*Séminaire Lotharingien de Combinatoire* **59** (2008), Article B59d

# COUNTING MULTIDERANGEMENTS BY EXCEDANCES

CHAK-ON CHOW

Abstract. We consider the enumeration of multiderangements of a multiset $\mathbf{n} = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$ by the number of excedances. We prove several properties, including the invariance under permutations of $\{n_1, n_2, \ldots, n_m\}$, the symmetry, recurrence relations, the real-rootedness, and a combinatorial expansion, of the generating function $d_{\mathbf{n}}(x)$ of multiderangements by excedances, thus generalizing the corresponding results for the classical derangements. By a further extension, the generating function for multipermutations by numbers of excedances and fixed points is also given.

## 1. Introduction

In [3] Brenti considered a class of derangement polynomials defined for $n \geqslant 1$ by

$$d_n(x) = \sum_{w \in \mathscr{D}_n} x^{e(w)},$$

and conjectured that $d_n(x)$ has only real zeros, where $e(w) = \#\{i \in [n] : w_i > i\}$ is the number of excedances of $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ and $\mathscr{D}_n$ is the set of derangements in $\mathfrak{S}_n$. Brenti remarked in [4] that this conjecture had been settled by E. Rodney Canfield (unpublished). A published proof of this conjecture later appeared in the work of Zhang [16, 17] which involved the recurrence relation for the $d_n(x)$'s, namely,

$$d_{n+1}(x) = nx[d_n(x) + d_{n-1}(x)] + x(1-x)d_n'(x). \tag{1}$$

There are a number of possible lines of generalizations of the above mentioned results. For instance, one may consider generalizations to other Coxeter families. See, e.g., [5] for the type $B$ case. Another line of generalization, which is the focus of this work, is to consider multiderangements. Let $\mathbf{n} = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$ be a multiset and $n := n_1 + \cdots + n_m$. A multipermutation $w = w_1 w_2 \cdots w_n$ of $\mathbf{n}$ is called a *multiderangement* if $w_i \neq p_i$ for $i = 1, 2, \ldots, n$, where the word $\delta(w) = p_1 p_2 \cdots p_n$ is the nondecreasing rearrangement of $w$. An integer $i \in [n]$ is an *excedance* of $w = w_1 w_2 \cdots w_n$ if $w_i > p_i$. Denote by $\textsc{exc}(w) := \{i \in [n] \colon w_i > p_i\}$ the excedance set, and by $e(w) := \# \textsc{exc}(w)$ the number of excedances, of $w$. Every multiderangement $w = w_1 w_2 \cdots w_n$, regarded as a multipermutation of $\mathbf{n}$, can be represented by the two-line representation $\binom{\delta(w)}{w}$ or by the one-line representation $w_1 w_2 \cdots w_n$. For further details on multipermutations, their representations, cycle factorizations and related algorithms, see [7]. Denote by $\mathscr{D}(\mathbf{n})$ the set of all multiderangements of $\mathbf{n}$.

---

2000 *Mathematics Subject Classification.* 05A05, 05A15.

*Key words and phrases.* Multisets, multiderangements, excedances, symmetric functions.

| $\{n_1, n_2, \ldots, n_m\}$ | $d_{\mathbf{n}}(x)$ |
|---|---|
| $\{1,1\}$ | $x$ |
| $\{1,1,1\}$ | $x+x^2$ |
| $\{2,2\}$ | $x^2$ |
| $\{2,1,1\}$ | $2x^2$ |
| $\{1,1,1,1\}$ | $x+7x^2+x^3$ |
| $\{2,2,1\}$ | $2x^2+2x^3$ |
| $\{2,1,1,1\}$ | $6x^2+6x^3$ |
| $\{1,1,1,1,1\}$ | $x+21x^2+21x^3+x^4$ |
| $\{3,3\}$ | $x^3$ |
| $\{3,2,1\}$ | $3x^3$ |
| $\{3,1,1,1\}$ | $6x^3$ |
| $\{2,2,2\}$ | $x^2+8x^3+x^4$ |
| $\{2,2,1,1\}$ | $4x^2+21x^3+4x^4$ |
| $\{2,1,1,1,1\}$ | $14x^2+56x^3+14x^4$ |
| $\{1,1,1,1,1,1\}$ | $x+51x^2+161x^3+51x^4+x^5$ |

TABLE 1. $d_{\mathbf{n}}(x)$ for $n = 2, 3, \ldots, 6$.

The generating function of multiderangements by excedances is defined as

$$d_{\mathbf{n}}(x) = \sum_{w \in \mathscr{D}(\mathbf{n})} x^{e(w)} = \sum_{k \geqslant 0} \Theta(\mathbf{n}, k) x^k,$$

where $\Theta(\mathbf{n}, k) = \#\{w \in \mathscr{D}(\mathbf{n}) : e(w) = k\}$. The goal of this work is to establish several properties of $d_{\mathbf{n}}(x)$. In the next section, we prove the invariance under permutations of $\{n_1, n_2, \ldots, n_m\}$, and the symmetry, of $d_{\mathbf{n}}(x)$. In Section 3, we compute the recurrence relations which $d_{\mathbf{n}}(x)$ satisfies. In Section 4, we prove the real-rootedness of $d_{\mathbf{n}}(x)$ from which we deduce the unimodality and log-concavity of the coefficients of $d_{\mathbf{n}}(x)$. In Section 5, we give a combinatorial expansion of $d_{\mathbf{n}}(x)$, which parallels the one for $d_n(x)$. In the final section, we extend the generating function to the one counting multipermutations by the numbers of excedances and fixed points.

## 2. Basic properties

We establish some basic properties of $d_{\mathbf{n}}(x)$ in this section. The first few non-zero $d_{\mathbf{n}}(x)$'s are listed in Table 1. If $n_1 = n_2 = \cdots = n_m = 1$, then $\mathbf{n} = \{1, 2, \ldots, m\}$ so that $d_{\mathbf{n}}(x) = d_m(x)$, whose properties are known [3, 16, 17].

PROPERTY 2.1. The following hold:

(i) If $\max\limits_{1 \leqslant i \leqslant m} n_i > n/2$, then $\mathscr{D}(\mathbf{n}) = \varnothing$ so that $d_{\mathbf{n}}(x) \equiv 0$.

(ii) If $n_i = n/2$ for some $i \in [m]$, then $d_{\mathbf{n}}(x) = \binom{n/2}{n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_m} x^{n/2}$.

(iii) Let $\mathbf{n} = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$ and $\mathbf{n}' = \{1^{n_{i_1}}, 2^{n_{i_2}}, \ldots, m^{n_{i_m}}\}$ be two multisets such that $(i_1, i_2, \ldots, i_m)$ is a permutation of $(1, 2, \ldots, m)$. Then $d_{\mathbf{n}}(x) \equiv d_{\mathbf{n}'}(x)$.

Properties 2.1(i)–(ii) are easily proved. Using MacMahon's Master Theorem [9, p. 97–98], Askey and Ismail [1] showed that

$$\sum_{n_1,\ldots,n_m\geqslant 0} d_{\mathbf{n}}(x)x_1^{n_1}\cdots x_m^{n_m} = \frac{1}{1-xe_2-(x+x^2)e_3-\cdots-(x+x^2+\cdots+x^{m-1})e_m}, \tag{2}$$

where $e_i$ $(2 \leqslant i \leqslant m)$ is the $i$th elementary symmetric function in the commuting indeterminates $x_1,\ldots,x_m$. See [13, Chapter 7] for the definitions of undefined terms concerning symmetric functions. The identity (2) was also obtained by Kim and Zeng using their $U$- and $V$-decompositions of derangements [8] and Foata's factorization of multipermutations [6, 7]. Again by using MacMahon's Master Theorem, Zeng enumerated in [15] multipermutations and multiderangements by the numbers of cycles and excedances.

