Recurrences for a Class of Ising Integrals

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Main Points

- The Ising Integrals $C_{n,k}$
- A convolution theorem
- Using `MultiSum` to obtain recurrences
- A class of multiple Barnes integrals
- From summation to integration
- Conclusion

The Problem: [1]

Find recurrences in $k \geq 0$ for Ising-class integrals of the form

$$C_{n,k} := \frac{1}{n!} \int_0^\infty \cdots \int_0^\infty \frac{dx_1\, dx_2 \ldots dx_n}{(\cosh x_1 + \cdots + \cosh x_n)^{k+1}}.$$

Renormalization:

$$c_{n,k} := \frac{n!}{2^n} \Gamma(k+1)\, C_{n,k}.$$

Closed forms for $n = 1, 2$:

$$C_{1,k} = \frac{2^k \Gamma\left(\frac{k+1}{2}\right)^2}{\Gamma(k+1)}, \quad C_{2,k} = \frac{\sqrt{\pi}\Gamma\left(\frac{k+1}{2}\right)^3}{2\Gamma\left(\frac{k}{2}+1\right)\Gamma(k+1)}.$$

[1] D. H. Bailey, D. Borwein, J. M. Borwein and R. E. Crandall. Hypergeometric forms for Ising-class integrals. *Experimental Mathematics*

Conjectured recurrences (D. H. Bailey and J. M. Borwein)

$$
\begin{aligned}
0 =&(k+1)^3 C_{3,k} - 2(k+2)(5(k+2)^2+1) C_{3,k+2} \\
&+ 9(k+2)(k+3)(k+4) C_{3,k+4}
\end{aligned}
$$

$$
\begin{aligned}
0 =&(k+1)^4 C_{4,k} - 4(k+2)^2(5(k+2)^2+3) C_{4,k+2} \\
&+ 64(k+2)(k+3)^2(k+4) C_{4,k+4}
\end{aligned}
$$

$$
\begin{aligned}
0 \stackrel{?}{=}&(k+1)^5 C_{5,k} - (k+2)(35k^4+280k^3+882k^2+1288k+731) C_{5,k+2} \\
&+ (k+2)(k+3)(k+4)(259k^2+1554k+2435) C_{5,k+4} \\
&- 225(k+2)(k+3)(k+4)(k+5)(k+6) C_{5,k+6}
\end{aligned}
$$

$$
\begin{aligned}
0 \stackrel{?}{=}&(k+1)^6 C_{6,k} - 8(k+2)^2(7k^4+56k^3+182k^2+280k+171) C_{6,k+2} \\
&+16(k+2)(k+3)^2(k+4)(49k^2+294k+500) C_{6,k+4} \\
&-2304(k+2)(k+3)(k+4)^2(k+5)(k+6) C_{6,k+6}
\end{aligned}
$$

The convolution theorem[2] (W. Zudilin)

For $k > 0$, $n \geq 1$ and $1 \leq q \leq n-1$ we have

$$c_{n,k} = \frac{1}{2\pi i} \int_{\mathcal{C}} c_{n-q,k+s} c_{q,-1-s} ds$$

where the contour $\mathcal{C}$ runs over $(-\lambda - i\infty, -\lambda + i\infty)$ with $\lambda \in \mathbb{R}$ such that $-1-k < -\lambda < -1$.

Proof: If $A := \sum_{i=1}^{n-q} \cosh x_i$ and $B := \sum_{i=q+1}^{n} \cosh x_i$ then

$$C_{n,k} = \frac{1}{n!} \int_0^\infty (A+B)^{-k-1} \, dx = \frac{1}{n!} \int_0^\infty A^{-k-1} \left(1 + \frac{B}{A}\right)^{-k-1} dx$$

and use the binomial theorem:

$$(1 + B/A)^{-k-1} = \frac{1}{\Gamma(k+1)} \frac{1}{2\pi i} \int_{\mathcal{C}} \Gamma(k+1+s) \Gamma(-s) (B/A)^s \, ds$$

[2] D. H. Bailey, D. Borwein, J. M. Borwein and R. E. Crandall. Hypergeometric forms for Ising-class integrals. *Experimental Mathematics*

