

# On the shape of Bruhat intervals

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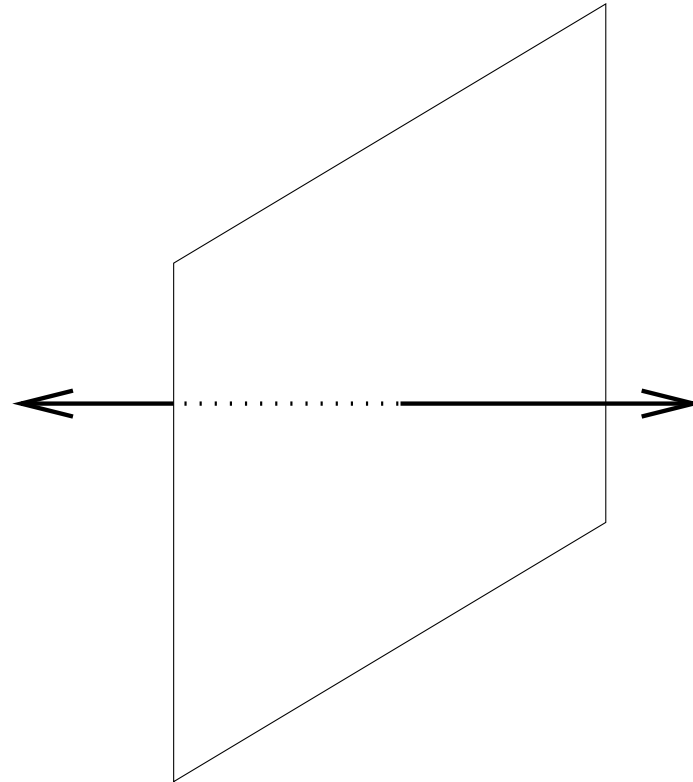
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Séminaire Lotharingien de Combinatoire 61  
Curia, September, 2008

## Coxeter groups

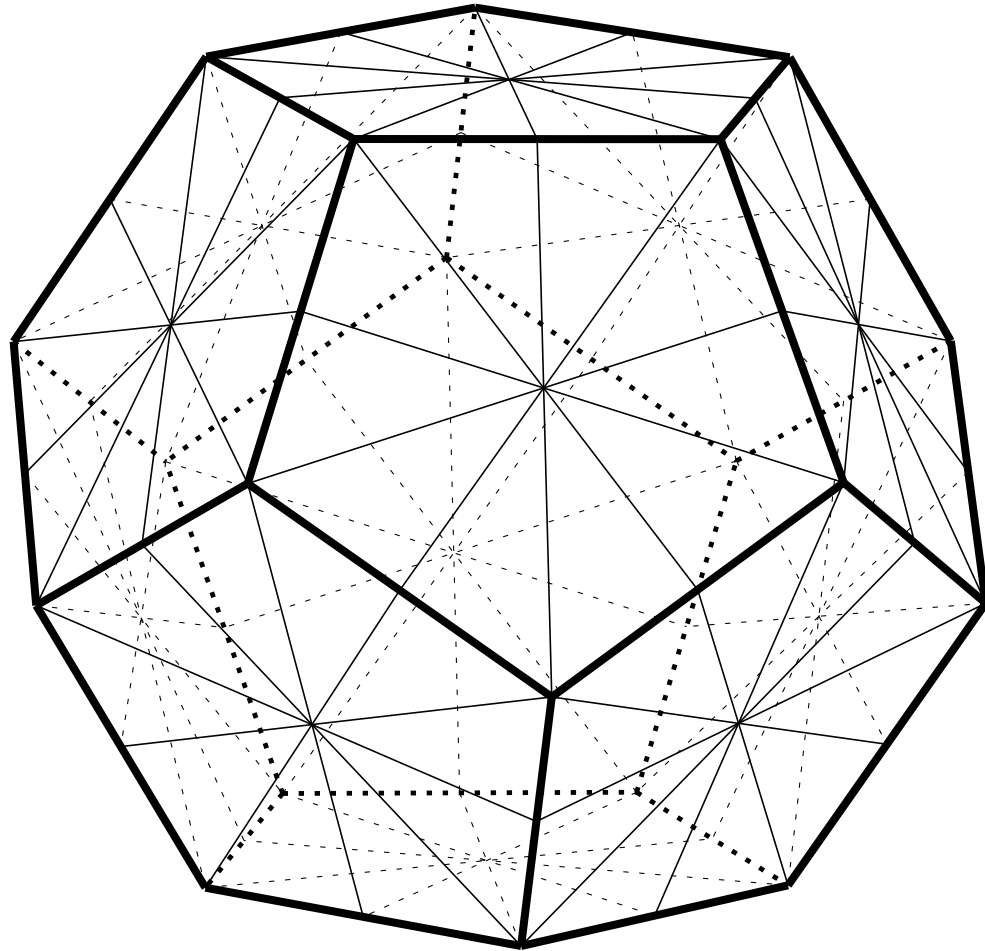
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**Finite Coxeter groups**  $\longleftrightarrow$  Finite reflection groups (i.e., groups generated by orthogonal reflections in hyperplanes)

