Rational expressions : Two applications in Combinatorial Physics

Gérard H. E. Duchamp

LIPN, Université de Paris XIII, France

Collaborators : **Jacob Katriel**, ***Technion - Israel Inst. of Techn., Haifa, Israel.***

Karol A. Penson, ***LPTMC, Université de Paris VI, France***

Allan I. Solomon, ***The Open University, United Kingdom***

Pawel Blasiak, ***Instit. of Nucl. Phys., Krakow, Pologne***

Andrzej Horzela, ***Instit. of Nucl. Phys., Krakow, Pologne***

Christophe Tollu, ***LIPN, Université de Paris XIII, France***

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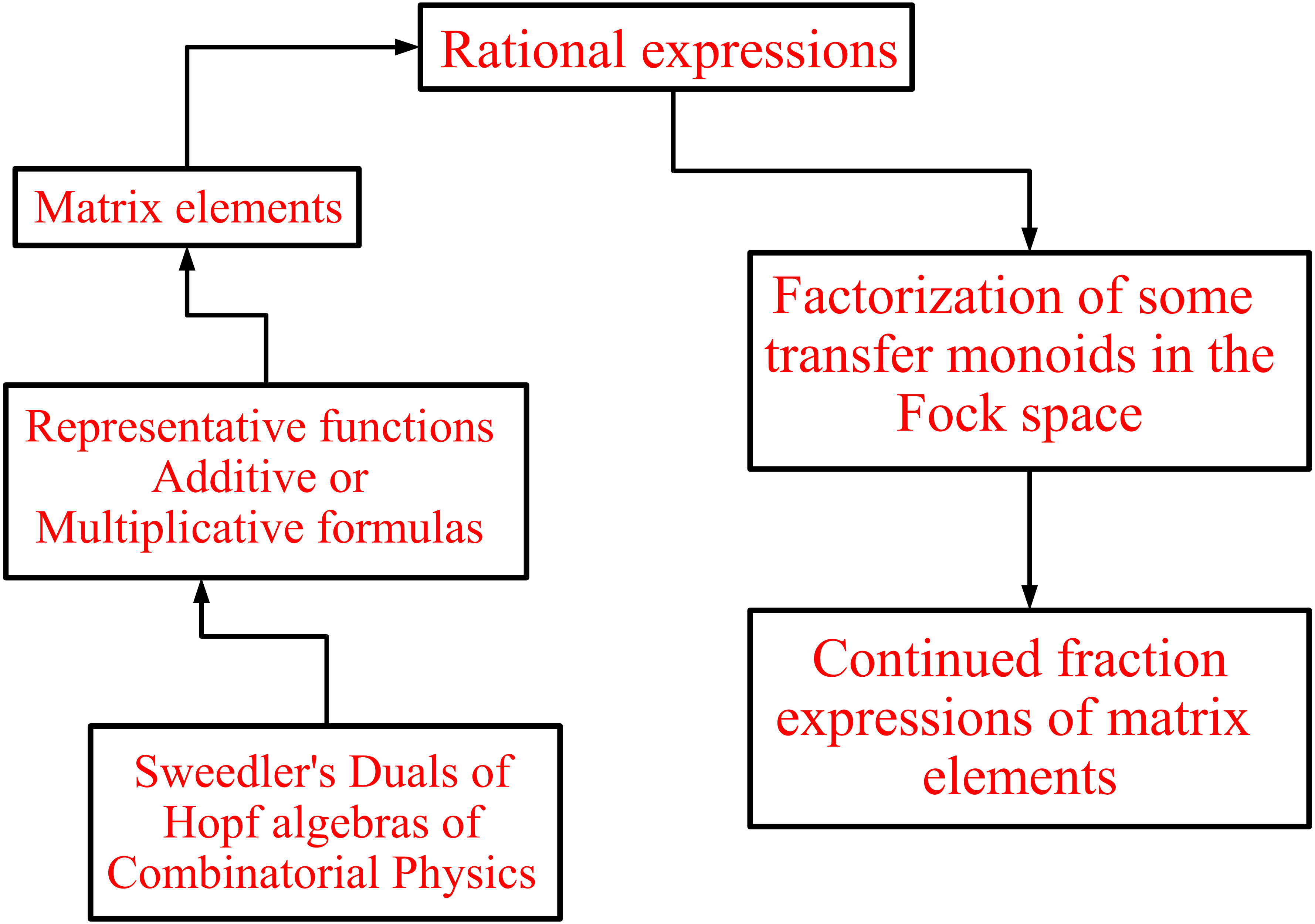

Rational expressions
Matrix elements
Factorization of some transfer monoids in the Fock space
Representative functions Additive or Multiplicative formulas
Continued fraction expressions of matrix elements
Sweedler's Duals of Hopf algebras of Combinatorial Physics

Content of talk

- **First application**
- **Classical Fock space**
- **Transfer packet and transfer value**
- **Solution as double continued fractions**

- **Second application**
- **Calculus in Sweedler's duals**

- **Conclusion**

Classical Fock space for bosons and q-ons

- Heisenberg-Weyl (two-dimensional) algebra is defined by two generators (a^+, a) which fulfill the relation

$$[a\ ,\ a^+]=aa^+ - a^+a=1$$

- Known to have no (faithful) representation by bounded operators in a Banach space.

There are many « combinatorial » (faithful) representations by operators. The most famous one is the Bargmann-Fock representation

$$a \dashrightarrow d/dx\ ;\ a+ \dashrightarrow x$$

where a has degree -1 and a^+ has degree 1.