Property 2.1(iii) is implicit in (2). To wit, recall that [13, Proposition 7.4.1] if $\lambda\vdash n$, then

$$e_\lambda = \sum_{\mu\vdash n} M_{\lambda,\mu} m_\mu, \tag{3}$$

where $e_\lambda$ is the elementary symmetric function indexed by $\lambda$, $m_\mu$ the monomial symmetric function indexed by $\mu$, $M_{\lambda,\alpha}$ the number of $(0,1)$-matrices $A=(a_{ij})_{i,j\geqslant 1}$ with row sum vector $\mathrm{row}(A)=\lambda$ and column sum vector $\mathrm{col}(A)=\alpha$, and $\alpha=(\alpha_1,\alpha_2,\ldots)$ a weak composition of $n$. Expanding the right side of (2), we have

$$\begin{aligned}
&\sum_{n_1,\ldots,n_m\geqslant 0} d_{\mathbf{n}}(x)x_1^{n_1}\cdots x_m^{n_m} \\
&\qquad = 1+\sum_{l\geqslant 1}\left(\sum_{k=2}^{m}(x+x^2+\cdots+x^{k-1})e_k\right)^l \\
&\qquad = 1+\sum_{l\geqslant 1}\sum_{2\leqslant k_1,\ldots,k_l\leqslant m} e_{k_1}\cdots e_{k_l}\prod_{j=1}^{l}(x+x^2+\cdots+x^{k_j-1}) \\
&\qquad = 1+\sum_{l\geqslant 1}\sum_{2\leqslant k_1,\ldots,k_l\leqslant m}\sum_{\mu\vdash k_1+\cdots+k_l} M_{(k_1,\ldots,k_l),\mu}m_\mu\prod_{j=1}^{l}(x+x^2+\cdots+x^{k_j-1}),
\end{aligned} \tag{4}$$

where the last equality follows from (3). Since $m_\mu$ is a sum of distinct permutations of monomials $x_1^{n_1}\cdots x_m^{n_m}$ having $\mu$ equal to the weakly decreasing rearrangement of its exponents, equating the coefficients of $x_1^{n_1}\cdots x_m^{n_m}$, we have that $d_{\mathbf{n}}(x)\equiv d_{\mathbf{n}'}(x)$, where $\mathbf{n}'$ is a multiset obtained by a permutation of $\mathbf{n}$.

If $n_j=n/2$ for some $j\in[m]$, then $n_i\leqslant n/2$ for all $i\in[m]$ so that $\max\limits_{1\leqslant i\leqslant m} n_i = n/2$ and Property 2.1(ii) implies that $d_{\mathbf{n}}(x)$ is symmetric of degree $n-\max\limits_{1\leqslant i\leqslant m} n_i$ and with center of symmetry at $n/2$. This symmetry result actually holds for arbitrary multiset $\mathbf{n}=\{1^{n_1},2^{n_2},\ldots,m^{n_m}\}$ such that $\max\limits_{1\leqslant i\leqslant m} n_i\leqslant n/2$. See Proposition 2.3 below.

We need some notations and results from [7] for the proof of Proposition 2.3.

Let $A$ be a totally order alphabet and $A^*$ be the free monoid generated by $A$. A word $w = w_1w_2\cdots w_n \in A^*$ is said to be *dominated* if $w_1 > w_i$ for $i = 2, 3, \ldots, n$. Let $w$ and $w'$ be $A$-words of the same length. The two-row matrix $\binom{w'}{w}$ is called a *flow*. If $w'$ is a rearrangement of $w$, then $\binom{w'}{w}$ is called a *circuit*. Denote by $\delta w := w_2w_3\cdots w_nw_1$ the *cyclic shift* of the word $w = w_1w_2\cdots w_n$. A circuit $c$ is said to be *dominated* if it is of the form $\binom{\delta w}{w}$ for some dominated word $w$. If $c = \binom{\delta w}{w}$ is a dominated circuit, then let $Fc := Fw$, the first letter of $w$. A *dominated circuit factorization* of a circuit $c$ is a sequence $(d_1, d_2, \ldots, d_r)$ of dominated circuits with the property that $c = d_1d_2\cdots d_r$ and $Fd_1 \leqslant Fd_2 \leqslant \cdots \leqslant Fd_r$. The next result is due to Foata [7, Theorem 10.4.1].

Theorem 2.2. *Every nonempty circuit admits exactly one dominated circuit factorization.*

For instance, for the word $w = 31514226672615$, its nondecreasing rearrangement $\delta(w) = 11122234556667$ and the circuit $\binom{\delta(w)}{w}$ admits the following dominated circuit factorization:

$$\begin{aligned}\begin{pmatrix}\delta(w)\\ w\end{pmatrix} &= \begin{pmatrix}1&1&1&2&2&2&3&4&5&5&6&6&6&7\\ 3&1&5&1&4&2&2&6&6&7&2&6&1&5\end{pmatrix}\\ &= \begin{pmatrix}1&1&2&3\\3&1&1&2\end{pmatrix}\begin{pmatrix}4&2&2&6\\6&4&2&2\end{pmatrix}\begin{pmatrix}6\\6\end{pmatrix}\begin{pmatrix}5&1&6\\6&5&1\end{pmatrix}\begin{pmatrix}5&7\\7&5\end{pmatrix}.\end{aligned}$$

Observe that the columns of the dominated circuits in the factorization are precisely those of $\binom{\delta(w)}{w}$, and that in each dominated circuit

$$\begin{pmatrix}a_i & a_{i-1} & \cdots & a_{j-1} & a_{j-2} & \cdots & a_1 & a_0\\ a_0 & a_i & \cdots & a_j & a_{j-1} & \cdots & a_2 & a_1\end{pmatrix},$$

each (vertical) occurrence of excedance corresponds to a (horizontal) occurrence of descent. It is clear that each dominated circuit is uniquely determined by the bottom word of the circuit. Thus, when we talk about dominated circuits in the sequel, we mean the bottom word of the circuit. Moreover, reversing the bottom word turns descents into non-descents, hence excedances into non-excedances, and vice versa.

Proposition 2.3. *For each multiset* $\mathbf{n} = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$, $d_{\mathbf{n}}(x)$ *is a symmetric polynomial of degree* $n - \max\limits_{1\leqslant i\leqslant m} n_i$ *and with center of symmetry at* $n/2$ *if it is not identically zero.*

*Proof.* Let $w \in \mathscr{D}(\mathbf{n})$. Denote by $(d_1, d_2, \ldots, d_r)$ the dominated circuit factorization of $w$. For $i = 1, 2, \ldots, r$, let $\tilde{d}_i$ be the dominated circuit obtained by cyclic permutation of the reversal of $d_i$. Let $\tilde{w} \in \mathscr{D}(\mathbf{n})$ be the multiderangement whose dominated circuit factorization is $(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_r)$. By the remark preceding the present proposition, we have $e(\tilde{w}) = n - e(w)$. The map $\Phi\colon \mathscr{D}(\mathbf{n}) \longrightarrow \mathscr{D}(\mathbf{n})$ defined by $\Phi(w) = \tilde{w}$ for $w \in \mathscr{D}(\mathbf{n})$ is thus a bijection sending $\mathscr{D}(\mathbf{n})$ onto itself such that $e(\Phi(w)) = e(\tilde{w}) = n - e(w)$. This establishes the symmetry of $d_{\mathbf{n}}(x)$.

That $d_{\mathbf{n}}(x)$ has center of symmetry at $n/2$ is clear.

It remains to prove that $\deg d_{\mathbf{n}}(x) = n - \max_{1\leqslant i\leqslant m} n_i$. By virtue of Property 2.1(iii), we may assume that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_m$ so that $n_1 = \max_{1\leqslant i\leqslant m} n_i$. To show that $\deg d_{\mathbf{n}}(x) = n - n_1$, we demonstrate explicitly a $w \in \mathscr{D}(\mathbf{n})$ with $e(w) = n - n_1$. Let $w$ be the word obtained by concatenating $n_2$ copies of 2's, $n_3$ copies of 3's, $\ldots, n_m$ copies of $m$'s, and $n_1$ copies of 1's. Also, let $\delta(w) = p_1p_2\cdots p_n$ be the nondecreasing rearrangement of $w$. Since $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_m$, it is clear that $w_i \neq p_i$ for all $i \in [n]$ so that $w \in \mathscr{D}(\mathbf{n})$ with the first $n - n_1$ positions being excedances and the remaining $n_1$ positions non-excedances. $\square$

It is worth mentioning that in the classical derangement case, i.e., $\mathbf{n} = \{1, 2, \ldots, m\}$, the (unique) derangement $w \in \mathscr{D}(\mathbf{n})$ with $e(w) = m - 1$ is $w = 23\cdots m1$; also, the bijection $\Phi\colon \mathscr{D}(\mathbf{n}) \longrightarrow \mathscr{D}(\mathbf{n})$ in the above proof is precisely the inversion map $\Phi(w) = w^{-1}$.