$C_{n,k}$ becomes a nested Barnes integral

$$c_{3,k} = \frac{1}{2\pi i}\int_{\mathcal{C}} c_{2,k+s}c_{1,-1-s}ds$$

$$c_{4,k} = \frac{1}{2\pi i}\int_{\mathcal{C}} c_{2,k+s}c_{2,-1-s}ds$$

$$c_{5,k} = \frac{-1}{4\pi^2}\int_{\mathcal{C}_s}\int_{\mathcal{C}_t} c_{2,k+s}c_{2,-1-s+t}c_{1,-1-t}dtds$$

$$c_{6,k} = \frac{-1}{4\pi^2}\int_{\mathcal{C}_s}\int_{\mathcal{C}_t} c_{2,k+s}c_{2,-1-s+t}c_{2,-1-t}dtds$$

$$\cdots\cdots$$

$$c_{n,k} = \frac{1}{(2\pi i)^q}\int_{\mathcal{C}_{t_1}}\cdots\int_{\mathcal{C}_{t_q}} c_{2,k+t_1}\left(\prod_{j=1}^{q-1} c_{2,-1-t_j+t_{j+1}}\right)c_{\epsilon,-1-t_q}dt_1\ldots dt_q$$

where $q := \left\lceil \frac{n}{2} \right\rceil - 1$ and $\epsilon := n - 2q$.

For example $C_{6,k}$

$$C_{6,k} = \frac{-1}{720\sqrt{\pi}} \int_{\mathcal{C}_s} \int_{\mathcal{C}_t} F(k,t,s) \; ds \, dt.$$

where

$$F(k,t,s) := \frac{\Gamma\left(s+\frac{k+1}{2}\right)^3 \Gamma\left(t-s\right)^3 \Gamma\left(-t\right)^3}{\Gamma\left(k+1\right)\Gamma\left(s+\frac{k}{2}+1\right)\Gamma\left(t-s+\frac{1}{2}\right)\Gamma\left(-t+\frac{1}{2}\right)}.$$

Prove the following conjectured recurrence:

$$\begin{aligned}0 =& (k+1)^6 C_{6,k} - 8(k+2)^2(7k^4+56k^3+182k^2+280k+171)C_{6,k+2}\\ &+16(k+2)(k+3)^2(k+4)(49k^2+294k+500)C_{6,k+4}\\ &-2304(k+2)(k+3)(k+4)^2(k+5)(k+6)C_{6,k+6}\end{aligned}$$

A Computer Proof- How to proceed?

Wegschaider's Algorithm - an extension of Fasenmyer/WZ summation - computes recurrences in elements of $\mu = (\mu_1, \ldots, \mu_l)$ for the multiple sum:

$$\boxed{Sum(\mu, \alpha) := \sum_{\kappa_1} \cdots \sum_{\kappa_r} \mathcal{F}(\mu, \kappa_1, \ldots, \kappa_r, \alpha)}$$

where $\mathcal{F}(\mu, \kappa, \alpha)$ is a hypergeometric term in the variables $\mu = (\mu_1, \ldots, \mu_l)$ and $\kappa = (\kappa_1, \ldots, \kappa_r)$ with additional parameters $\alpha = (\alpha_1, \ldots, \alpha_s)$.

<u>Remark:</u> for $r = 1$ and $l = 1$ we have <u>Zeilberger's Algorithm</u>

How to prove that the computed recurrences hold?

Wegschaider's Algorithm first delivers certificate recurrences for the summand $\mathcal{F}(\mu, \kappa, \alpha)$:

$$\boxed{\sum_{(m,k)\in\mathbb{S}} a_{m,k}(\mu,\alpha)\,\mathcal{F}(\mu - m, \kappa - k, \alpha) = \sum_{j=1}^{r} \Delta_{\kappa_j}\left(r_j(\mu,\kappa)\,\mathcal{F}(\mu,\kappa,\alpha)\right)}$$

where the forward shift operators are defined as

$$\Delta_{\kappa_j}\mathcal{F}(\mu,\kappa,\alpha) := \mathcal{F}(\mu,\kappa_1,\ldots,\kappa_j+1,\ldots,\kappa_r,\alpha) - \mathcal{F}(\mu,\kappa,\alpha).$$