- These were bosons, there are also fermions. The relation for fermions is

$$aa^+ + a^+a = 1$$

- This provides a framework for the q-analogue which is defined by

$$[a\ ,\ a^+]_q = aa^+ - qa^+a = 1$$

- For which Bargmann-Fock representation reads

$$a \dashrightarrow D_q\ ;\ a+ \dashrightarrow x$$

where a has degree -1 and a^+ has degree 1 and D_q is the (classical) q-derivative.

• For a faithful representation, one needs an infinite-dimensional space. The smallest, called Fock space, has a countable basis $(e_n)_{n\geq 0}$ (the actions are described below, each e_n is represented by a circled state « n »).

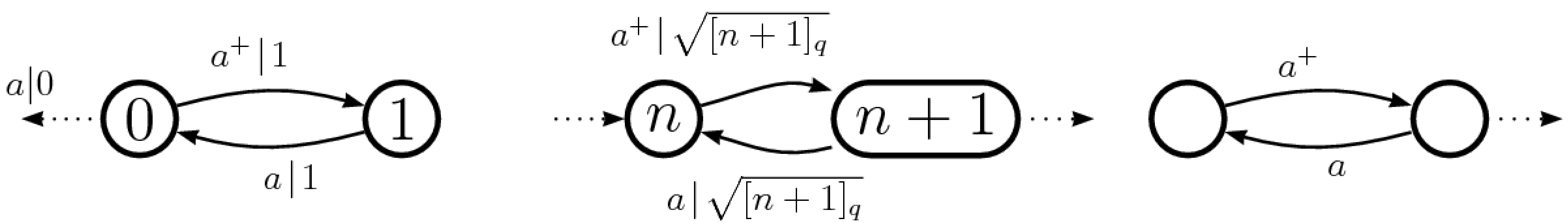


Figure 1: Classical Fock space

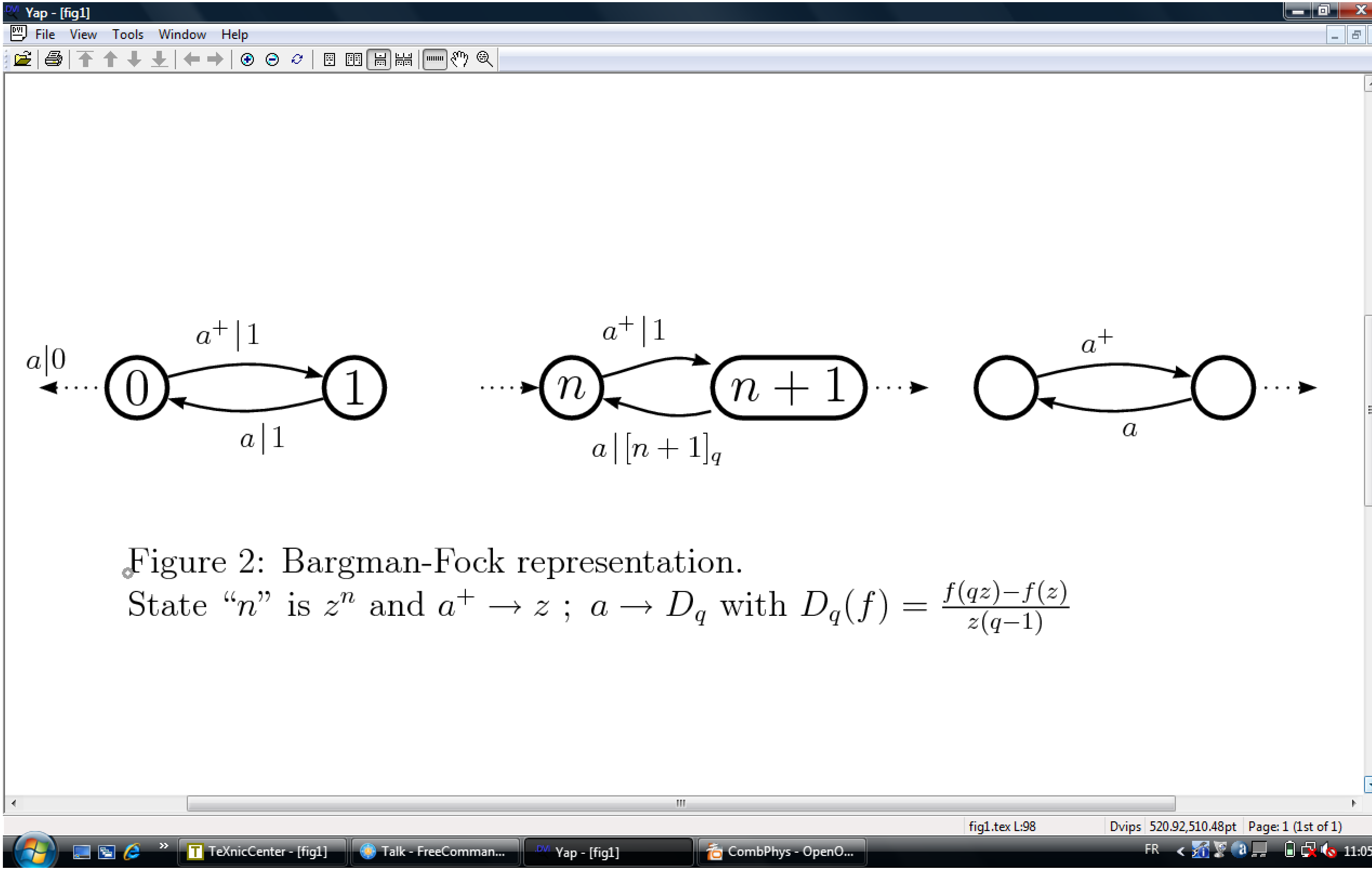


Figure 2: Bargman-Fock representation.
State "n" is z^n and $a^+ \to z$; $a \to D_q$ with $D_q(f) = \frac{f(qz)-f(z)}{z(q-1)}$