## 3. Recurrence relations

We derive in this section the recurrence relations for $d_{\mathbf{n}}(x)$. The multi-analogue of (1) is the following.

**Proposition 3.1.** *Let $\mathbf{n} = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$. The polynomial $d_{\mathbf{n}}(x)$ satisfies*

$$d_{\mathbf{n}+\mathbf{e}_{m+1}}(x) = x[n_1 d_{\mathbf{n}-\mathbf{e}_1}(x) + \cdots + n_m d_{\mathbf{n}-\mathbf{e}_m}(x)] + nxd_{\mathbf{n}}(x) + x(1-x)d'_{\mathbf{n}}(x) \tag{5}$$

*and for $j = 1, 2, \ldots, m$,*

$$\begin{aligned}(n_j+1)d_{\mathbf{n}+\mathbf{e}_j}(x) = x[&n_1 d_{\mathbf{n}-\mathbf{e}_1}(x) + \cdots + n_{j-1}d_{\mathbf{n}-\mathbf{e}_{j-1}}(x) + n_{j+1}d_{\mathbf{n}-\mathbf{e}_{j+1}}(x) \\ &+ \cdots + n_m d_{\mathbf{n}-\mathbf{e}_m}(x)] + [(n-n_j)x - n_j]d_{\mathbf{n}}(x) + x(1-x)d'_{\mathbf{n}}(x),\end{aligned} \tag{6}$$

*where $\mathbf{n}+\mathbf{e}_j$ (respectively $\mathbf{n}-\mathbf{e}_j$) denotes the multisets obtained from $\mathbf{n}$ by adjoining an additional copy (respectively by removing a copy) of the letter $j$.*

*Proof.* There are three cases to consider.

Case 1: Let $w = w_1\cdots w_nw_{n+1} \in \mathscr{D}(\mathbf{n}+\mathbf{e}_{m+1})$. Since $w_{n+1} \neq m+1$ and $m+1$ occurs only once in $w$, there exists exactly one $i \in [n]$ such that $w_i = m+1$. It is clear that $i$ is an excedance of $w$. Consider the word $w' = w_1\cdots w_{i-1}w_{n+1}w_{i+1}\cdots w_n(m+1)$ obtained by swapping the $i$th and the $(n+1)$st letters of $w$. If $w_{n+1} \neq p_i$, then $w_1\cdots w_{i-1}w_{n+1}w_{i+1}\cdots w_n \in \mathscr{D}(\mathbf{n})$ and

$$e(w) = e(w_1\cdots w_{i-1}w_{n+1}w_{i+1}\cdots w_n) + \chi(i \notin \textsc{exc}(w_1\cdots w_{i-1}w_{n+1}w_{i+1}\cdots w_n)),$$

where $\chi(P) = 1$ if $P$ is true, and 0 otherwise. If $w_{n+1} = p_i$, then $w_1\cdots w_{i-1}w_{i+1}\cdots w_n \in \mathscr{D}(\mathbf{n}-\mathbf{e}_{p_i})$ and $e(w) = e(w_1\cdots w_{i-1}w_{i+1}\cdots w_n)+1$, where $\mathbf{n}-\mathbf{e}_{p_i}$ is the multiset obtained from $\mathbf{n}$ by removing one copy of the $p_i$th type element. Putting pieces together, we have

$$
\begin{aligned}
d_{\mathbf{n}+\mathbf{e}_{m+1}}(x) &= \sum_{i=1}^{n} \sum_{w\in\mathscr{D}(\mathbf{n})} x^{e(w)+\chi(i\notin \mathrm{EXC}(w))} + \sum_{i=1}^{n} \sum_{w\in\mathscr{D}(\mathbf{n}-\mathbf{e}_{p_i})} x^{e(w)+1} \\
&= \sum_{w\in\mathscr{D}(\mathbf{n})} \left[e(w)x^{e(w)} + (n-e(w))x^{e(w)+1}\right] + \sum_{i=1}^{n} \sum_{w\in\mathscr{D}(\mathbf{n}-\mathbf{e}_{p_i})} x^{e(w)+1} \\
&= xd'_{\mathbf{n}}(x) + nxd_{\mathbf{n}}(x) - x^2 d'_{\mathbf{n}}(x) + x\sum_{i=1}^{n} d_{\mathbf{n}-\mathbf{e}_{p_i}}(x) \\
&= x[n_1 d_{\mathbf{n}-\mathbf{e}_1}(x) + \cdots + n_m d_{\mathbf{n}-\mathbf{e}_m}(x)] + nxd_{\mathbf{n}}(x) + x(1-x)d'_{\mathbf{n}}(x).
\end{aligned}
$$

Case 2: Consider now the case that $j = m$. Let $w = w_1 \cdots w_n w_{n+1} \in \mathscr{D}(\mathbf{n}+\mathbf{e}_m)$. There exist $i_1, \ldots, i_{n_m+1} \in [n-n_m]$ such that $w_{i_1} = \cdots = w_{i_{n_m+1}} = m$. It is clear that $i_1, \ldots, i_{n_m+1}$ are excedances of $w$. For $i \in \{i_1, \ldots, i_{n_m+1}\}$, consider the word $w_1 \cdots w_{i-1} w_{n+1} w_{i+1} \cdots w_n m$ obtained by swapping the $i$th and the $(n+1)$st letters of $w$. If $w_{n+1} \neq p_i$, then $w_1 \cdots w_{i-1} w_{n+1}\ w_{i+1} \cdots w_n \in \mathscr{D}(\mathbf{n})$ and

$$
e(w) = e(w_1 \cdots w_{i-1} w_{n+1} w_{i+1} \cdots w_n) + \chi(i \notin \mathrm{EXC}(w_1 \cdots w_{i-1} w_{n+1} w_{i+1} \cdots w_n)).
$$

If $w_{n+1} = p_i$, then $w_1 \cdots w_{i-1} w_{i+1} \cdots w_n \in \mathscr{D}(\mathbf{n}-\mathbf{e}_{p_i})$ and $e(w) = e(w_1 \cdots w_{i-1}\ w_{i+1} \cdots w_n) + 1$. The map

$$
w \mapsto \begin{cases} w_1 \cdots w_{i-1} w_{n+1} w_{i+1} \cdots w_n & \text{if } w_{n+1} \neq p_i, \\ w_1 \cdots w_{i-1} w_{i+1} \cdots w_n & \text{if } w_{n+1} = p_i, \end{cases}
$$

is a $(n_m+1)$-to-one correspondence between $\mathscr{D}(\mathbf{n}+\mathbf{e}_m)$ and $\mathscr{D}(\mathbf{n}) \cup \mathscr{D}(\mathbf{n}-\mathbf{e}_{p_i})$. Identifying now the indices $i_1, \ldots, i_{n_m+1}$, there are $e(w_1 \cdots w_{i-1} w_{n+1} w_{i+1} \cdots w_n) - n_m$ "distinct" excedances and $n - n_m - e(w_1 \cdots w_{i-1} w_{n+1} w_{i+1} \cdots w_n)$ "distinct" non-excedances in $[n-n_m]$. Consequently, we have

$$
\begin{aligned}
(n_m+1)d_{\mathbf{n}+\mathbf{e}_m}(x) &= \sum_{i=1}^{n-n_m} \left( \sum_{w\in\mathscr{D}(\mathbf{n})} x^{e(w)+\chi(i\notin \mathrm{EXC}(w))} + \sum_{w\in\mathscr{D}(\mathbf{n}-\mathbf{e}_{p_i})} x^{e(w)+1} \right) \\
&= \sum_{w\in\mathscr{D}(\mathbf{n})} \left[(e(w)-n_m)x^{e(w)} + (n-n_m-e(w))x^{e(w)+1}\right] + x\sum_{i=1}^{n-n_m} d_{\mathbf{n}-\mathbf{e}_{p_i}}(x) \\
&= xd'_{\mathbf{n}}(x) - n_m d_{\mathbf{n}}(x) + (n-n_m)xd_{\mathbf{n}}(x) - x^2 d'_{\mathbf{n}}(x) + x\sum_{i=1}^{n-n_m} d_{\mathbf{n}-\mathbf{e}_{p_i}}(x) \\
&= x[n_1 d_{\mathbf{n}-\mathbf{e}_1}(x) + \cdots + n_{m-1} d_{\mathbf{n}-\mathbf{e}_{m-1}}(x)] + [(n-n_m)x - n_m]d_{\mathbf{n}}(x) \\
&\qquad + x(1-x)d'_{\mathbf{n}}(x).
\end{aligned}
$$