## Coxeter groups

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The dodecahedron as a reflection group

## Coxeter groups

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The pair  $(W, S)$  is a *Coxeter group* (Coxeter system) if  $W$  is a group with presentation

Generators:  $S$ , such that

$$s^2 = e, \text{ for all } s \in S,$$

Relations: for  $s, s' \in S$

$$\underbrace{ss's's' \dots}_{m(s,s')} = \underbrace{s's's's' \dots}_{m(s,s')}$$

# Coxeter groups

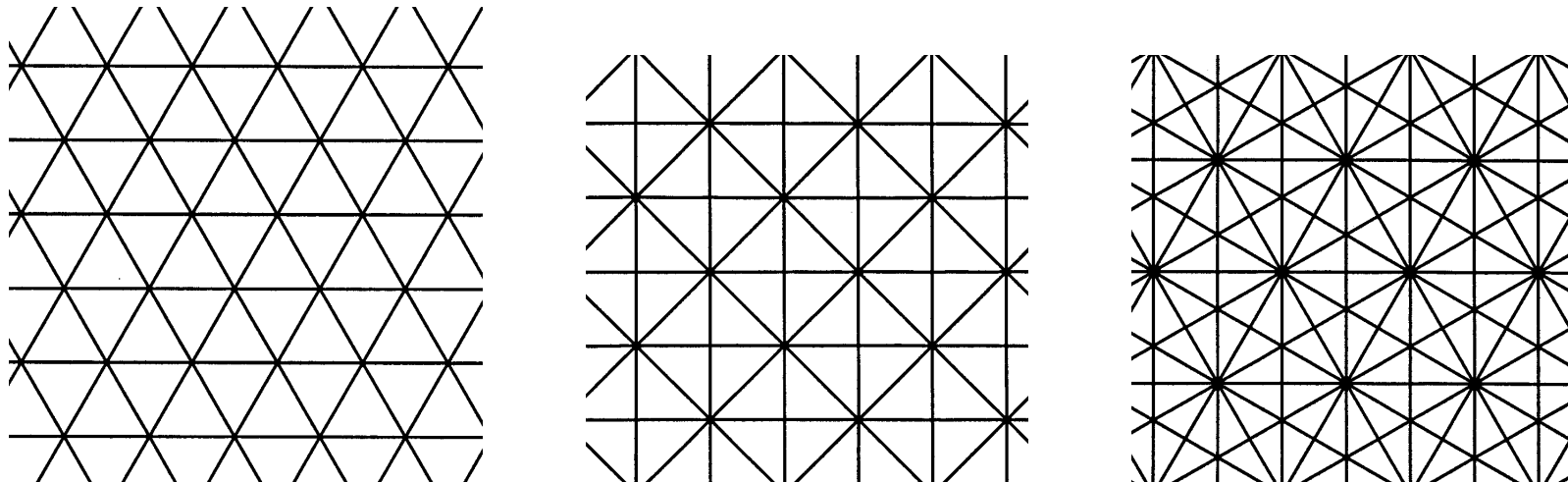
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## Examples

1. The symmetric group  $S_n$ .

Coxeter generators = Adjacent transpositions  $(i, i + 1)$

2. Affine reflection groups



The  $\tilde{A}_2$ ,  $\tilde{C}_2$  and  $\tilde{G}_2$  tessellations of the affine plane.

## Coxeter groups

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### ∃ **classifications**

*finite* Coxeter groups: type  $A_n, B_n, \dots$  etc.