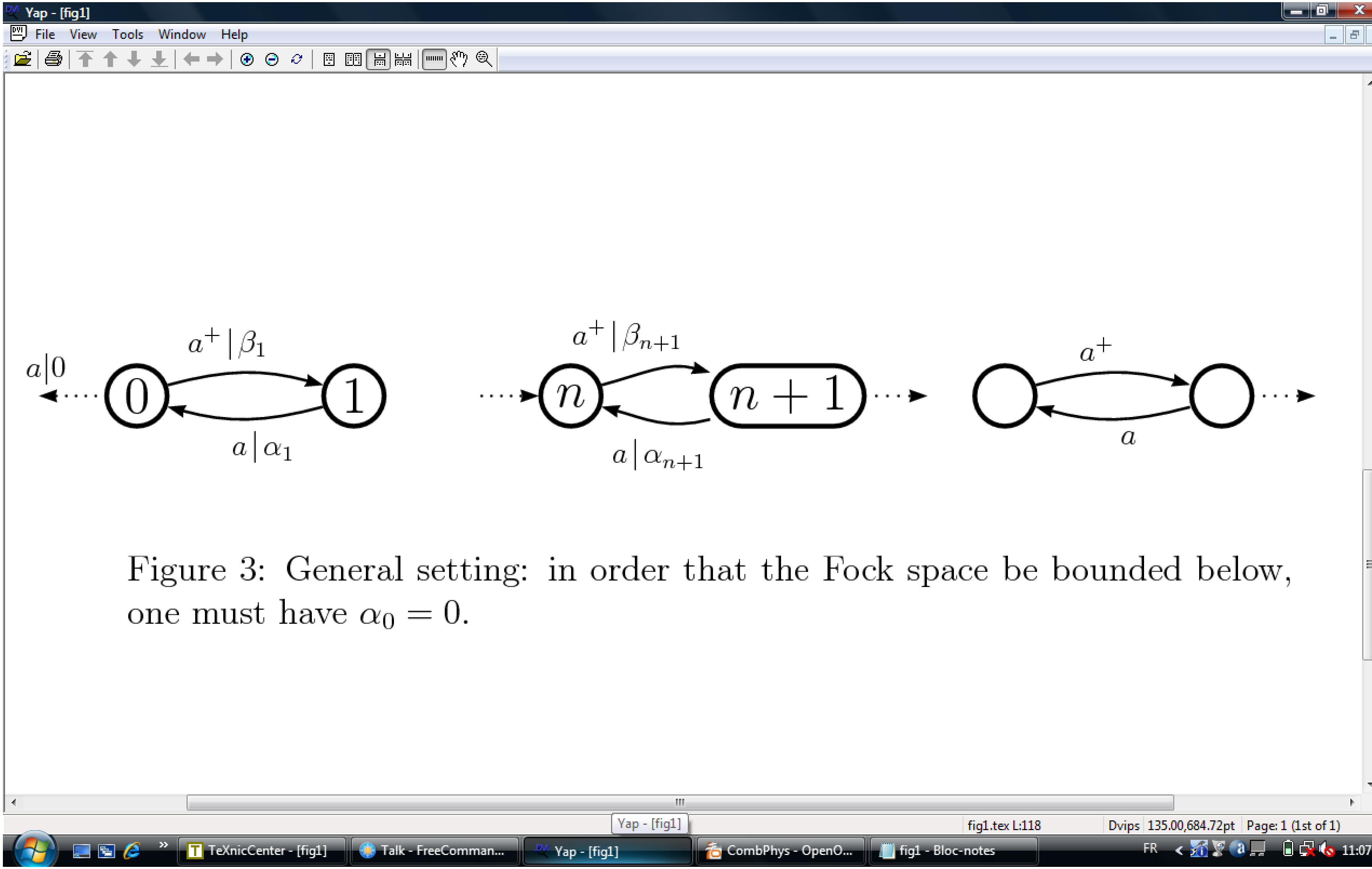


Figure 3: General setting: in order that the Fock space be bounded below, one must have $\alpha_0 = 0$.

• Physicists need to know the sum of all weights created when one passes from level « n » to level « m ». This problem has been called the « transfer packet problem » and is at once rephrased by combinatorists as the computation of a formal power series.

