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Invariants, coinvariants, and the multivariate Robinson-Schensted correspondence

60th SLC
Strobl (Austria)

Preliminaries

We denote by W the symmetric group S_n.

Given a permutation $\sigma \in W$ we denote by

$$\mathrm{Des}(\sigma) = \{i | \sigma(i) > \sigma(i+1)\}$$

the set of the descents of σ and its major index by

$$\mathrm{maj}(\sigma) = \sum_{i \in \mathrm{Des}(\sigma)} i$$

If $\sigma = 35241$ we have $\mathrm{Des}(\sigma) = \{2, 4\}$.

Theorem (Mac Mahon)

$$\begin{aligned} W(q) &= \sum_{\sigma \in W} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma \in W} q^{\mathrm{inv}(\sigma)} \\ &= \prod_{i=1}^{n} (1 + q + q^2 + \cdots + q^i), \end{aligned}$$

where $\mathrm{inv}(\sigma) = |\{(i, j) : i < j \text{ and } \sigma(i) > \sigma(j)\}|$.

The coinvariant algebra associated to W is

$$R^W := \mathbb{C}[X]/(\mathbb{C}[X]_+^W),$$

where $\mathbb{C}[X]$ is the algebra of polynomial functions in n indeterminates, i.e. $X = (x_1, \ldots, x_n)$.

Given a module multigraded in $\mathbb{N}^k$

$$R = \bigoplus_{(a_1,\ldots,a_k)\in\mathbb{N}^k} R_{a_1,\ldots,a_k}$$

we define its Hilbert series by

$$\text{Hilb}(R) = \sum (\dim R_{a_1,\ldots,a_k}) q_1^{a_1} \cdots q_k^{a_k}.$$

The algebra R^W is graded in $\mathbb{N}$ and

Theorem. We have

$$W(q) = \text{Hilb}(R^W).$$

Given an irreducible representation λ of W let $f^{\lambda}(q)$ be the polynomial whose coefficient of q^i is the multiplicity of the representation λ in the homogeneous component of degree i in R^W, i.e.

$$f^{\lambda}(q) = \sum \langle \chi^{\lambda}, \chi(R_k^W) \rangle q^k.$$

There is an explicit combinatorial interpretation of the polynomials $f^{\lambda}(q)$ in terms of standard tableaux.

We say that i is a descent of a tableau T if i appears strictly above $i+1$ and we define $\mathrm{maj}(T)$ as the sum of its descents.

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array}$$

In this case $\mathsf{Des}(T) = \{1, 3, 5, 6\}$ and so $\mathsf{maj}(T) = 15$.
Its shape is denoted by $\lambda(T) = (3, 2, 1, 1)$.

Theorem (Lusztig et al.). We have

$$f^{\lambda}(q) = \sum_{\{T:\lambda(T)=\lambda\}} q^{\mathsf{maj}(T)}$$

The 2-dimensional case

Since R^W is isomorphic as a W-module to the group algebra $\mathbb{C}W$ we deduce the identity

$$W(q) = \sum_\lambda f^\lambda(1) f^\lambda(q).$$

If we consider the natural generalization

$$W(q,t) = \sum_\lambda f^\lambda(q) f^\lambda(t)$$

we fall again in a Hilbert polynomial:

Theorem (Barcelo, Reiner, Stanton). $W(q,t)$ is the Hilbert polynomial of

$$(\mathbb{C}[X] \otimes \mathbb{C}[X])^{\Delta W} / ((\mathbb{C}[X] \otimes \mathbb{C}[X])_+^{W \times W}).$$

The polynomial $W(q,t)$ is known as the bimahonian distribution.