Case 3: Let $\omega$ be the cyclic permutation of $(1,2,\dots,m)$, i.e., $\omega(i)=i+1$ for $i=1,2,\dots,m-1$ and $\omega(m)=1$. For $j=1,2,\dots,m-1$, denote by

$$\begin{aligned}\omega^{m-j}\mathbf{n} &= \{\omega^{m-j}(1)^{n_1},\omega^{m-j}(2)^{n_2},\dots,\omega^{m-j}(m)^{n_m}\}\\ &= \{1^{n_{j+1}},2^{n_{j+2}},\dots,(m-j)^{n_m},(m-j+1)^{n_1},\dots,(m-1)^{n_{j-1}},m^{n_j}\}\end{aligned}$$

the multiset obtained by applying $\omega^{m-j}$ to the elements of $\mathbf{n}$. It is clear that $\mathbf{n}+\mathbf{e}_j=\omega^{-(m-j)}(\omega^{m-j}\mathbf{n}+\mathbf{e}_m)$. By Property 2.1(iii) and Case 2, we have

$$\begin{aligned}(n_j+1)d_{\mathbf{n}+\mathbf{e}_j}(x) &= (n_j+1)d_{\omega^{-(m-j)}(\omega^{m-j}\mathbf{n}+\mathbf{e}_m)}(x)\\ &= (n_j+1)d_{\omega^{m-j}\mathbf{n}+\mathbf{e}_m}(x)\\ &= x[n_{j+1}d_{\omega^{m-j}\mathbf{n}-\mathbf{e}_1}(x)+\cdots+n_{j-1}d_{\omega^{m-j}\mathbf{n}-\mathbf{e}_{m-1}}(x)]\\ &\qquad +[(n-n_j)x-n_j]d_{\omega^{m-j}\mathbf{n}}(x)+x(1-x)d'_{\omega^{m-j}\mathbf{n}}(x)\\ &= x[n_{j+1}d_{\omega^{-(m-j)}(\omega^{m-j}\mathbf{n}-\mathbf{e}_1)}(x)+\cdots+n_{j-1}d_{\omega^{-(m-j)}(\omega^{m-j}\mathbf{n}-\mathbf{e}_{m-1})}(x)]\\ &\qquad +[(n-n_j)x-n_j]d_{\mathbf{n}}(x)+x(1-x)d'_{\mathbf{n}}(x)\\ &= x[n_1d_{\mathbf{n}-\mathbf{e}_1}(x)+\cdots+n_{j-1}d_{\mathbf{n}-\mathbf{e}_{j-1}}(x)+n_{j+1}d_{\mathbf{n}-\mathbf{e}_{j+1}}(x)\\ &\qquad\quad +\cdots+n_md_{\mathbf{n}-\mathbf{e}_m}(x)]+[(n-n_j)x-n_j]d_{\mathbf{n}}(x)+x(1-x)d'_{\mathbf{n}}(x),\end{aligned}$$

as desired. □

In case $n_1=n_2=\cdots=n_m=1$, (5) reduces to (1) with $n=n_1+\cdots+n_m=m$. Note that if we regard $\mathbf{n}=\{1^{n_1},\dots,m^{n_m}\}$ as the multiset $\{1^{n_1},\dots,m^{n_m},(m+1)^{n_{m+1}}\}$, where $n_{m+1}:=0$, then (6) with $m+1$ in place of $m$ and $j=m+1$ becomes (5).

By exploiting (2), an alternative proof of the recurrence relation (6) can be given.

Towards this end, we first note the following properties of the elementary symmetric functions $e_k=e_k(x_1,\dots,x_m)$:

(i) $e_k=\widehat{e}_{j,k}+x_j\widehat{e}_{j,k-1}$,
(ii) $\frac{\partial}{\partial x_j}e_k=\widehat{e}_{j,k-1}$,
(iii) $\sum_{j=1}^m\widehat{e}_{j,k}=(m-k)e_k$,
(iv) $\sum_{j=1}^m x_j\widehat{e}_{j,k}=(k+1)e_{k+1}$,

where $\widehat{e}_{j,k}=e_k(x_1,\dots,\hat{x}_j,\dots,x_m)$ denotes the $k$th elementary symmetric function in the alphabet $\{x_1,\dots,x_m\}\setminus\{x_j\}$, and $j,k=1,2,\dots,m$. We have

$$\begin{aligned}I &= \sum_{n_1,\dots,n_m\geqslant 0}(n_j+1)d_{\mathbf{n}+\mathbf{e}_j}(x)x_1^{n_1}\cdots x_m^{n_m}\\ &= \frac{\partial}{\partial x_j}\sum_{n_1,\dots,n_m\geqslant 0}d_{\mathbf{n}+\mathbf{e}_j}(x)x_1^{n_1}\cdots x_j^{n_j+1}\cdots x_m^{n_m}\\ &= \frac{\partial}{\partial x_j}\Bigg(\sum_{n_1,\dots,n_j,\dots,n_m\geqslant 0}d_{\mathbf{n}}(x)x_1^{n_1}\cdots x_m^{n_m}\\ &\qquad -\sum_{n_1,\dots,n_{j-1},n_{j+1},\dots,n_m\geqslant 0}d_{\hat{\mathbf{n}}}(x)x_1^{n_1}\cdots x_{j-1}^{n_{j-1}}x_{j+1}^{n_{j+1}}\cdots x_m^{n_m}\Bigg)\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial x_j}\left(\frac{1}{1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k}\right)\\
&= \frac{\sum_{k=2}^m (x+\cdots+x^{k-1})\widehat{e}_{j,k-1}}{(1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k)^2},\\
II &= \sum_{n_1,\dots,n_m\geqslant 0}\sum_{i\neq j} x n_i d_{\mathbf{n}-\mathbf{e}_i}(x) x_1^{n_1}\cdots x_m^{n_m}\\
&= \sum_{i\neq j}\Bigg[x x_i^2 \sum_{\substack{n_1,\dots,n_{i-1},n_{i+1},\dots,n_m\geqslant 0\\ n_i\geqslant 1}} (n_i-1) d_{\mathbf{n}-\mathbf{e}_i}(x) x_1^{n_1}\cdots x_i^{n_i-2}\cdots x_m^{n_m}\\
&\qquad + x x_i \sum_{\substack{n_1,\dots,n_{i-1},n_{i+1},\dots,n_m\geqslant 0\\ n_i\geqslant 1}} d_{\mathbf{n}-\mathbf{e}_i}(x) x_1^{n_1}\cdots x_i^{n_i-1}\cdots x_m^{n_m}\Bigg]\\
&= \sum_{i\neq j}\left[x x_i^2 \frac{\partial}{\partial x_i}\left(\frac{1}{1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k}\right) + \frac{x x_i}{1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k}\right]\\
&= \sum_{i\neq j}\frac{x x_i^2 \sum_{k=2}^m (x+\cdots+x^{k-1})\widehat{e}_{i,k-1}}{(1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k)^2} + \frac{x\widehat{e}_{j,1}}{1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k}\\
&= \frac{\sum_{i\neq j} x_i \sum_{k=2}^m (x^2+\cdots+x^k)(e_k-\widehat{e}_{i,k}) + x\widehat{e}_{j,1}(1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k)}{(1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k)^2}\\
&= \frac{x\widehat{e}_{j,1} - \sum_{k=2}^m (x^2+\cdots+x^k)\sum_{i\neq j} x_i \widehat{e}_{i,k}}{(1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k)^2}\\
&= \frac{x\widehat{e}_{j,1} - \sum_{k=2}^m (x^2+\cdots+x^k)(k e_{k+1}+\widehat{e}_{j,k+1})}{(1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k)^2},\\
III &= \sum_{n_1,\dots,n_m\geqslant 0} [(n-n_j)x-n_j] d_{\mathbf{n}}(x) x_1^{n_1}\cdots x_m^{n_m}\\
&= \sum_{i\neq j} x \sum_{n_1,\dots,n_m\geqslant 0} n_i d_{\mathbf{n}}(x) x_1^{n_1}\cdots x_m^{n_m} - \sum_{n_1,\dots,n_m\geqslant 0} n_j d_{\mathbf{n}}(x) x_1^{n_1}\cdots x_m^{n_m}\\
&= \sum_{i\neq j} x x_i \frac{\partial}{\partial x_i}\left(\frac{1}{1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k}\right)\\
&\qquad\qquad - x_j \frac{\partial}{\partial x_j}\left(\frac{1}{1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k}\right)\\
&= \frac{\sum_{i\neq j} x x_i \sum_{k=2}^m (x+\cdots+x^{k-1})\widehat{e}_{i,k-1} - x_j\sum_{k=2}^m (x+\cdots+x^{k-1})\widehat{e}_{j,k-1}}{(1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k)^2}\\
&= \frac{\sum_{k=2}^m (x^2+\cdots+x^k)\sum_{i\neq j} x_i\widehat{e}_{i,k-1} - \sum_{k=2}^m (x+\cdots+x^{k-1}) x_j \widehat{e}_{j,k-1}}{(1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sum_{k=2}^m (x^2+\cdots+x^k)ke_k - \sum_{k=2}^m (x+2x^2+\cdots+2x^{k-1}+x^k)x_j\widehat{e}_{j,k-1}}{(1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k)^2}\\
&= \frac{\sum_{k=2}^m (x^2+\cdots+x^k)ke_k - \sum_{k=2}^m (x+2x^2+\cdots+2x^{k-1}+x^k)(e_k-\widehat{e}_{j,k})}{(1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k)^2}\\
&= \frac{\sum_{k=2}^m (-x+(k-2)x^2+\cdots+(k-2)x^{k-1}+(k-1)x^k)e_k}{(1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k)^2}\\
&\qquad + \frac{\sum_{k=2}^m (x+2x^2+\cdots+2x^{k-1}+x^k)\widehat{e}_{j,k}}{(1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k)^2},
\end{aligned}
$$