MultiSum for Summation

For the summation problem

$$\boxed{Sum(\mu,\alpha) := \sum_{\kappa_1}\cdots\sum_{\kappa_r}\mathcal{F}\left(\mu,\kappa_1,\ldots,\kappa_r,\alpha\right)}$$

MultiSum delivers a certificate recurrence

$$\boxed{\sum_{(m,k)\in\mathbb{S}} a_{m,k}\left(\mu,\alpha\right)\mathcal{F}(\mu-m,\kappa-k,\alpha) = \sum_{j=1}^{r}\Delta_{\kappa_j}\left(r_j\left(\mu,\kappa\right)\mathcal{F}(\mu,\kappa,\alpha)\right)}$$

Assuming $\mathcal{F}\left(\mu,\kappa,\alpha\right)$ has finite support, we get a homogeneous recurrence for the sum

$$\boxed{\sum_{(m,k)\in\mathbb{S}} a_{m,k}\left(\mu\right) Sum\left(\mu-m,\alpha\right) = 0}$$

MultiSum for Integration

For the integration problem

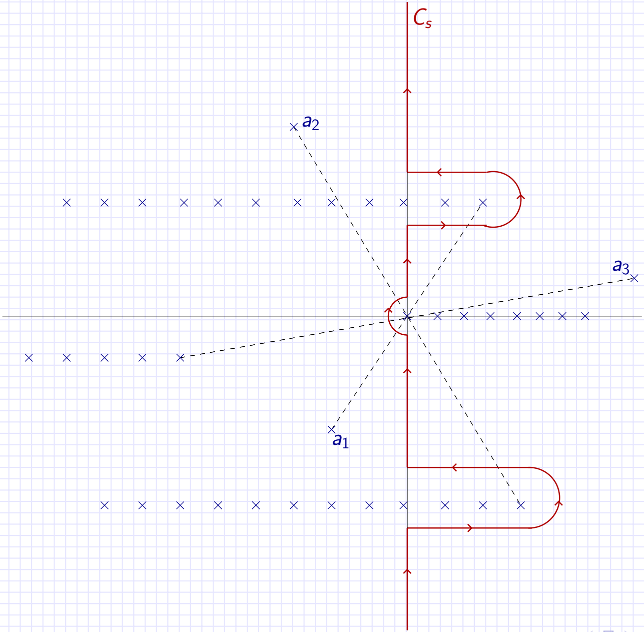
$$\boxed{Int\,(\mu,\alpha) = \int_{\mathcal{C}_{\kappa_1}} \dots \int_{\mathcal{C}_{\kappa_r}} \mathcal{F}\,(\mu,\kappa_1,\dots,\kappa_r,\alpha)\,d\kappa_1 \dots d\kappa_r}$$

MultiSum delivers a certificate recurrence

$$\boxed{\sum_{(m,k)\in\mathbb{S}} a_{m,k}\,(\mu,\alpha)\,\mathcal{F}(\mu-m,\kappa-k,\alpha) = \sum_{j=1}^{r} \Delta_{\kappa_j}\,(r_j\,(\mu,\kappa)\,\mathcal{F}(\mu,\kappa,\alpha))}$$

Assuming $\mathcal{F}\,(\mu,\kappa,\alpha)$ has appropriate asymptotics, we get a homogeneous recurrence for the integral

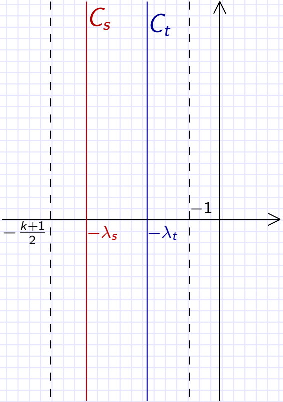
$$\boxed{\sum_{(m,k)\in\mathbb{S}} a_{m,k}\,(\mu)\,Int\,(\mu-m,\alpha) = 0}$$



A Barnes path of integration separating the poles of $\Gamma\left(a_i + s\right); i = \overline{1,3}$ from the poles of $\Gamma\left(-s\right)$.

