_1+1,\dots,k_{j-1}+1,0,k_j,\dots,k_{l-1}).$$

This completes our proof of the lemma. $\square$

*Remark.* The identity of Lemma 2 belongs to a family of identities in the style of Littlewood [Li], p. 85:

$$\prod_{j=1}^{l}\frac{1}{x_j} = \sum_{\sigma\in\mathfrak{S}_l}\frac{1}{x_{\sigma(1)}(x_{\sigma(1)}+x_{\sigma(2)})\cdots(x_{\sigma(1)}+x_{\sigma(2)}+\cdots+x_{\sigma(l)})}.$$

Identities of similar type and their applications may be found in [Me], Section 10.9, Appendix A.23. Lassalle in [La] gives formulae, where monomial symmetric functions are specialized; taking as variables the parts of the partition, which indexes the function (i.e., replacing $x_j$ by $q^{j-1}$ for $j = 1, \dots, l$), one obtains curious generalizations of Littlewood-type identities.

*Proof of Theorem* 2. From Lemma 2 we deduce that

$$\begin{aligned}
\ell_q(z)^l &= \biggl(\sum_{m=1}^{\infty} \frac{z^m}{p^m - 1}\biggr)^l = \sum_{\substack{m_i=1\\ i=1,\dots,l}}^{\infty} \frac{z^{m_1+\dots+m_l}}{(p^{m_1}-1)\cdots(p^{m_l}-1)} \\
&= \frac{1}{l!} \sum_{\substack{m_i=1\\ i=1,\dots,l}}^{\infty} \sum_{\sigma\in\mathfrak{S}_l} \frac{\Phi_l(x_{\sigma(1)},\dots,x_{\sigma(l)})}{\prod_{k=1}^{l}(x_{\sigma(1)}\cdots x_{\sigma(k)}-1)}\bigg|_{x_i=p^{m_i},\ i=1,\dots,l} \\
&= \sum_{\substack{m_i=1\\ i=1,\dots,l}}^{\infty} \frac{\Phi_l(x_1,\dots,x_l)}{\prod_{k=1}^{l}(x_1\cdots x_k-1)}\bigg|_{x_i=p^{m_i},\ i=1,\dots,l},
\end{aligned}$$

and after the change $n_i = m_1 + \dots + m_i$, $i = 1, \dots, l$, we arrive at the claimed formula (3). $\square$

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Department of Mechanics and Mathematics
Moscow Lomonosov State University
Vorobiovy Gory, GSP-2
119992 Moscow, RUSSIA
*URL*: `http://wain.mi.ras.ru/index.html`
*E-mail address*: `wadim@ips.ras.ru`