$$
\begin{aligned}
IV &= \sum_{n_1,\ldots,n_m\geqslant 0} x(1-x)d'_{\mathbf{n}}(x)x_1^{n_1}\cdots x_m^{n_m}\\
&= x(1-x)\frac{\partial}{\partial x}\left(\frac{1}{1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k}\right)\\
&= \frac{x(1-x)\sum_{k=2}^m (1+2x+3x^2+\cdots+(k-1)x^{k-2})e_k}{(1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k)^2}\\
&= \frac{\sum_{k=2}^m (x+\cdots+x^{k-1}-(k-1)x^k)e_k}{(1-\sum_{k=2}^m (x+\cdots+x^{k-1})e_k)^2}.
\end{aligned}
$$

It is not hard to see that

$$I = II + III + IV,$$

so that

$$
\begin{aligned}
\sum_{n_1,\ldots,n_m\geqslant 0} d_{\mathbf{n}+\mathbf{e}_j}(x)x_1^{n_1}\cdots x_m^{n_m} = \sum_{n_1,\ldots,n_m\geqslant 0} &\{x[n_1 d_{\mathbf{n}+\mathbf{e}_1}(x)+\cdots+n_{j-1}d_{\mathbf{n}+\mathbf{e}_{j-1}}(x)\\
&+ n_{j+1}d_{\mathbf{n}+\mathbf{e}_{j+1}}(x)+\cdots+n_m d_{\mathbf{n}+\mathbf{e}_m}(x)]\\
&+ [(n-n_j)x-n_j]d_{\mathbf{n}}(x)+x(1-x)d'_{\mathbf{n}}(x)\}x_1^{n_1}\cdots x_m^{n_m}.
\end{aligned}
$$

Equating the coefficients of $x_1^{n_1}\cdots x_m^{n_m}$, (6) follows.

## 4. Real-rootedness

A real polynomial $f(x)$ is *standard* if its leading coefficient is positive; it is *simply real-rooted* if all its zeros are real and simple. Let $f(x)$ and $g(x)$ be simply real-rooted standard polynomials of degrees $n$ and $n-1$, respectively. Then $g(x)$ is said to *interlace* $f(x)$ if

$$\theta_1 \leqslant \xi_1 \leqslant \theta_2 \leqslant \xi_2 \leqslant \cdots \leqslant \theta_{n-1} \leqslant \xi_{n-1} \leqslant \theta_n, \tag{7}$$

where $\theta_1,\ldots,\theta_n$ and $\xi_1,\ldots,\xi_{n-1}$ are the zeros of $f(x)$ and $g(x)$, respectively. If all the inequalities in (7) are strict, then $g(x)$ is said to *strictly interlace* $f(x)$.

We shall need the following easily established fact about interlacing polynomials:

(*) If $f(x)$ and $g(x)$ are two simply real-rooted standard polynomials of degrees $n$ and $n-1$, respectively, and $g(x)$ strictly interlaces $f(x)$, then $\operatorname{sgn} g(\theta_i) = (-1)^{n-i}$ for $i = 1,2,\ldots,n$, and $\operatorname{sgn} f(\xi_j) = (-1)^{n-j}$ for $j = 1,2,\ldots,n-1$, where $\theta_1 < \theta_2 < \cdots < \theta_n$ and $\xi_1 < \xi_2 < \cdots < \xi_{n-1}$ are the simple real zeros of $f(x)$ and $g(x)$, respectively.

Theorem 4.1. *For each multiset* $\mathbf{n} = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$, *the polynomial* $d_{\mathbf{n}}(x)$ *is real-rooted. The multiplicity of* $0$ *as a zero of* $d_{\mathbf{n}}(x)$ *is equal to* $\max_{1\leqslant i\leqslant m} n_i$ *if* $d_{\mathbf{n}}(x)$ *is not identically zero.*

*Proof.* For the sake of simplicity, we let $N(\mathbf{n}) := \max_{1\leqslant i\leqslant m} n_i$. By virtue of Proposition 2.3, $d_{\mathbf{n}}(x)$ is a polynomial in $x$ of degree $n - N(\mathbf{n})$ and with center of symmetry at $n/2$. The symmetry of $d_{\mathbf{n}}(x)$ then implies that the lowest order term of $d_{\mathbf{n}}(x)$ has degree $N(\mathbf{n})$. Thus, $d_{\mathbf{n}}(x) = x^{N(\mathbf{n})}\tilde{d}_{\mathbf{n}}(x)$ for some symmetric polynomial $\tilde{d}_{\mathbf{n}}(x)$ of degree $n - 2N(\mathbf{n})$ and with positive constant term. The multiplicity of 0 as a zero of $d_{\mathbf{n}}(x)$ being equal to $N(\mathbf{n})$ follows.

To prove the real-rootedness of $d_{\mathbf{n}}(x)$, we proceed by proving that $\tilde{d}_{\mathbf{n}}(x)$ (respectively, $\tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x)$) strictly interlaces $\tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x)$ (respectively, $\tilde{d}_{\mathbf{n}}(x)$) if $\deg \tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x) = \deg \tilde{d}_{\mathbf{n}}(x)+1$ (respectively, $\deg \tilde{d}_{\mathbf{n}}(x) = \deg \tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x)+1$) by induction on $n = n_1+\cdots+n_m$, the cases $n = 2, 3, 4$ being clear from the first eight entries of Table 1. Assume now that the result holds for any multiset $\mathbf{n}'$ such that $n' \leqslant n$, where $n \geqslant 4$. Let $\mathbf{n}''$ be a multiset such that $n'' = n + 1$, which can be obtained from a multiset $\mathbf{n} = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$ with $n_1 + \cdots + n_m = n$ by adjoining an additional copy of the $j$th type element for some $j \in [m + 1]$ (see the paragraph following the proof of Proposition 3.1). If $N(\mathbf{n} + \mathbf{e}_j) > (n+1)/2$, then Property 2.1(i) implies that $d_{\mathbf{n}+\mathbf{e}_j}(x) \equiv 0$. If $N(\mathbf{n}+\mathbf{e}_j) = (n+1)/2$, then Property 2.1(i) implies that $d_{\mathbf{n}+\mathbf{e}_j}(x)$ is a monomial having only the trivial zero $x = 0$ of multiplicity $N(\mathbf{n} + \mathbf{e}_j)$. If $N(\mathbf{n} + \mathbf{e}_j) < (n + 1)/2$, there are two cases to consider, namely

(a) $N(\mathbf{n} + \mathbf{e}_j) = N(\mathbf{n}) + 1$,
(b) $N(\mathbf{n} + \mathbf{e}_j) = N(\mathbf{n})$.

