Isometry classes of Generalized Associahedra

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(joint work with C. Hohlweg, C. Lange and H. Thomas)

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Associahedron (Stasheff polytope)
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Loday’s construction

Permutahedron \rightarrow Associahedron

\{\alpha_1, \alpha_2\} is a basis of the root system of type \textit{A}_2
(\(W, S\)) a finite Coxeter system acting on \((V, \langle \cdot, \cdot \rangle)\).

\(\Phi\) root system with simple roots \(\Delta = \{\alpha_s \mid s \in S\}\).

\(\Delta^* = \{v_s \mid s \in S\}\) be the dual simple roots of \(\Delta\).
Generalized Associahedra
[Fomin, Zelevinski + Chapoton + Reading + HLT]

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$\Delta^* = \{ v_s | s \in S \}$ be the dual simple roots of $\Delta$.

$$v = \sum_{s \in S} v_s$$

The permutahedron: $\text{Perm}(W) = \text{convex hull} \{ w(v) | w \in W \}$.
Generalized Associahedra

[ Fomin, Zelevinski + Chapoton + Reading + HLT ]

\( \text{Perm}(W) = \text{convex hull} \{ w(v) \mid w \in W \} \) where \( v = \sum_{s \in S} v_s \).
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\[ \text{Perm}(W) = \text{convex hull} \{w(v) \mid w \in W\} \text{ where } v = \sum_{s \in S} v_s. \]

Fix a coxeter element \( c \) of \((W, S)\). \( c = \prod_{s \in S} s \) in some order.

example: for \( W = A_3 \) and \( S = \{s_1, s_2, s_3\} \) we can choose

\[
\begin{align*}
c &= s_1 s_2 s_3 \\
c &= s_1 s_3 s_2 = s_3 s_1 s_2 \\
c &= s_2 s_1 s_3 = s_2 s_3 s_1 \\
c &= s_3 s_2 s_1 
\end{align*}
\]
**Generalized Associahedra**

[Fomin, Zelevinski + Chapoton + Reading + HLT]

\[
\text{Perm}(W) = \text{convex hull} \left\{ w(v) \mid w \in W \right\} \text{ where } v = \sum_{s \in S} v_s.
\]

Fix a coxeter element \( c \) of \((W, S)\). \( c = \prod_{s \in S} s \) in some order.

Let \( w_0 = c_{K_1} c_{K_2} \cdots c_{K_p} \) (unique) reduced factorization such that

\[
K_1 \supseteq K_2 \supseteq \cdots \supseteq K_p \quad \text{and} \quad c_K = \prod_{s \in K} s
\]

example: for \( W = A_3 \) and \( S = \{s_1, s_2, s_3\} \), if we choose

\[
c = s_1 s_2 s_3 \quad \rightarrow \quad w_0 = s_1 s_2 s_3 s_1 s_2 s_1 = c_{\{1,2,3\}} c_{\{1,2\}} c_{\{1\}}
\]

\[
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Fix a coxeter element \( c \) of \((W, S)\).

\[ c = \prod_{s \in S} s \text{ in some order.} \]

Let \( w_0 = c_{K_1}c_{K_2} \cdots c_{K_p} \) (unique) reduced factorization
\[ T_c = \{ u \in W : u \text{ is a prefix of } c_{K_1}c_{K_2} \cdots c_{K_p} \text{ up to commutations} \} \]

Using only the allowed commutation \( s_is_j = s_js_i \).

example: for \( W = A_3 \) and \( S = \{ s_1, s_2, s_3 \} \), with \( c = s_1s_3s_2 \) we have
\( w_0 = s_1s_3s_2 \cdot s_1s_3s_2 \) and
\[ T_c = \{ e, s_1, s_1s_3, s_1s_3s_2, s_1s_3s_2s_1, s_1s_3s_2s_3, w_0, s_3, s_1s_3s_2s_3 \} \]
**Generalized Associahedra**

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\[ \text{Perm}(W) = \text{convex hull} \{ w(v) \mid w \in W \} \text{ where } v = \sum_{s \in S} v_s. \]

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\[ \text{Ass}_c(W) \]

is the polytope defined by the hyperplanes of \( \text{Perm}(W) \) that contains elements \( u(v) \) for \( u \in T_c \).
Generalized Associahedra: $A_2$ and $c = s_2s_1$

\[ w_0 = s_2s_1 \cdot s_2 \text{ and } T_c = \{ e, s_2, s_2s_1, w_0 \} \]
Generalized Associahedra: $A_3$ and $c = s_1 s_2 s_3$

$w_0 = s_1 s_2 s_3 \cdot s_1 s_2 \cdot s_1$ and $T_c = \{ e, s_1, s_1 s_2, c, s_1 s_2 s_1, c s_1, c s_1 s_2, w_0 \}$
Generalized Associahedra: $A_3$ and $c = s_1s_3s_2$

$$w_0 = s_1s_3s_2 \cdot s_1s_3s_2$$ and $$T_c = \{ e, s_1, s_3, s_1s_3, c, cs_1, cs_3, cs_1s_3, w_0, \}$$
Some questions

$T_c$ is known to be a lattice, but what is $|T_c|$ (even for type $A$)?
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How many distinct polytope do we get (up to isometry)?

**Theorem** [BHLT] For $(W, S)$ irreducible finite Coxeter system and $c, c'$ Coxeter elements:

$$\text{Ass}_c(W) \cong \text{Ass}_{c'}(W) \iff c' = \mu(c)^{\pm 1}$$

where $\mu$ is an automorphism of the Coxeter graph of $W$. 

The Main Theorem

**Theorem** [BHLT] For \((W, S')\) irreducible finite Coxeter system and \(c, c'\) Coxeter elements:

\[
\text{Ass}_c(W) \cong \text{Ass}_{c'}(W) \iff c' = \mu(c)^{\pm 1}
\]

where \(\mu\) is an automorphism of the Coxeter graph of \(W\).

In type \(A\), an isometry class contains 1, 2 or 4 Coxeter elements.

In type \(D\), an isometry class contains 1, 2 or 4 Coxeter elements (except for \(D_4\) which has a class of 12 elements).
Idea of proof

1. An isometry $\text{Ass}_c(W) \rightarrow \text{Ass}_{c'}(W)$ must fix the set $\{e, w_0\}$ and $\text{Perm}(W)$.

2. Such isometry send coxeter elements $c$ to $c' = \mu(c)^{\pm 1}$.

3. Conversely, there is such an isometry for any $\mu$ and the map $w \mapsto ww_0$ induces an isometry $\text{Ass}_c(W) \rightarrow \text{Ass}_{c^{-1}}(W)$.

For more details, see paper...[ArXive]