*affine* Coxeter groups: type  $\tilde{A}_n, \tilde{B}_n, \dots$  etc.

*hyperbolic* Coxeter groups



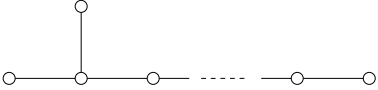
**Definition:**  $(W, S)$  is *crystallographic* if  $m(s, t) \in \{2, 3, 4, 6, \infty\}$  for all distinct generators  $s$  and  $t$ .

E.g., *finite and affine Weyl groups* are crystallographic.

# Coxeter groups

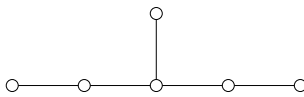
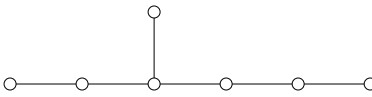
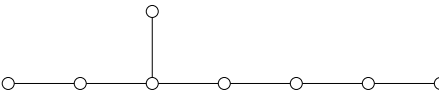
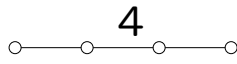
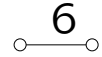
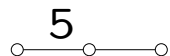
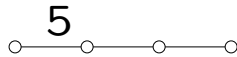
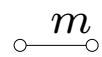
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## The finite irreducible Coxeter systems

Name	Diagram	Order	$ T $	Exponents
$A_n$ ( $n \geq 1$ )		$(n + 1)!$	$\binom{n + 1}{2}$	$1, 2, \dots, n$
$B_n$ ( $n \geq 2$ )		$2^n n!$	$n^2$	$1, 3, \dots, 2n - 1$
$D_n$ ( $n \geq 4$ )		$2^{n-1} n!$	$n^2 - n$	$1, 3, \dots, 2n - 3, n - 1$

## Coxeter groups

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$E_6$		$2^7 3^4 5$	36	1, 4, 5, 7, 8, 11
$E_7$		$2^{10} 3^4 5 7$	63	1, 5, 7, 9, 11, 13, 17
$E_8$		$2^{14} 3^5 5^2 7$	120	1, 7, 11, 13, 17, 19, 23, 29
$F_4$		1152	24	1, 5, 7, 11
$G_2$		12	6	1, 5
$H_3$		120	15	1, 5, 9
$H_4$		14400	60	1, 11, 19, 29
$I_2(m)$ ( $m \geq 3$ )		$1$ $2m$	$m$	$1, m - 1$

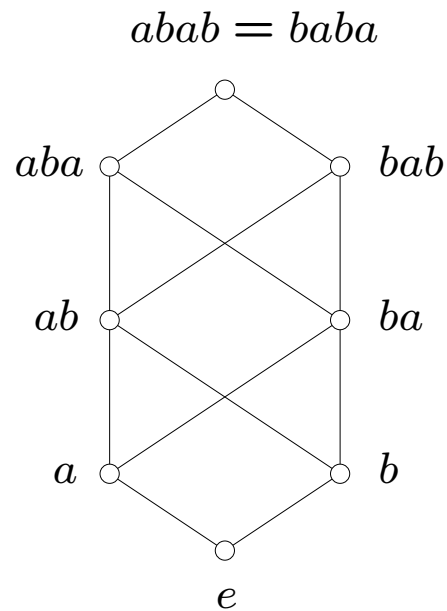


## Bruhat order

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**Bruhat order:** For  $u, w \in W$ :

$u \leq w \stackrel{\text{def}}{\iff}$  for  $\forall$  reduced expression  $w = s_1 s_2 \dots s_q$   
 $\exists$  a reduced subexpression  $u = s_{i_1} s_{i_2} \dots s_{i_k}$ ,  
 $1 \leq i_1 < \dots < i_k \leq q$ .