Note that in Case (a),

$$\deg \tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x) = n + 1 - 2N(\mathbf{n} + \mathbf{e}_j) = n - 2N(\mathbf{n}) - 1 = \deg \tilde{d}_{\mathbf{n}}(x) - 1,$$

and in Case (b)

$$\deg \tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x) = n + 1 - 2N(\mathbf{n} + \mathbf{e}_j) = n - 2N(\mathbf{n}) + 1 = \deg \tilde{d}_{\mathbf{n}}(x) + 1.$$

Let $x_1 < x_2 < \cdots < x_{n-2N(\mathbf{n})} < 0$ be the real zeros of $\tilde{d}_{\mathbf{n}}(x)$. Let also $x_0 := -\infty$ and $x_{n-2N(\mathbf{n})+1} := 0$.

Since $d_{\mathbf{n}}(x) = x^{N(\mathbf{n})}\tilde{d}_{\mathbf{n}}(x)$, differentiation with respect to $x$ gives that

$$d'_{\mathbf{n}}(x) = N(\mathbf{n})x^{N(\mathbf{n})-1}\tilde{d}_{\mathbf{n}}(x) + x^{N(\mathbf{n})}\tilde{d}'_{\mathbf{n}}(x).$$

Since $\tilde{d}'_{\mathbf{n}}(x)$ strictly interlaces $\tilde{d}_{\mathbf{n}}(x)$, (*) above implies that

$$\operatorname{sgn} d'_{\mathbf{n}}(x_i) = (-1)^{N(\mathbf{n})}(-1)^{n-2N(\mathbf{n})-i} = (-1)^{n-N(\mathbf{n})-i} \tag{8}$$

for $i = 1, 2, \ldots, n - 2N(\mathbf{n})$.

Let $k \in [m + 1] \setminus \{j\}$. If $N(\mathbf{n} - \mathbf{e}_k) = N(\mathbf{n})$, then

$$\deg \tilde{d}_{\mathbf{n}-\mathbf{e}_k}(x) = n - 1 - 2N(\mathbf{n}) = \deg \tilde{d}_{\mathbf{n}}(x) - 1$$

so that by the induction hypothesis, $\tilde{d}_{\mathbf{n}-\mathbf{e}_k}(x)$ strictly interlaces $\tilde{d}_{\mathbf{n}}(x)$. Fact (*) above then yields that $\operatorname{sgn} \tilde{d}_{\mathbf{n}-\mathbf{e}_k}(x_i) = (-1)^{n-2N(\mathbf{n})-i}$ for $i = 1, 2, \ldots, n - 2N(\mathbf{n})$. Since $d_{\mathbf{n}-\mathbf{e}_k}(x) = x^{N(\mathbf{n})}\tilde{d}_{\mathbf{n}-\mathbf{e}_k}(x)$, we have

$$\operatorname{sgn} d_{\mathbf{n}-\mathbf{e}_k}(x_i) = (-1)^{N(\mathbf{n})}(-1)^{n-2N(\mathbf{n})-i} = (-1)^{n-N(\mathbf{n})-i} \tag{9}$$

for $i = 1, 2, \ldots, n - 2N(\mathbf{n})$.

If $N(\mathbf{n} - \mathbf{e}_k) = N(\mathbf{n}) - 1$, then

$$\deg \tilde{d}_{\mathbf{n}-\mathbf{e}_k}(x) = n - 1 - 2(N(\mathbf{n}) - 1) = \deg \tilde{d}_{\mathbf{n}}(x) + 1$$

so that the induction hypothesis then implies that $\tilde{d}_{\mathbf{n}}(x)$ strictly interlaces $\tilde{d}_{\mathbf{n}-\mathbf{e}_k}(x)$. Fact (*) then yields that $\operatorname{sgn} \tilde{d}_{\mathbf{n}-\mathbf{e}_k}(x_i) = (-1)^{n+1-2N(\mathbf{n})-i}$ for $i = 1, 2, \ldots, n - 2N(\mathbf{n})$. Since $d_{\mathbf{n}-\mathbf{e}_k}(x) = x^{N(\mathbf{n})-1}\tilde{d}_{\mathbf{n}-\mathbf{e}_k}(x)$, we have

$$\operatorname{sgn} d_{\mathbf{n}-\mathbf{e}_k}(x_i) = (-1)^{N(\mathbf{n})-1}(-1)^{n+1-2N(\mathbf{n})-i} = (-1)^{n-N(\mathbf{n})-i} \tag{10}$$

for $i = 1, 2, \ldots, n - 2N(\mathbf{n})$.

Setting now $x = x_i$ in (6), we have

$$\begin{aligned}(n_j + 1)d_{\mathbf{n}+\mathbf{e}_j}(x_i) = x_i[&n_1 d_{\mathbf{n}-\mathbf{e}_1}(x_i) + \cdots + n_{j-1}d_{\mathbf{n}-\mathbf{e}_{j-1}}(x_i) \\ &+ n_{j+1}d_{\mathbf{n}-\mathbf{e}_{j+1}}(x_i) + \cdots + n_m d_{\mathbf{n}-\mathbf{e}_m}(x_i)] + x_i(1 - x_i)d'_{\mathbf{n}}(x_i).\end{aligned}$$

Since $x_i < 0$ and $x_i(1 - x_i) < 0$, by (8), (9) and (10), all terms on the right side have the same sign so that

$$\operatorname{sgn} d_{\mathbf{n}+\mathbf{e}_j}(x_i) = (-1)^{n-N(\mathbf{n})+1-i}$$

for $i = 1, 2, \ldots, n - 2N(\mathbf{n})$.

In Case (a), $d_{\mathbf{n}+\mathbf{e}_j}(x) = x^{N(\mathbf{n})+1}\tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x)$ so that

$$\operatorname{sgn} \tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x_i) = (-1)^{n-2N(\mathbf{n})-i}$$

for $i = 1, 2, \ldots, n - 2N(\mathbf{n})$. Thus, there exist $x_i^* \in (x_i, x_{i+1})$ for which $\tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x_i^*) = 0$ for $i = 1, 2, \ldots, n - 2N(\mathbf{n}) - 1$. These $n - 2N(\mathbf{n}) - 1$ simple real zeros of $\tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x)$, together with the trivial zero $x = 0$ of multiplicity $N(\mathbf{n}) + 1$, account for the $n + 1 - N(\mathbf{n} + \mathbf{e}_j) = n - N(\mathbf{n})$ real zeros of $d_{\mathbf{n}+\mathbf{e}_j}(x)$.

In Case (b), $d_{\mathbf{n}+\mathbf{e}_j}(x) = x^{N(\mathbf{n})}\tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x)$ so that

$$\operatorname{sgn} \tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x_i) = (-1)^{n+1-2N(\mathbf{n})-i}$$

for $i = 1, 2, \ldots, n - 2N(\mathbf{n})$. Also,

$$\operatorname{sgn} \tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x_0) = (-1)^{n+1-2N(\mathbf{n})} \quad \text{and} \quad \operatorname{sgn} \tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x_{n-2N(\mathbf{n})+1}) = +1$$

(since the constant term of $\tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x)$ is positive). Thus, there exist $x_i^* \in (x_{i-1}, x_i)$ for which $\tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x_i^*) = 0$ for $i = 1, 2, \ldots, n - 2N(\mathbf{n}) + 1$. These $n - 2N(\mathbf{n}) + 1$ simple real zeros of $\tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x)$, together with the trivial zero $x = 0$ of multiplicity $N(\mathbf{n})$, account for the $n + 1 - N(\mathbf{n} + \mathbf{e}_j) = n - N(\mathbf{n}) + 1$ real zeros of $d_{\mathbf{n}+\mathbf{e}_j}(x)$.