Recall that $(\mathbb{C}[X]\otimes\mathbb{C}[X])^{\Delta W}$ is a Cohen-Macauley algebra and in particular it is a free module on the subalgebra $(\mathbb{C}[X]\otimes\mathbb{C}[X])^{W\times W}$. It follows another interpretation $W(q,t)$

$$W(q,t) = \frac{\mathsf{Hilb}((\mathbb{C}[X]\otimes\mathbb{C}[X])^{\Delta W})}{\mathsf{Hilb}((\mathbb{C}[X]\otimes\mathbb{C}[X])^{W\times W})}.$$

The two Hilbert series appearing in the preceding formula can be studied using the theory of bipartite partitions and results of Garsia and Gessel.

Theorem (Garsia-Gessel).

$$W(q,t) = \sum_{w\in W} q^{\mathsf{maj}(w)} t^{\mathsf{maj}(w^{-1})}$$

We can summarize these facts in the following sequence of identities:

$$
\begin{aligned}
W(q,t) &= \sum_{\{S,T:\lambda(S)=\lambda(T)\}} q^{\mathsf{maj}(S)} t^{\mathsf{maj}(T)} \\
\text{(Lusztig)} &= \sum_{\lambda} f^{\lambda}(q) f^{\lambda}(t) \\
\text{(Bar., Rei., Sta.)} &= \mathsf{Hilb}\left(\frac{\mathbb{C}[X]\otimes\mathbb{C}[X])^{\Delta W}}{((\mathbb{C}[X]\otimes\mathbb{C}[X])_{+}^{W\times W})}\right) \\
\text{(Cohen-Macauley)} &= \frac{\mathsf{Hilb}((\mathbb{C}[X]\otimes\mathbb{C}[X])^{\Delta W})}{\mathsf{Hilb}((\mathbb{C}[X]\otimes\mathbb{C}[X])^{W\times W})} \\
\text{(Garsia-Gessel)} &= \sum_{\sigma\in W} q^{\mathsf{maj}(\sigma)} t^{\mathsf{maj}(\sigma^{-1})}
\end{aligned}
$$

The equality between the first and the last line follows also immediately from the Robinson-Schensted correspondence.

Let $\sigma = 31542$. We construct

$$\begin{array}{c} 3 \end{array} \qquad \begin{array}{c} 1 \\ 3 \end{array} \qquad \begin{array}{cc} 1 & 5 \\ 3 & \end{array} \qquad \begin{array}{cc} 1 & 4 \\ 3 & 5 \end{array} \qquad \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & \end{array}$$

The last tableau is denoted by $P(\sigma)$.
The recording tableau $Q(\sigma)$ is

$$\begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 5 & \end{array}$$

Theorem (Robinson-Schensted).

- The correspondence $\sigma \to (Q(\sigma), P(\sigma))$ is a bijection between W and pairs of tableaux having the same shape.

- We have Des(σ)=Des$(Q(\sigma))$ and Des(σ^{-1})=Des$(P(\sigma))$.

Multimahonian distributions

How to generalize to the k-dimensional case? We need another ingredient. Given k irreducible representations $\lambda^{(1)}, \ldots, \lambda^{(k)}$ of W we define

$$d_{\lambda^{(1)},\ldots,\lambda^{(k)}} := \frac{1}{|W|} \sum_{w \in W} \chi^{\lambda^{(1)}}(w) \cdots \chi^{\lambda^{(k)}}(w).$$

In other words $d_{\lambda^{(1)},\ldots,\lambda^{(k)}}$ is the multiplicity of $\lambda^{(k)}$ in $\lambda^{(1)} \otimes \cdots \otimes \lambda^{(k-1)}$. These numbers have been deeply studied by Bessenrodt, Kleshchev, Dvir, Regev.

Theorem (C.). We have

$$\mathrm{Hilb}\left(\frac{\mathbb{C}[X_1, \ldots, X_k]^{\Delta W}}{(\mathbb{C}[X_1, \ldots, X_k]_+^{W^k})}\right) =$$

$$\sum_{\lambda^{(1)},\ldots,\lambda^{(k)}} d_{\lambda^{(1)},\ldots,\lambda^{(k)}} f^{\lambda^{(1)}}(q_1) \cdots f^{\lambda^{(k)}}(q_k).$$

Here X_i stands for the n-tuple of variables $(x_{i,1}, \ldots, x_{i,n})$. We call this polynomial the multimahonian distribution.