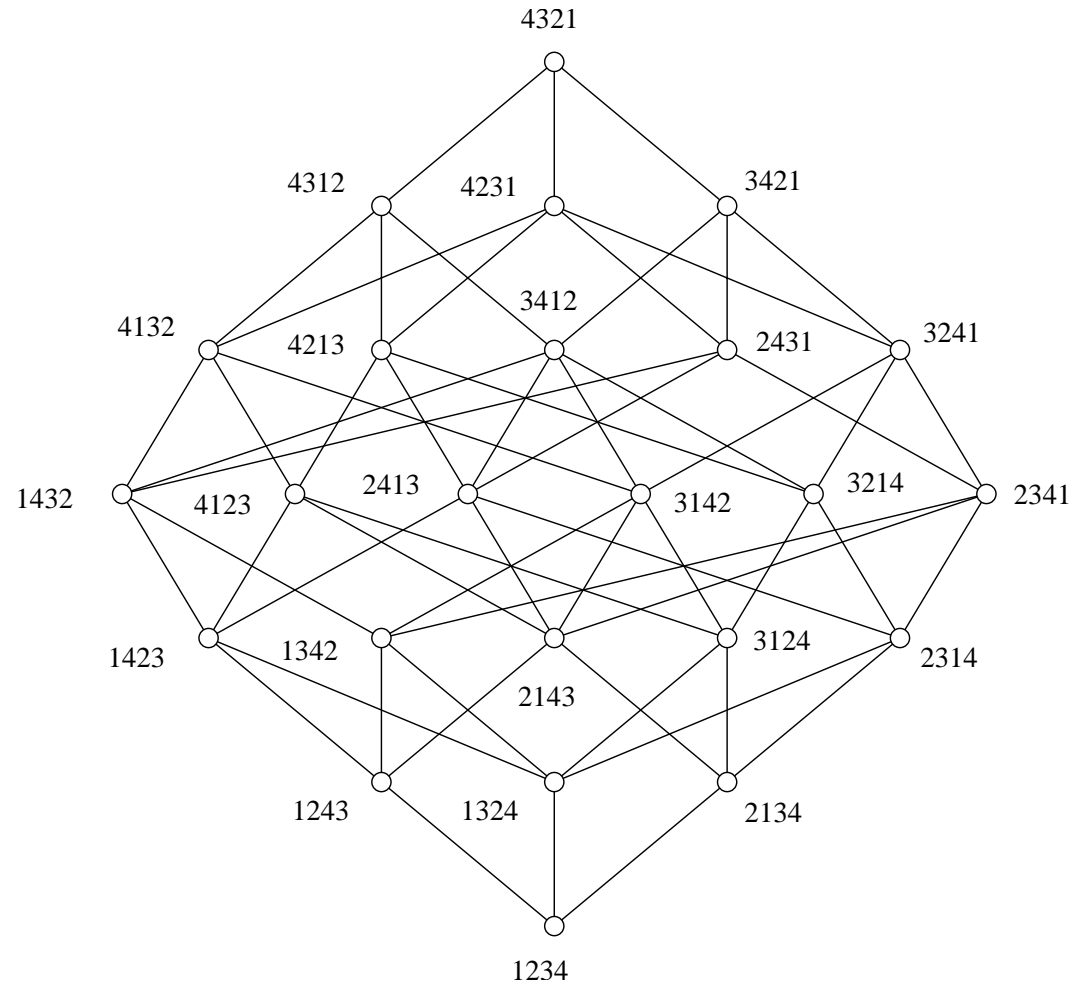


Bruhat order of  $B_2$

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# Bruhat order

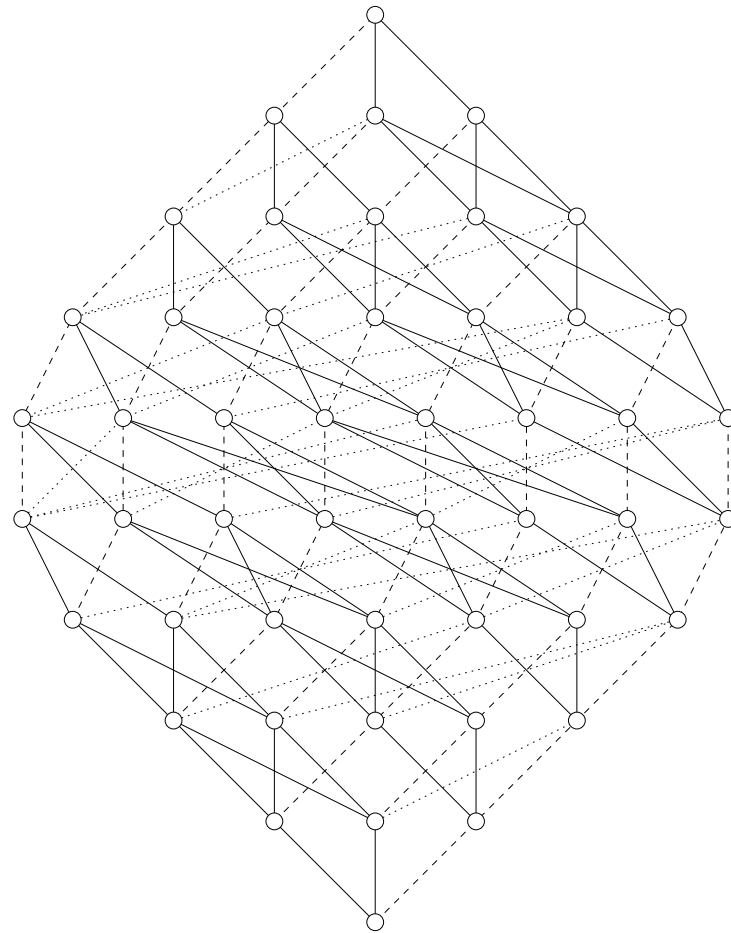
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Bruhat order of  $S_4$ .