Note finally that in Case (a), $\deg \tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x) = \deg \tilde{d}_{\mathbf{n}}(x) - 1$ and $\tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x)$ strictly interlaces $\tilde{d}_{\mathbf{n}}(x)$, and that in Case (b), $\deg \tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x) = \deg \tilde{d}_{\mathbf{n}}(x) + 1$ and $\tilde{d}_{\mathbf{n}}(x)$ strictly interlaces $\tilde{d}_{\mathbf{n}+\mathbf{e}_j}(x)$. This completes the induction and the proof of the theorem. □

Theorem 4.1 establishes that $\{d_{\mathbf{n}}(x)\}$ is a multi-indexed Sturm sequence. Similar Sturm sequences had been studied previously by Simion [11], who proved that $f_{\mathbf{n}}(x)$ has all its zeros in the interval $[-1, 0]$ and that $f_{\mathbf{n}}(x)$ and $f_{\mathbf{n}+\mathbf{e}_j}(x)$ have interlaced zeros, where $f_{\mathbf{n}}(x) := \sum_{k \geqslant 0} \mathscr{O}(\mathbf{n}, k) x^k$ is the generating function of the number of compositions $\mathscr{O}(\mathbf{n}, k)$ of the multiset $\mathbf{n}$ into exactly $k$ parts.

A sequence $\{a_0, a_1, \ldots, a_d\}$ of real numbers is called *log-concave* if $a_{i-1}a_{i+1} \leqslant a_i^2$ for $i = 1, 2, \ldots, d-1$. It is *unimodal* if there exists an index $0 \leqslant j \leqslant d$ such that $a_i \leqslant a_{i+1}$ for $i = 0, 1, \ldots, j-1$ and $a_i \geqslant a_{i+1}$ for $i = j, j+1, \ldots, d-1$. It has *no internal zeros* if there are not three indices $0 \leqslant i < j < k \leqslant d$ such that $a_i, a_k \neq 0$ but $a_j = 0$. It is *symmetric* if $a_i = a_{d-i}$ for $i = 0, \ldots, \lfloor d/2 \rfloor$. It is a *Pólya frequency sequence of order* $r$ (or a $PF_r$ *sequence*) if any minor of order $r$ of the matrix $M = (M_{ij})_{i,j \in \mathbb{N}}$ defined by $M_{ij} = a_{j-i}$ for all $i, j \in \mathbb{N}$ (where $a_k = 0$ if $k < 0$ or $k > d$) is non-negative. It is a *Pólya frequency sequence of infinite order* (or a *PF sequence*) if it is a $PF_r$ sequence for all $r \geqslant 1$.

It is clear that a positive sequence is $PF_1$, and a log-concave (which is also unimodal and internal-zero free) sequence is $PF_2$.

A polynomial $\sum_{i=0}^{d} a_i x^i$ is symmetric (respectively, unimodal, log-concave, with no internal zeros) if the sequence $\{a_0, a_1, \ldots, a_d\}$ has the corresponding property. If $p(x)$ is a symmetric unimodal polynomial, then its *center of symmetry* $C(p) = (\deg(p) + \mathrm{mult}(0, p))/2$, where $\mathrm{mult}(0, p)$ is the multiplicity of 0 as a zero of $p$. If we write $p(x) = x^n p(x^{-1})$, then $C(p) = n/2$. An elementary property of symmetric unimodal polynomials is the following. See, e.g., [4, Proposition 2.1] and [12, Proposition 1].

Proposition 4.2. *Let $p(x)$ and $q(x)$ be two symmetric unimodal polynomials. Then $p(x)q(x)$ is a symmetric unimodal polynomial and $C(pq) = C(p) + C(q)$.*

An important classical result concerning *PF* sequences and polynomials having only real zeros is the following [2, Theorem 2.2.4].

Theorem 4.3 (Aissen–Schoenberg–Whitney). *Let $p(x) = \sum_{i=0}^{d} a_i x^i \in \mathbb{R}[x]$ have non-negative coefficients. Then $p(x)$ has only real zeros if and only if $\{a_0, a_1, \ldots, a_d\}$ is a PF sequence.*

Corollary 4.4. *For each multiset $\mathbf{n} = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$, the sequence*

$$\{\Theta(\mathbf{n}, k)\}_{k = N(\mathbf{n}), \ldots, n - N(\mathbf{n})}$$

*is a PF sequence, where $N(\mathbf{n}) := \max_{1 \leqslant i \leqslant m} n_i$. In particular, it is unimodal and log-concave.*

*Proof.* Combine Theorem 4.1 and Theorem 4.3 to conclude. □

## 5. A combinatorial expansion

We give in this section an expansion formula of $d_{\mathbf{n}}(x)$. See Proposition 5.1 below.

By virtue of Property 2.1(iii), we may write

$$\sum_{n_1,\ldots,n_m\geqslant 0} d_{\mathbf{n}}(x)x_1^{n_1}\cdots x_m^{n_m} = 1 + \sum_{1\leqslant l(\mathbf{n})\leqslant m} d_{\mathbf{n}}(x)m_{\mu(\mathbf{n})}, \tag{11}$$

where the sum on the right ranges over all multisets $\mathbf{n} = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$ such that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_m \geqslant 0$, $\mu(\mathbf{n}) = (n_1, n_2, \ldots, n_m)$, $l(\mathbf{n}) = \#\{i \in [m] \colon n_i > 0\}$ (which is the number of positive parts of $\mu(\mathbf{n})$) and $m_\mu = m_\mu(x_1, x_2, \ldots, x_m)$ is the monomial symmetric function in $x_1, x_2, \ldots, x_m$ indexed by $\mu$.

Combining (2) and (11), we have the following symmetric function identity

$$1 + \sum_{1\leqslant l(\mathbf{n})\leqslant m} d_{\mathbf{n}}(x)m_{\mu(\mathbf{n})} = \frac{1}{1 - xe_2 - (x+x^2)e_3 - \cdots - (x+x^2+\cdots+x^{m-1})e_m}$$

in $\Lambda_{\mathbb{Z}[x]}(x_1, \ldots, x_m)$, the ring of symmetric functions in $x_1, \ldots, x_m$ over $\mathbb{Z}[x]$.

Proposition 5.1. *We have*

$$d_{\mathbf{n}}(x) = \sum_{\substack{k_1+\cdots+k_l=n\\ 2\leqslant k_1,\ldots,k_l\leqslant m}} M_{(k_1,\ldots,k_l),\mu(\mathbf{n})} \prod_{j=1}^{l}(x + x^2 + \cdots + x^{k_j-1}), \tag{12}$$

*where the sum ranges over all compositions* $(k_1, \ldots, k_l)$ *of* $n$ *such that* $2 \leqslant k_1, \ldots, k_l \leqslant m$.

*Proof.* Comparing (11) and (4), and invoking the linear independence of $m_\mu$. □

The identity (12) is a combinatorial expansion of $d_{\mathbf{n}}(x)$ in terms of the polynomials $\prod_{j=1}^{l}(x + x^2 + \cdots + x^{k_j-1})$ each of which is symmetric unimodal and has center of symmetry at $\sum_{j=1}^{l} k_j/2 = n/2$ by Proposition 4.2, thus refining Proposition 2.3 and Corollary 4.4.

On the other hand, (12) is a multi-analogue of the one [3, Proposition 6] for $d_n(x)$, namely,

$$d_n(x) = \sum_{\lambda\vdash n} f^\lambda R_\lambda(x), \tag{13}$$

where

$$R_\lambda(x) = \sum_{\substack{k_1+\cdots+k_l=n\\ 2\leqslant k_1,\ldots,k_l\leqslant n}} K_{\lambda,(k_1,\ldots,k_l)} \prod_{j=1}^{l}(x + x^2 + \cdots + x^{k_j-1}), \tag{14}$$

$f^\lambda$ is the number of standard Young tableaux of shape $\lambda$ and $K_{\lambda,\mu}$ is a Kostka number. The identity (14) is obtained by expanding [3, eqn (6)], that is,

$$\sum_\lambda R_\lambda(x)s_\lambda = \frac{1}{1 - \sum_{k\geqslant 2}(x + x^2 + \cdots + x^{k-1})s_k},$$

where $s_k$ and $s_\lambda$ are Schur functions indexed, respectively, by the partitions $(k)$ and $\lambda$. A combinatorial proof of (13) and (14) had been given by Stembridge in [14, p. 319].