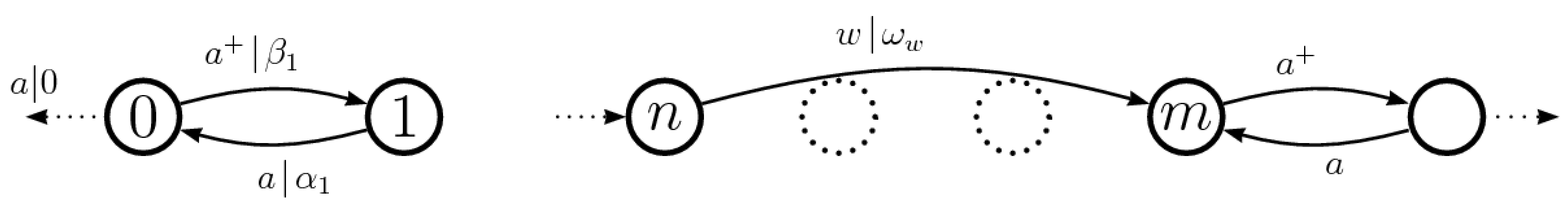
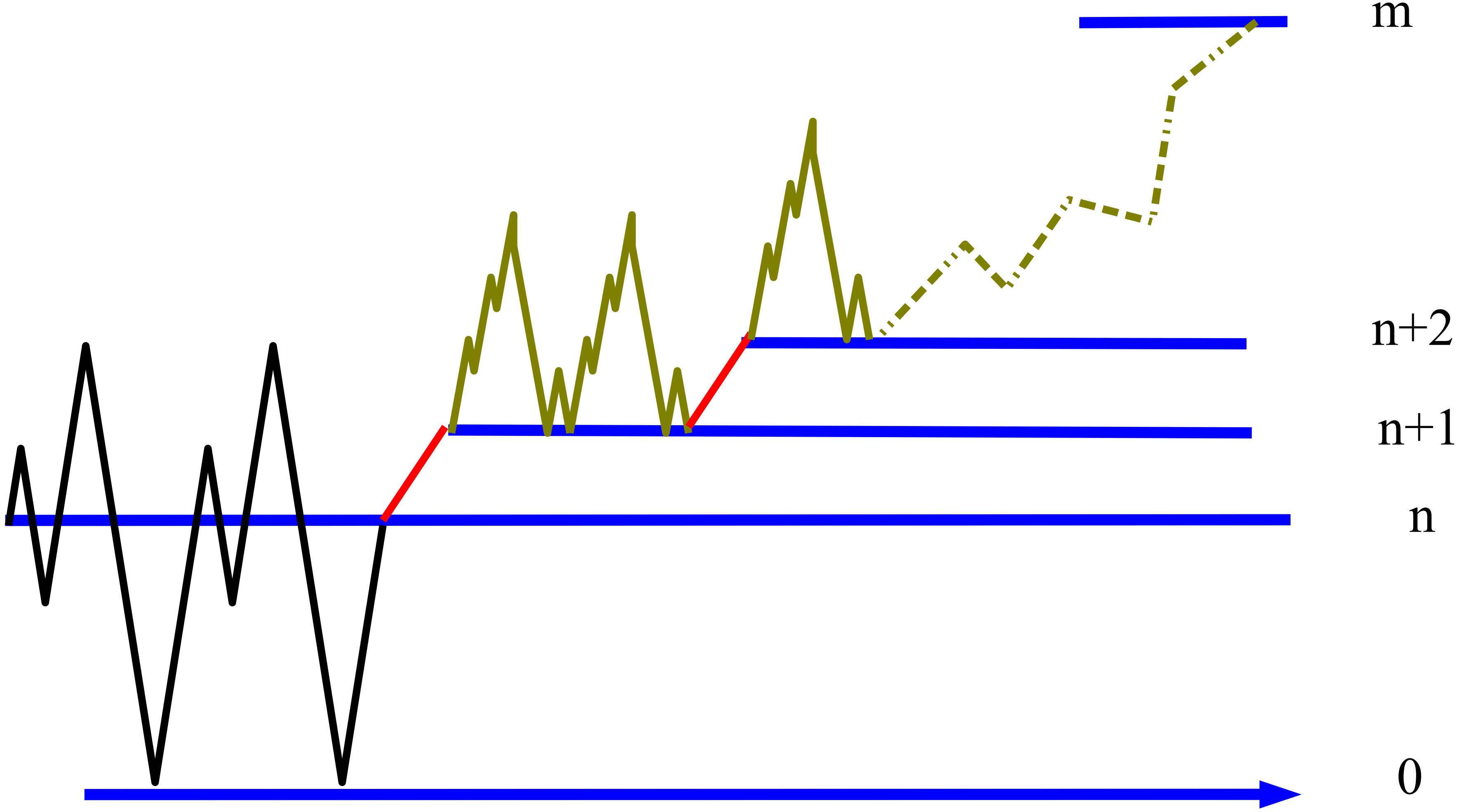