One can also generalize the other equalities of the 2-dimensional case.

Corollary. We have

$$\sum_{T_1,\dots,T_k} d_{\lambda(T_1),\dots,\lambda(T_k)} q_1^{\mathrm{maj}(T_1)} \dots q_k^{\mathrm{maj}(T_k)}$$
$$= \sum_{\sigma_1 \cdots \sigma_k = 1} q_1^{\mathrm{maj}(\sigma_1)} \dots q_k^{\mathrm{maj}(\sigma_k)}$$

It follows the existence of a "multivariate Robinson-Schensted correspondence" in the following sense:

Corollary. There exists a map that associates to every k-tuple of permutations whose product is the identity a k-tuple of standard tableaux of size n such that:

• Every k-tuple of tableaux $(T_1, \dots, T_k)$ is obtained exactly $d_{\lambda(T_1),\dots,\lambda(T_k)}$ times;

• If $(T_1, \dots, T_k)$ corresponds to $(\sigma_1, \dots, \sigma_k)$ then $\mathrm{maj}(T_i) = \mathrm{maj}(\sigma_i)$ for all i.

Refined multimahonian distributions

We can do something more from a combinatorial point of view. We have a further decomposition of the coinvariant algebra

Theorem(Adin-Brenti-Roichman)
We have

$$R_k^W \cong \sum_{|\lambda|=k} R_\lambda,$$

as W-modules, by means of a canonical isomorphism.

We can use this result to define a multidegree on the coinvariant algebra. We say that R_λ is the homogeneous component of R^W of multidegree λ and we consider its Hilbert series

$$\mathrm{Hilb}(R^W)(q_1, \ldots, q_n) = \sum_\lambda (\dim R_\lambda) q_1^{\lambda_1} \cdots q_n^{\lambda_n}.$$

Putting $q = q_1 = q_2 = \cdots = q_n$ we reobtain the polynomial $W(q)$.

By means of this decomposition of R^W we can also decompose the algebra

$$\frac{\mathbb{C}[X_1,\ldots,X_k]^{\Delta W}}{(\mathbb{C}[X_1,\ldots,X_k]_+^{W^k})}$$

in homogeneous components whose degrees are k-tuples of partitions with at most n parts. (Recall $X_i = (x_{i,1},\ldots,x_{i,n})$).
Therefore the Hilbert series will depend on k n-tuples of variables $Q_1,\ldots,Q_k$, where $Q_i = (q_{i,1},\ldots,q_{i,n})$.

We define $f^\lambda(q_1,\ldots,q_k)$ as the polynomial whose coefficient of $q_1^{\mu_1}\cdots q_k^{\mu_k}$ is the multiplicity of the representation λ in R_μ. This is our main result.

Theorem (C). We have

$$\mathrm{Hilb}\left(\frac{\mathbb{C}[X_1,\ldots,X_k]^{\Delta W}}{(\mathbb{C}[X_1,\ldots,X_k]_+^{W})}\right)(Q_1,\ldots,Q_k) =$$

$$\sum_{\lambda^{(1)},\ldots,\lambda^{(k)}} d_{\lambda^{(1)},\ldots,\lambda^{(k)}} f^{\lambda^{(1)}}(Q_1)\cdots f^{\lambda^{(k)}}(Q_k).$$

Given a permutation or a tableau X we define the partition $\mu(X)$ by putting

$$(\mu(X))_i = |\mathrm{Des}(X) \cap \{i, \ldots, n\}|.$$

The following is an extension of Lusztig decomposition theorem

Theorem (Adin-Brenti-Roichman).
The multiplicity of the representation μ in R_λ is

$$|\{T \text{ tableau} : \lambda(T) = \lambda \text{ e } \mu(T) = \mu\}|$$

and hence

$$f^\lambda(q_1, \ldots, q_n) = \sum_{\{T : \lambda(T) = \lambda\}} Q^{\mu(T)}.$$