Similar expansions of the generating function for multiderangements by the numbers of cycles and excedances has already been given by Zeng [15, Eq. (2.14)].

## 6. Extension to multipermutations

Let $\mathbf{n} = \{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}$ be a multiset, where $n := n_1 + \cdots + n_m$. Denote by $\mathscr{R}(\mathbf{n})$ the set of all multipermutations of $\mathbf{n}$. Let $w = w_1 w_2 \cdots w_n \in \mathscr{R}(\mathbf{n})$ and $\delta(w) = p_1 p_2 \cdots p_n$ be its non-decreasing rearrangement. Denote by $f(w) := \#\{i \in [n] \colon w_i = p_i\}$ the number of fixed points of $w$, and define

$$r_{\mathbf{n}}(x, Y) = \sum_{w \in \mathscr{R}(\mathbf{n})} x^{e(w)} Y^{f(w)}.$$

The identity (2) can be extended to count multipermutations by the numbers of excedances and fixed points by using the classical MacMahon Master Theorem [9, pp. 97–98], which states that the coefficient of $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ in the expansion of $1/V_n$ is the same as the coefficient of the same term in $(a(1,1)x_1 + \cdots + a(1,n)x_n)^{k_1} \cdots (a(n,1)x_1 + \cdots + a(n,n)x_n)^{k_n}$, where $V_n = \det(\delta_{i,j} - a(i,j)x_j)$.

Theorem 6.1. *We have*

(15)
$$\sum_{n_1,\ldots,n_m \geqslant 0} r_{\mathbf{n}}(x, Y) x_1^{n_1} \cdots x_m^{n_m} = \left\{ \left[ 1 - \sum_{k=2}^{m} (x + x^2 + \cdots + x^{k-1}) e'_k \right] \sum_{0 \leqslant k \leqslant m} (-1)^k e_k Y^k \right\}^{-1},$$

*where $x, Y, x_1, \ldots, x_m$ are commuting indeterminates and $e_k$ and $e'_k$ $(1 \leqslant k \leqslant m)$ are the $k$th elementary symmetric function in $x_1, \ldots, x_m$ and in $\frac{x_1}{1-Yx_1}, \ldots, \frac{x_m}{1-Yx_m}$, respectively.*

*Proof.* For $i, j = 1, 2, \ldots, m$, let

$$a(i,j) = \begin{cases} x & \text{if } i < j, \\ Y & \text{if } i = j, \\ 1 & \text{if } i > j. \end{cases}$$

Each $w = w_1 \cdots w_n \in \mathscr{R}(\mathbf{n})$ is assigned the weight

$$x^{e(w)} Y^{f(w)} = \prod_{i=1}^{n} a(p_i, w_i),$$

where $\delta(w) = p_1 \cdots p_n$. The MacMahon Master Theorem then yields that

$$\sum_{n_1,\ldots,n_m \geqslant 0} r_{\mathbf{n}}(x, Y) x_1^{n_1} \cdots x_m^{n_m} = V_m^{-1},$$

where the determinant of MacMahon $V_m$ is given by

$$
\begin{aligned}
V_m &= \det(\delta_{i,j} - a(i,j)x_j) \\
&= \begin{vmatrix} 1-Yx_1 & -xx_2 & \cdots & -xx_m \\ -x_1 & 1-Yx_2 & \cdots & -xx_m \\ \vdots & \vdots & \ddots & \vdots \\ -x_1 & -x_2 & \cdots & 1-Yx_m \end{vmatrix} \\
&= (-1)^m x_1 \cdots x_m \begin{vmatrix} -(\frac{x_1}{1-Yx_1})^{-1} & x & \cdots & x \\ 1 & -(\frac{x_2}{1-Yx_2})^{-1} & \cdots & x \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -(\frac{x_m}{1-Yx_m})^{-1} \end{vmatrix} \\
&= (-1)^m \prod_{i=1}^{m} (\frac{x_i}{1-Yx_i}) \begin{vmatrix} -(\frac{x_1}{1-Yx_1})^{-1} & x & \cdots & x \\ 1 & -(\frac{x_2}{1-Yx_2})^{-1} & \cdots & x \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -(\frac{x_m}{1-Yx_m})^{-1} \end{vmatrix} \\
&\quad \times \prod_{i=1}^{m} (1-Yx_i) \\
&= \left[ 1 - \sum_{k=2}^{m} (x + x^2 + \cdots + x^{k-1}) e'_k \right] \sum_{k=0}^{m} (-1)^k e_k Y^k,
\end{aligned}
$$

where the last equality follows from [10, p. 441]. □

## 7. Concluding remarks

We have considered in this work certain important properties, namely, the invariance under permutations of $\{n_1, \ldots, n_m\}$, symmetry, recurrence relations and real-rootedness of the generating function $d_{\mathbf{n}}(x)$ of multiderangements by the number of excedances. Extension to multipermutations by the numbers of excedances and fixed points is also given. Those properties presented help make the theory of multiderangements more parallel to its classical counterparts.

## 8. Acknowledgements

The author thanks the anonymous referee for helpful suggestions.

## References

[1] R. Askey and M. Ismail, Permutation problems and special functions, *Canad. J. Math.* **28** (1976) 853–874.

[2] F. Brenti, Unimodal, log-concave and Pólya frequency sequences in combinatorics, *Mem. Amer. Math. Soc.* **81** (1989), no. 413.

[3] F. Brenti, Unimodal polynomials arising from symmetric functions, *Proc. Amer. Math. Soc.* **108** (1990) 1133–1141.

[4] F. Brenti, Permutation enumeration, symmetric functions and unimodality, *Pacific J. Math.* **157** (1993) 1–28.

[5] C.-O. Chow, On derangement polynomials of type $B$. II, submitted, 2007.

[6] P. Cartier and D. Foata, Problèmes combinatoires de commutation et réarrangements, *Lecture Notes in Mathematics*, vol. 85, Springer-Verlag, Berlin, 1969.

[7] D. Foata, Rearrangements of words, in: *Combinatorics on Words*, M. Lothaire, ed., 184–212, Addison-Wesley, Reading, Massachusetts, 1983.

[8] D.S. Kim and J. Zeng, A new decomposition of derangements, *J. Combin. Theory Ser. A* **96** (2001) 192–198.

[9] P.A. MacMahon, *Combinatory Analysis*, vol. 1, Chelsea, 1960.

[10] T. Muir, *A Treatise on the Theory of Determinants*, Longmans, London, 1933; reprinted: Dover, New York, 1960.

[11] R. Simion, A multi-indexed Sturm sequence of polynomials and unimodality of certain combinatorial sequences, *J. Combin. Theory Ser. A* **36** (1984) 15–22.

[12] R.P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, *Ann. New York Acad. Sci.* **576** (1989) 500–535.

[13] R.P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, Cambridge, 1999.

[14] J.R. Stembridge, Eulerian numbers, tableaux, and the Betti numbers of a toric variety, *Discrete Math.* **99** (1992) 307–320.

[15] J. Zeng, Linéarisation de produits de polynômes de Meixner, Krawtchouk et Charlier, *SIAM J. Math. Anal.* **21** (1990) 1349–1368.

[16] X. Zhang, On $q$-derangement polynomials, in: *Combinatorics and Graph Theory* '95, vol. 1 (Hefei), 462–465, World Sci. Publishing, River Edge, NJ, 1995.

[17] X. Zhang, On a kind of sequence of polynomials, in: Computing and Combinatorics (Xi'an, 1995), 379–383, *Lecture Notes in Computer Science*, vol. 959, Springer, Berlin, 1995.

P.O. Box 91100, Tsimshatsui Post Office, Hong Kong
*E-mail address*: `cchow@alum.mit.edu`