Figure 4: The transfer packet problem

Change of level



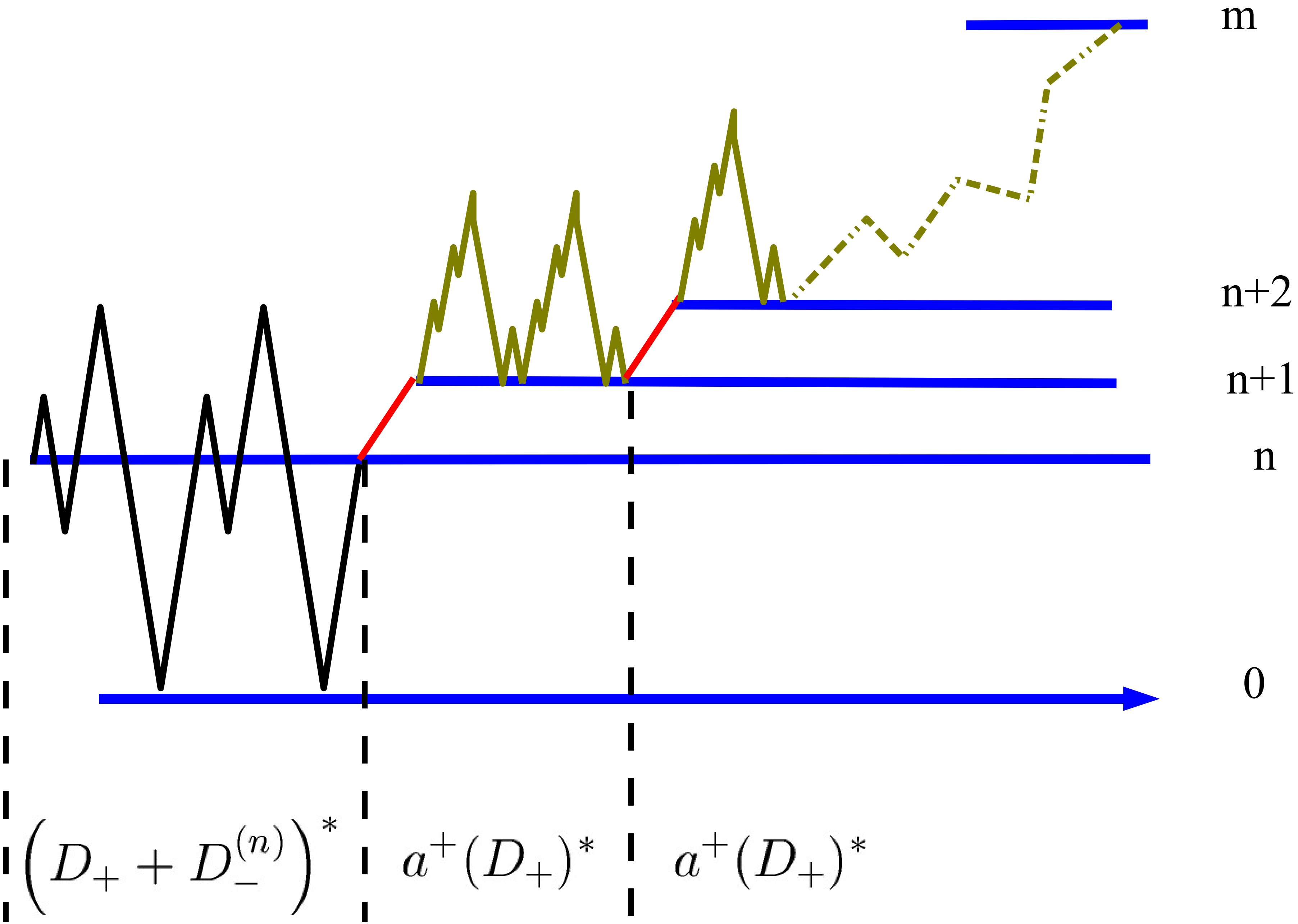
- The set of words which allow to pass from level « n » to level « m » in « i » steps is clearly.

$$W_k^{(i)} = \left\{ w \in \{a, a^+\}^* \middle| \pi_e(w) = k \text{ and } |w| = i \right\}$$

$$\text{with } k = m - n \text{ and } \pi_e(w) = |w|_{a^+} - |w|_a.$$

- The weight associated with this packet and the desired generating series are then

$$e_n.W_{m-n}^{(i)} = \omega_{n\to m}^{(i)}\ e_m\ ;\ T_{n\to n+k} := \sum_{i\geq 0} t^i \omega_{n\to n+k}^{(i)}$$


m
n+2
n+1
n
0
$\left(D_+ + D_-^{(n)}\right)^*$
$a^+(D_+)^*$
$a^+(D_+)^*$

The following selfreproducing formulas can be considered as noncommutative continued fraction expansions of the involved “D” codes

$$D_+ = a^+\left(D_+\right)^* a \ ; \ D_-^{(n)} = a\left(D_-^{(n-1)}\right)^* a^+ \ ; \ D_-^{(0)} = \emptyset$$

Using a bit of analysis to extend the right action of the words to series of words and the representation

a ---> ?.ta ; a+ ---> ?.ta+

We get

$$T_{n\to n}[t] = \cfrac{1}{1 - \cfrac{t^2\alpha_{n+1}\beta_{n+1}}{1-\cfrac{t^2\alpha_{n+2}\beta_{n+2}}{1-\cdots}} - \cfrac{t^2\alpha_n\beta_n}{1-\cfrac{t^2\alpha_{n-1}\beta_{n-1}}{1-\cdots}}}$$

And, if one allows only the positive loops

$$T^{+}_{n\to n}[t] = \cfrac{1}{1 - \cfrac{t^2\alpha_{n+1}\beta_{n+1}}{1-\cfrac{t^2\alpha_{n+2}\beta_{n+2}}{1-\cdots}}}$$