Back to $\quad C_{6,k} = \frac{-1}{720\sqrt{\pi}} \int_{\mathcal{C}_s} \int_{\mathcal{C}_t} F(k,t,s)\ ds\, dt \quad$ where

$$F(k,t,s) := \frac{\Gamma\left(s+\frac{k+1}{2}\right)^3 \Gamma(t-s)^3 \Gamma(-t)^3}{\Gamma(k+1)\Gamma\left(s+\frac{k}{2}+1\right)\Gamma\left(t-s+\frac{1}{2}\right)\Gamma\left(-t+\frac{1}{2}\right)}.$$



$\mathcal{C}_s := (-\lambda_s - i\infty, -\lambda_s + i\infty)$ separates the poles of $\Gamma\left(\frac{k+1}{2}+s\right)$ from the poles of $\Gamma(t-s)$ and similarly $\mathcal{C}_t := (-\lambda_t - i\infty, -\lambda_t + i\infty)$ separates the poles of $\Gamma(t-s)$ from the poles of $\Gamma(-t)$.

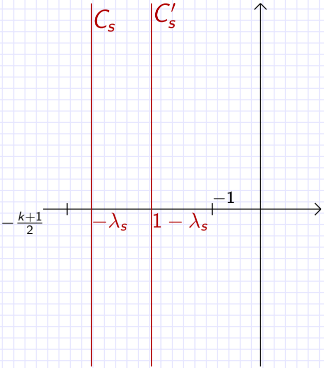
We choose λ_s, λ_t such that

$$\boxed{-\frac{1+k}{2} < -\lambda_s < -\lambda_t < -1}$$

Certificates recurrences for $C_{6,k} = \frac{-1}{720\sqrt{\pi}} \int_{\mathcal{C}_s} \int_{\mathcal{C}_t} F(k,t,s)\ \ ds\, dt$ are of the form:

$$\sum_{(i,j,l)\in\mathbb{S}} a_{i,j,l}(k)\, F(k-i, s-j, t-l) =$$

$$= \Delta_s\left(rat_1(k,s,t)\, F(k,s,t)\right) + \Delta_t\left(rat_2(k,s,t)\, F(k,s,t)\right).$$



We consider

$$I := \int_{\mathcal{C}_s} \Delta_s F(k,s,t)\, ds$$
$$= \int_{\mathcal{C}_s'} F(k,s,t)\, ds - \int_{\mathcal{C}_s} F(k,s,t)\, ds.$$

For the integral

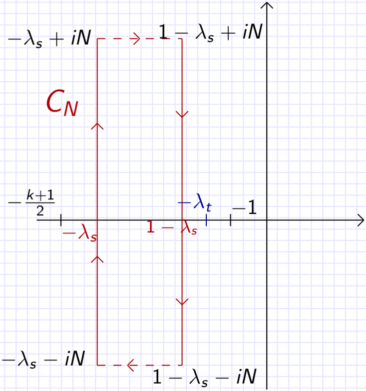
$$J := \int_{\mathcal{C}_t} \Delta_t F(k,s,t)\, dt$$

one proceedes similarly.

It only remains to show that

$$I_N := \int_{\mathcal{C}_N} F(k,s,t)\,ds \to I \text{ as } N \to \infty.$$

For this we need to prove that the integrals



$$\int_{-\lambda_1+iN}^{1-\lambda_1+iN} F(k,s,t)\,ds \quad \text{and}$$

$$\int_{-\lambda_1-iN}^{1-\lambda_1-iN} F(k,s,t)\,ds$$

tend to zero as $N \to \infty$.

In any such rectangular region of integration, we have the asymptotic behavior $\boxed{F(k,s,t) = \mathcal{O}\left(e^{-c|s|}\right)}$

as $|s| \to \infty$ with $c > 0$.

Conclusions

- We present an algorithmic method to prove and compute recurrences for members of the Ising-class integrals.
- We describe an algorithmic approach to obtain recurrences for a general class of multiple contour integrals of Barnes' type.
- If the input of Wegschaider's algorithm is too involved, computations become time consuming.
- J. M. Borwein and B. Salvy. A proof of a recursion for Bessel moments. *Computing Research Repository*, June 2007.
- http://www.risc.uni-linz.ac.at/research/combinat/software