Lefschetz algebras and Eulerian numbers

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Outline

1. Basic definitions and background
2. Combinatorial g-Theorems: Classical and new results
3. Inequalities for the refined Eulerian statistics on permutations

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1. **Basic definitions and background**

2. Combinatorial g-Theorems: Classical and new results

3. Inequalities for the refined Eulerian statistics on permutations

Simplicial Complexes

A simplicial complex Δ on vertex set Ω is a collection of subsets of Ω such that
$F \in \Delta,\ G \subseteq F \Rightarrow G \in \Delta$.

The elements of Δ are called faces of Δ and for a face $F \in \Delta$ $\dim(F) := \#F - 1$ is the dimension of F.

$\dim \Delta := \max\{\ \dim(F) \mid F \in \Delta\}$ is the dimension of Δ.

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Combinatorial invariants for simplicial complexes

The vector $\mathfrak{f}^{\Delta} = (f^{\Delta}_{-1}, f^{\Delta}_{0}, \ldots, f^{\Delta}_{\dim\Delta})$,
where $f^{\Delta}_{i} = \#\{\, F \in \Delta \mid \dim(F) = i\}$, is called the $\mathfrak{f}$-vector of Δ.

The h-vector of Δ is the vector $\mathfrak{h}^{\Delta} = (h^{\Delta}_{0}, h^{\Delta}_{1}, \ldots, h^{\Delta}_{\dim\Delta+1})$,
where
$\sum_{0\leq i\leq \dim\Delta+1} h^{\Delta}_{i} x^{\dim\Delta+1-i} = \sum_{0\leq i\leq d} f^{\Delta}_{i-1}(x-1)^{\dim\Delta+1-i}$.

We call $\mathfrak{g}^{\Delta} := (g^{\Delta}_{0}, g^{\Delta}_{1}, \ldots, g^{\Delta}_{\lfloor\frac{\dim\Delta+1}{2}\rfloor})$ the g-vector of Δ, where
$g^{\Delta}_{0} = 1$ and $g^{\Delta}_{i} = h^{\Delta}_{i} - h^{\Delta}_{i-1}$ for $1 \leq i \leq \lfloor\frac{\dim\Delta+1}{2}\rfloor$.

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The barycentric subdivision

The barycentric subdivision $\mathrm{sd}(\Delta)$ of a simplicial complex Δ is the simplicial complex on vertex set $\mathring{\Delta} := \Delta \setminus \{\emptyset\}$ whose simplices are flags

$$A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_t$$

of elements $A_j \in \mathring{\Delta}$ for $0 \leq j \leq t$.

M-sequence

Given an integer $d > 0$ any $a \in \mathbb{N}$ can uniquely be written in the form

$$a = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_j}{j},$$

where $k_d > k_{d-1} > \cdots > k_j \geq j \geq 1$.

We define $a^{<d>} := \binom{k_d+1}{d+1} + \binom{k_{d-1}+1}{d} + \cdots + \binom{k_j+1}{j+1}$ and set $0^{<d>} = 0$.

A sequence $(a_0, \ldots, a_t) \in \mathbb{N}^{t+1}$ is called an *M*-sequence if $a_0 = 1$ and $a_{i+1} \leq a_i^{<i>}$ for $1 \leq i \leq t-1$.

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The classical g-theorem and the g-conjecture

Theorem (Stanley, Billera, Lee)

$(h_0, \ldots, h_d)$ is the h-vector of a d-dimensional simplicial polytope if and only if $h_i = h_{d-i}$ for all $0 \leq i \leq d$, $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor \frac{d}{2} \rfloor}$ and the vector $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1})$ is an M-sequence.

Conjecture (McMullen)

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g-Theorems for barycentric subdivisions of certain classes of simplicial complexes

Theorem (K., Nevo)

Let Δ be a $(d-1)$-dimensional Cohen-Macaulay simplicial complex. Then the g-vector of its barycentric subdivision $\mathrm{sd}(\Delta)$ is an M-sequence.

In particular, the g-conjecture holds for barycentric subdivisions of simplicial spheres, of homology spheres and of doubly Cohen-Macaulay complexes.

Furthermore, $h_i^{\mathrm{sd}(\Delta)} \leq h_{d-1-i}^{\mathrm{sd}(\Delta)}$ for any $0 \leq i \leq \lfloor \frac{d-2}{2} \rfloor$.

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The refined Eulerian statistics on permutations

Let S_d be the symmetric group on $\{1, \ldots, d\}$.
For $\sigma \in S_d$ let $D(\sigma) := \{ i \in [d-1] \mid \sigma(i) > \sigma(i+1)\}$ be the descent set of σ and set $\mathrm{des}(\sigma) := \#D(\sigma)$.

For $d \geq 1$, $0 \leq i \leq d-1$ and $1 \leq j \leq d$ we set

$$A(d,i,j) := \#\{ \sigma \in S_d \mid \mathrm{des}(\sigma) = i,\ \sigma(1) = j\}.$$

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New Inequalities for the refined Eulerian statistics (1)

Corollary

(i) $A(d, j, r) \leq A(d, d-2-j, r)$

for $d \geq 1$, $1 \leq r \leq d$ and $0 \leq j \leq \lfloor \frac{d-3}{2} \rfloor$.

(ii) $A(d, 0, r) \leq A(d, 1, r) \leq \ldots \leq A(d, \lfloor \frac{d-1}{2} \rfloor, r)$

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for $d \geq 2$ and $1 \leq r \leq d$.

For d even, $A(d, \lfloor \frac{d-1}{2} \rfloor, r)$ may be larger or smaller than $A(d, \lceil \frac{d-1}{2} \rceil, r)$.

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Corollary

(i) $A(d,j,1) \leq A(d,j,2) \leq \ldots \leq A(d,j,d)$

for $\lceil \frac{d}{2} \rceil = \lfloor \frac{d+1}{2} \rfloor \leq j \leq d-1$.

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What the proof essentially relies on

- Theorem (Brenti, Welker):
  Let Δ be a $(d-1)$-dimensional simplicial complex and let $\mathrm{sd}(\Delta)$ be its barycentric subdivision. Then

$$h_j^{\mathrm{sd}(\Delta)} = \sum_{r=0}^{d} A(d+1, j, r+1) h_r^{\Delta}$$

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Example for the refined Eulerian numbers

$d = 6$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 26 & 16 & 8 & 4 & 2 & 1 \\ 66 & 66 & 60 & 48 & 36 & 26 \\ 26 & 36 & 48 & 60 & 66 & 66 \\ 1 & 2 & 4 & 8 & 16 & 26 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Thank you for your attention!

Questions?