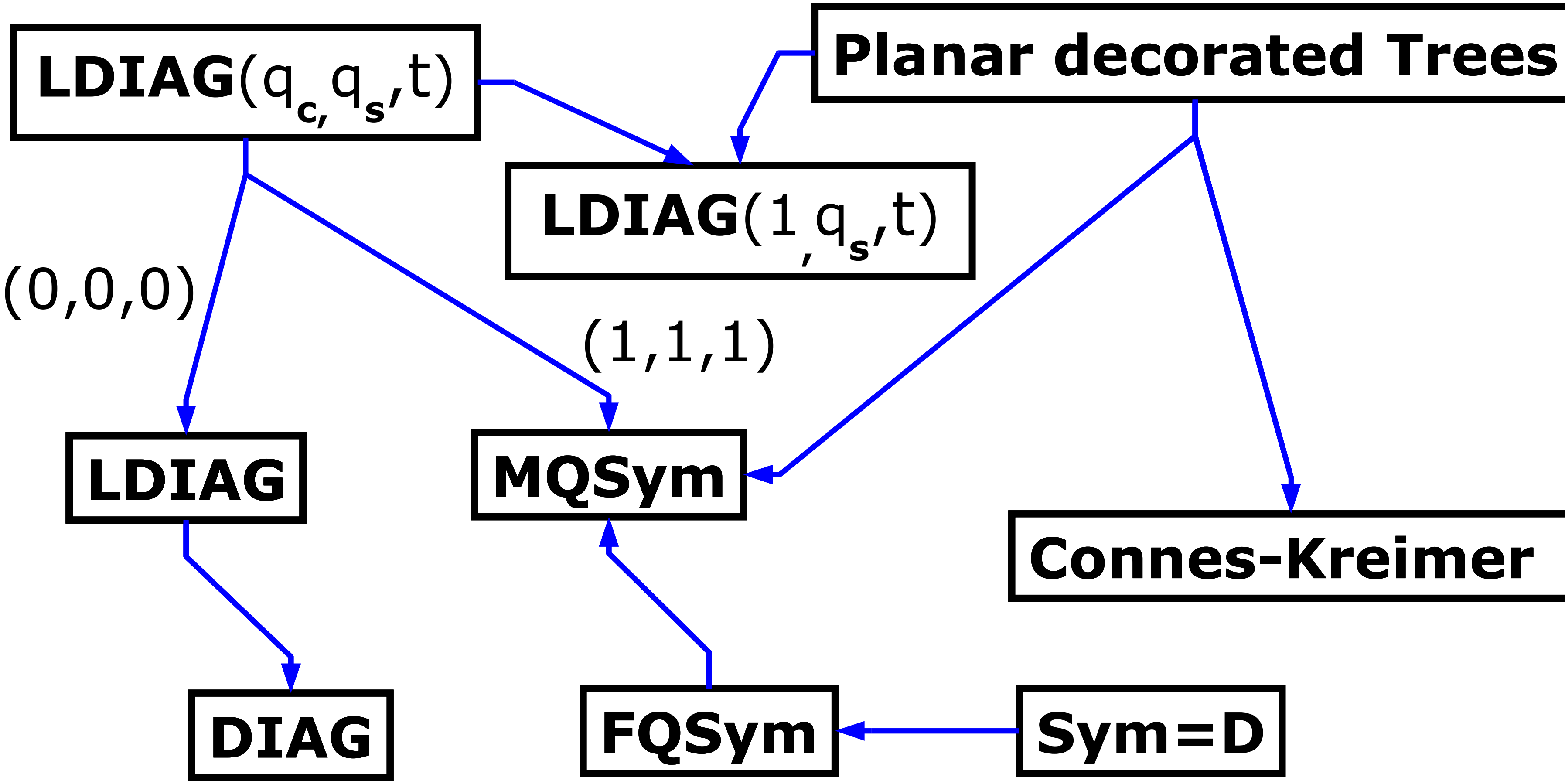
Which solves, with two cases, the problem of the transfer packet.

G. H. E. Duchamp, K.A. Penson, A.I. Solomon, A. Horzela and P. Blasiak,

One-parameter groups and combinatorial physics,

Third International Workshop on Contemporary Problems in Mathematical Physics (COPROMAPH3), Porto-Novo (Benin), November 2003.

arXiv : quant-ph/0401126.



Notes :

i) The arrow *Planar Dec. Trees* → *LDIAG(1,q_s,t)* is due to L. Foissy

ii) **LDIAG**, through a noncommutative alphabetic realization shows to be a bidendriform algebra (FPSAC07 paper by ParisXIII & Monge).

(A part of) The legacy of Schützenberger or how to compute efficiently in Sweedler's duals using Automata Theory

Sweedler's dual of a Hopf algebra

i) Multiplication

$$A \otimes A \to A$$

ii) By dualization one gets an arrow (comultiplication)

$$\Delta :\ A^* \to (A \otimes A)^*$$

but not a "stable calculus" as

$$A^* \otimes A^* \subset (A \otimes A)^*$$

(strict in general). We ask for elements $x \in A$ such that their coproduct be in $A^* \otimes A^*$

These elements are easily characterized as the "representative linear forms" (see also the Group-Theoretical formulation in recent talks by Pierre Cartier) .

Proposition : TFAE (the notations being as above)

i) $\Delta(c) \in A^* \otimes A^*$

ii) There are functions f_i, g_i $i=1,2..n$ such that

$$c(xy)=\sum_{i=1}^{n} f_i(x)\, g_i(y)$$

for all x,y in *A*.

iii) There is a morphism of algebras $\mu: A \to k^{n \times n}$ (square matrices of size n x n), a row λ in $k^{1 \times n}$ and a column ξ in $k^{n \times 1}$ such that, for all z in *A*,

$$c(z)=\lambda\mu(z)\xi$$

In many “Combinatorial” cases, we are concerned with the case $A=k<\Sigma>$ (non-commutative polynomials with coefficients in a field k).

Indeed, one has the following theorem (the beginning can be found in [ABE : Hopf algebras]) and the end is one of the starting points of Schützenberger's school of automata and language theory.

Theorem A: TFAE (the notations being as above)

i) $\Delta(c) \in A^* \otimes A^*$

ii) There are functions f_i, g_i $i=1,2..n$ such that

$$c(uv)=\sum_{i=1}^{n} f_i(u)\, g_i(v)$$

u,v words in Σ^* (the free monoid of alphabet Σ).

iii) There is a morphism of monoids $\mu: \Sigma^* \dashrightarrow k^{n \times n}$ (square matrices of size $n \times n$), a row λ in $k^{1 \times n}$ and a column ξ in $k^{n \times 1}$ such that, for all word w in A^*

$$c(w)=\lambda\mu(w)\xi$$

iv) (Schützenberger) (If Σ is finite) c lies in the rational closure of Σ within the algebra $k\langle\langle A\rangle\rangle$.