We can therefore write

$$\begin{aligned} & \sum_{\lambda^{(1)}, \ldots, \lambda^{(k)}} d_{\lambda^{(1)}, \ldots, \lambda^{(k)}} f^{\lambda^{(1)}}(Q_1) \cdots f^{\lambda^{(k)}}(Q_k) \\ = & \sum_{T_1, \ldots, T_k} d_{\lambda(T_1), \ldots, \lambda(T_k)} Q_1^{\mu(T_1)} \ldots Q_k^{\mu(T_k)} \\ = & \ \mathrm{Hilb} \frac{\mathbb{C}[X_1, \ldots, X_k]^{\Delta W}}{(\mathbb{C}[X_1, \ldots, X_k]_+^W)} (Q_1, \ldots, Q_k) \end{aligned}$$

And from the point of view of permutations? Consider the multidegree on $\mathbb{C}[X]$ for which a monomial is homogeneous of multidegree equal to the partition obtained by reordering its exponents.

$$\deg(x_1^3 x_2^5 x_3^4) = (5, 4, 3).$$

Consequently the algebra of polynomials in nk variables $\mathbb{C}[X_1 \ldots, X_k]$ is multigraded by k-tuples of partitions. Since the action of W^k respects this grading we can consider the Hilbert series of the invariants of $W^k \ominus$ and its diagonal subgroup ΔW. Their quotient is given by

Theorem (C.) We have

$$\frac{\mathrm{Hilb}(\mathbb{C}[X_1 \ldots, X_k]^{\Delta W})(Q_1, \ldots, Q_k)}{\mathrm{Hilb}(\mathbb{C}[X_1 \ldots, X_k]^{W^k})(Q_1, \ldots, Q_k)} =$$

$$\sum_{\sigma_1 \cdots \sigma_k = 1} Q_1^{\mu(\sigma_1)} \ldots Q_k^{\mu(\sigma_k)}$$

So similarly to the case of the total degree we have

$$
\begin{aligned}
&W(Q_1,\ldots,Q_k)\\
&\quad= \sum_{T_1,\ldots,T_k} d_{\lambda(T_1),\ldots,\lambda(T_k)} Q_1^{\mu(T_1)},\ldots,Q_k^{\mu(T_k)}\\
&\quad= \sum_{\lambda^{(1)},\ldots,\lambda^{(k)}} d_{\lambda^{(1)},\ldots,\lambda^{(k)}} f^{\lambda^{(1)}}(Q_1)\cdots f^{\lambda^{(k)}}(Q_k)\\
&\quad= \mathsf{Hilb}\Big(\frac{\mathbb{C}[X_1,\ldots,X_k]^{\Delta W}}{(\mathbb{C}[X_1,\ldots,X_k]_+^W)}\Big)(Q_1,\ldots,Q_k)\\
&\quad\overset{?}{=} \frac{\mathsf{Hilb}(\mathbb{C}[X_1\ldots,X_k]^{\Delta W})(Q_1,\ldots,Q_k)}{\mathsf{Hilb}(\mathbb{C}[X_1\ldots,X_k]^{W^k})(Q_1,\ldots,Q_k)}\\
&\quad= \sum_{\sigma_1\cdots\sigma_k=1} Q_1^{\mu(\sigma_1)}\ldots Q_k^{\mu(\sigma_k)}
\end{aligned}
$$

Corollary-Conjecture There exists a map that associates to every k-tuple of permutations whose product is the identity a k-tuple of standard tableaux of size n such that:

- Every k-tuple of tableaux $(T_1, \ldots, T_k)$ is obtained exactly $d_{\lambda(T_1),\ldots,\lambda(T_k)}$ times;
- If $(T_1, \ldots, T_k)$ corresponds to $(\sigma_1, \ldots, \sigma_k)$ then Des(T_i) =Des(σ_i) for all i.