We can safely apply the first three conditions of **Theorem A** to *Ldiag*. The monoid of labelled diagrams is free, but with an infinite alphabet, so we cannot keep Schützenberger's equivalence at its full strength and have to take more “basic” functions. The modification reads

iv) (Σ is infinite) c is in the rational closure of the weighted sums of letters

$$\sum_{a \in \Sigma} p(a)\, a$$

within the algebra k<<A>>.

(Joint work with C. Tollu).

arXiv:0802.0249v1 [quant-ph]

In this case, *Schützenberger's* theorem (known as the theorem of Kleene-Schützenberger) could be rephrased in saying that functions in a Sweedler's dual are behaviours of finite (state and alphabet) automata.

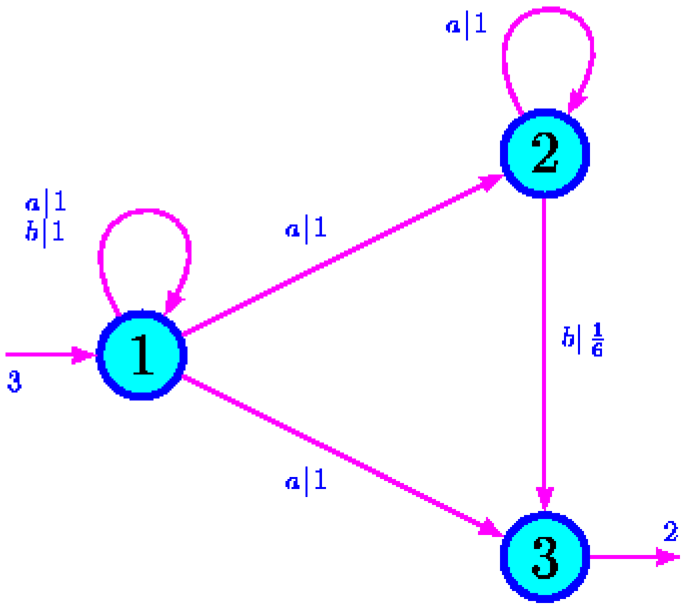
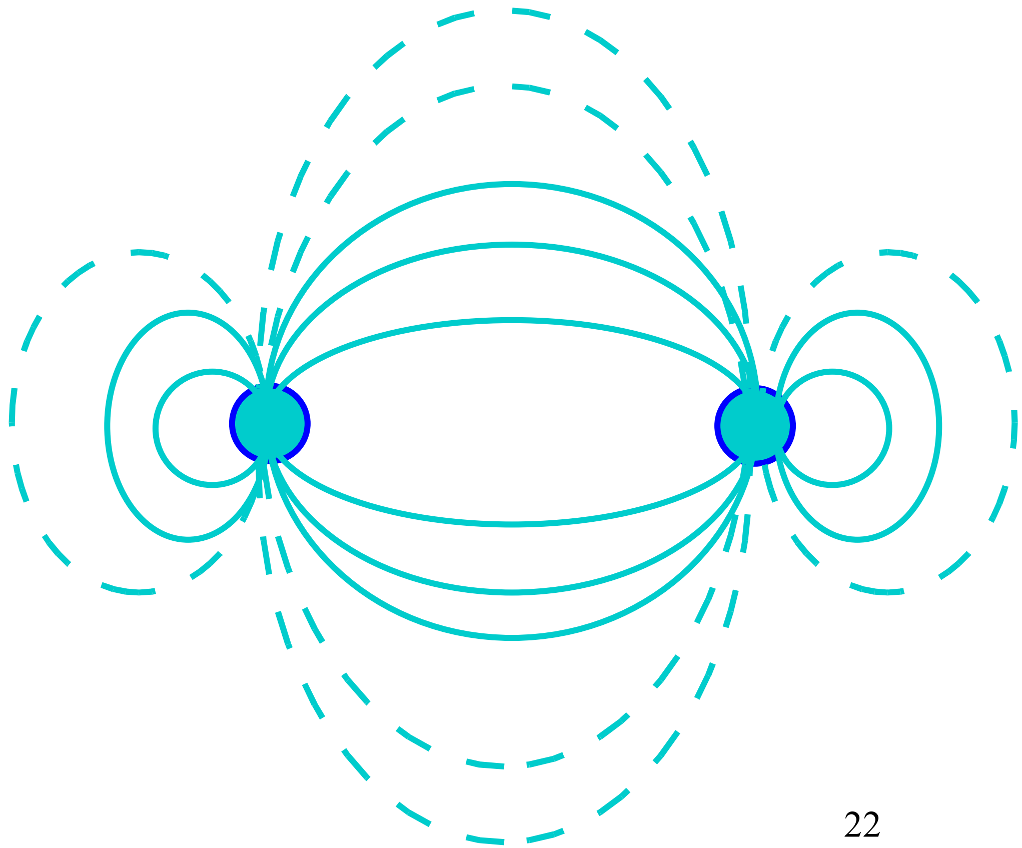


FIG. 1 – Un $\mathbb{Q}$-automate $\mathcal{A}$.

Le comportement de $\mathcal{A}$ est :

$$\text{comportement}(\mathcal{A}) = \sum_{a,b \in A} (a+b)^*(6 + a^*b).$$

In our case, we are obliged to allow infinitely many edges.



Computations in $K^{rat}\langle\langle A\rangle\rangle$, Sweedler's dual of $K\langle A\rangle$

Summability : We say that a family $(f_i)_{i\in I}$ (I finite or not, f_i in $K\langle\langle A\rangle\rangle$) is summable if, for each $w\in A^*$, the family $(\langle f_i|w\rangle)_{i\in I}$ is finitely supported and we set

$$(\sum_{i\in I} f_i) : w \to (\sum_{i\in I} \langle f_i|w\rangle)$$

Identifying each word with the Dirac linear form located at the word, one has then, for each $f\in K\langle\langle A\rangle\rangle$

$$f=\sum_{w\in A^*} f(w)w$$

If $f \in K^{rat}\langle\langle A \rangle\rangle$, i. e. if it fulfills the conditions of Theorem A it exists a morphism of monoids

$\mu: A^* \dashrightarrow K^{n \times n}$ (square matrices of size n x n), a row λ in $k^{1 \times n}$ and a column ξ in $k^{n \times 1}$ such that, for all word w in A^*, $f(w)=\lambda\mu(w)\xi$. Then

$$f= \sum_{w \in A^*} f(w)w=\sum_{w \in A^*} \lambda\mu(w)\xi\, w=\lambda\left(\sum_{w \in A^*} \mu(w)w\right)\xi=$$

$$\lambda\left(\sum_{w \in A^*} \mu(w)w\right)\xi=\lambda\left(\sum_{m \geq 0} \sum_{|w|=m} \mu(w)w\right)\xi$$

But, as words and scalars commute (it is so by construction of the convolution algebra $K^{n \times n}\langle\langle A\rangle\rangle$), one has

$$\sum_{m\geq 0} \sum_{|w|=m} \mu(w)w=\sum_{m\geq 0} \left(\sum_{a\in A} \mu(a)a\right)^m=\left(\sum_{a\in A} \mu(a)a\right)^*$$

hence

$$f=\lambda\left(\sum_{a\in A} \mu(a)a\right)^*\xi$$

where the "star" stands for the sum of the geometric series.

If Q is a finite set, the space $k^{Q\times Q}$ of square matrices with indices in Q and coefficients in k has a natural semiring structure with the usual operations (sum and product). A (right) star of $M \in k^{Q\times Q}$ (when it exists) is a solution of the equation $MY + 1_{Q\times Q} = Y$ (where $1_{Q\times Q}$ is the identity matrix). Let $M \in k^{Q\times Q}$ be given by

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{11} \in k^{Q_1\times Q_1}$, $a_{12} \in k^{Q_1\times Q_2}$, $a_{21} \in k^{Q_2\times Q_1}$ and $a_{22} \in k^{Q_2\times Q_2}$ such that $Q_1 + Q_2 = Q$. Let $N \in k^{Q\times Q}$ given by

$$N = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$A_{11} = (a_{11} + a_{12}a_{22}{}^{*}a_{21})^{*} \tag{1}$$

$$A_{12} = a_{11}{}^{*}a_{12}A_{22} \tag{2}$$

$$A_{21} = a_{22}{}^{*}a_{21}A_{11} \tag{3}$$

$$A_{22} = (a_{22} + a_{21}a_{11}{}^{*}a_{12})^{*} \tag{4}$$

A (short) word on automata theory.

- The formulas (for the star* of a matrix) above are sufficiently "expressive" to be the crucial fact in the resolution of a conjecture in Noncommutative Geometry.

- For applications, automata theory had to cope with spaces of coefficients much more general than that of a field ... even the "minus" operation of the rings had to disappear in order to allow to cope with problems like shortest path or the Noncommutative problem or the shortest path with list of minimal arcs .

The emerging structure is that of a semiring. Think of a ring without the "minus" operation, nevertheless "transfer" matrix computations can be performed.

The input alphabet being set by the automaton under consideration, we will here rather focus on the definition of semirings providing transition coefficients. For convenience, we first begin with various laws on $\mathbb{R}_+ := [0, +\infty[$ including

1. $+$ (ordinary sum)
2. $\times$ (ordinary product)
3. min (if over $[0,1]$, with neutral 1, otherwise must be extended to $[0,+\infty]$ and then, with neutral $+\infty$) or max
4. $+_a$ defined by $x +_a y := log_a(a^x + a^y)$ $(a > 0)$
5. $+_{[n]}$ (Hölder laws) defined by $x +_{[n]} y := \sqrt[n]{x^n + y^n}$
6. $+^s$ (shifted sum, $x +^c y := x + y - 1$, over whole $\mathbb{R}$, with neutral 1)
7. $\times^c$ (complemented product, $x + y - xy$, can be extended also to whole $\mathbb{R}$, stabilizes the range of probabilities or fuzzy $[0,1]$ and is distributive over the shifted sum)

As (useful) examples, one has ([0,+∞], min, +), ([0,+∞[, max, +) or its (commutative or not) variants.

What remains for K<A> ? (free algebra)

- K semiring :

  - Universal properties (comprising – little known - tensor products)

  - Complete semiring K<<A>>, summability is defined by pointwise convergence (see e. g. computation above).

  - Rational closures and Kleene-Schützenberger Thm

  - Rational expressions, Brzozowski theorem

  - Automata theory, theory of codes

  - Lazard's monoidal elimination

Concluding remarks and future

i) We can solve the problem of packet of monoidal actions on a “level” space by means of the theory of codes and a bit of operator analysis.

ii) In general, Hopf algebras of physics which are non-commutative are free on some alphabet (often, of diagrams) and computation on its Sweedler's dual (which is the biggest available for finite comultiplications) and we have seen that they can be performed via rational expressions.

Concluding remarks and future (cont'd)

iii) Kleene-Schützenberger's theorem still works in the general case (infinite alphabet) up to a slight modification. This will allow the development of a calculus in the general (i. e. out of the free) case.

Thank You