# Bruhat order

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Bruhat order of  $B_3$ .

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## Bruhat order

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Some global properties of Bruhat order of a finite  $W$ , as a poset:

\*\* Bottom element  $e$ , top element  $w_0$

\*\* Graded (all maximal chains of same size)

\*\* Poset rank = Group-theoretic length  $\ell(\cdot)$

\*\* Rank-generating function

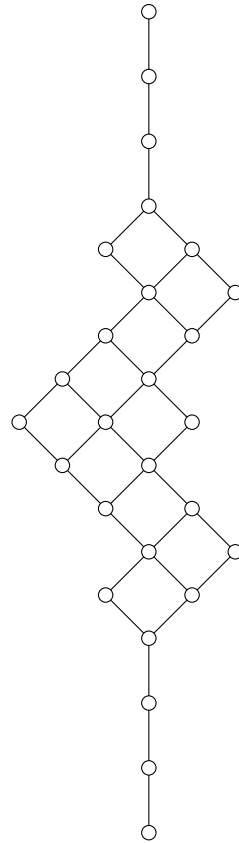
$$\sum_{w \in W} q^{\ell(w)} = \prod_{1 \leq i \leq d} (1 + q + q^2 + \cdots + q^{e_i})$$

\*\* Anti-automorphic under map  $w \mapsto ww_0$

## Bruhat order

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**Quotients  $W^J$ :** Minimal coset representatives modulo parabolic subgroups  $W_J = \langle J \rangle$ ,  $J \subseteq S$ , with induced order.



The Bruhat poset  $E_6$  modulo  $D_5$ .

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## Bruhat order

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Global poset properties of Bruhat order of finite quotients  $W^J$ :

\*\* Graded

\*\* Bottom element  $e$ , top element  $w_0^J$

\*\* Poset rank = Group-theoretic length  $\ell(\cdot)$

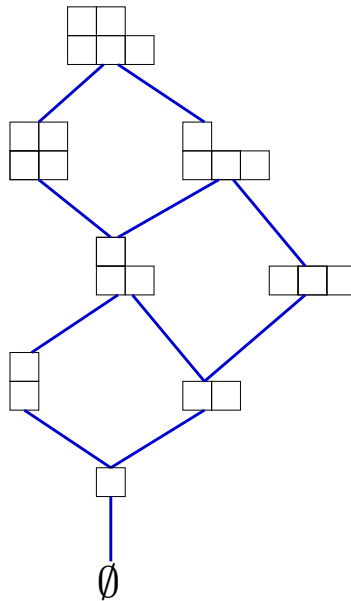
\*\* Rank-generating function  $\sum_{w \in W^J} q^{\ell(w)} = \frac{\sum_{w \in W} q^{\ell(w)}}{\sum_{w \in W_J} q^{\ell(w)}}$

\*\* Anti-automorphic under map  $w \mapsto w_{J,0} w w_0$

## Bruhat order

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A special case of quotient  $W^J$ : Young's lattice



Lower intervals  $[\emptyset, \lambda]$ : Ferrers' diagrams contained in shape  $\lambda$ , and ordered by containment

$\#$  maximal chains =  $\#$  standard Young tableaux of shape  $\lambda$

## Bruhat order

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### General Problem:

Study the combinatorial structure of intervals

$$[u, w]^J \stackrel{\text{def}}{=} \{z : u \leq z \leq w\} \cap W^J$$

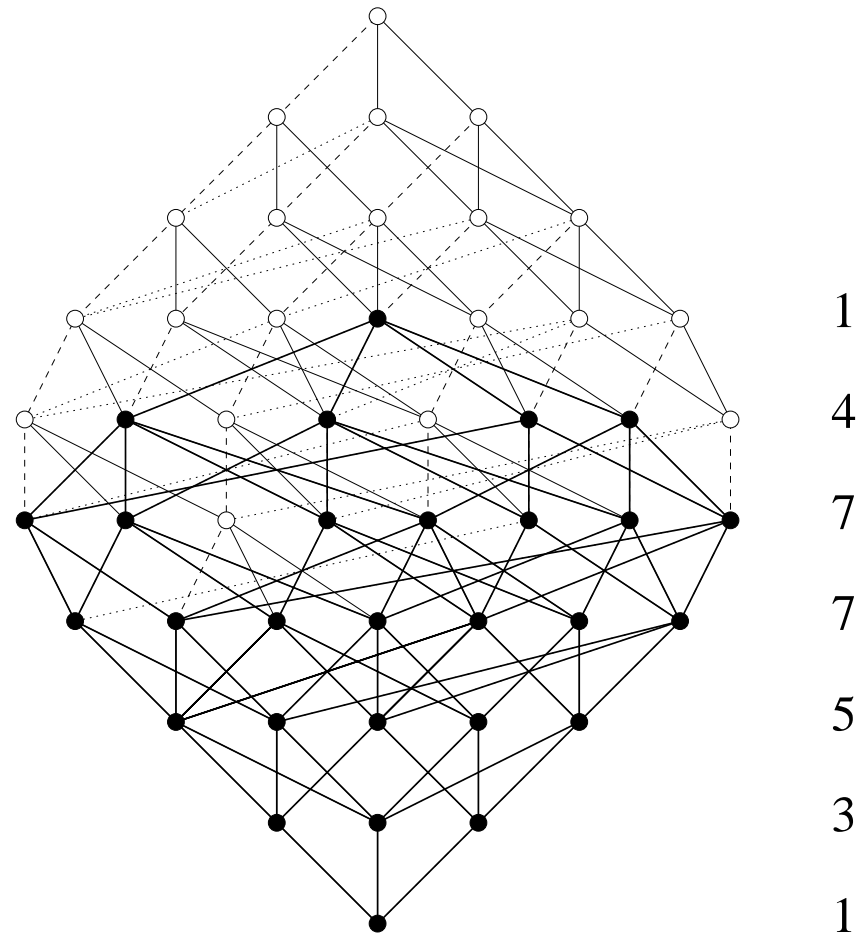
### TOPIC 1: $f$ -vectors of Bruhat intervals

– Joint work with T.Ekedahl

If asking for global interval structure is too hard, study the enumerative “shadow”.

# Bruhat interval $f$ -vectors

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$f^w$ -vector of Bruhat interval  $[e, w]$

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## Bruhat interval $f$ -vectors

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Shape (or  $f$ -vector) of lower interval  $[e, w]^J$ :

$$f^{w,J} = \{f_0^{w,J}, f_1^{w,J}, \dots, f_{\ell(w)}^{w,J}\},$$

$f_i^{w,J} \stackrel{\text{def}}{=} \text{number of elements } x \leq w, x \in W^J, \text{ of length } i.$

Special case of full group:

$$W = W^\emptyset$$

$$f^w \stackrel{\text{def}}{=} f^{w,\emptyset}$$

## Bruhat interval $f$ -vectors

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Another example of  $f^w$ -vector of Bruhat interval  $[e, w]$

Here  $w \in C_4$ ,  $\ell(w) = 13$ :

$$f^w = (1, 4, 9, 16, 24, 32, 39, 44, 46, 42, 31, 17, 6, 1)$$

## Bruhat interval $f$ -vectors

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Another example of  $f^w$ -vector of Bruhat interval  $[e, w]$

Here  $w \in C_4$ ,  $\ell(w) = 13$ :

$$f^w = (1, 4, 9, 16, 24, 32, 39 \mid 44, 46, 42, 31, 17, 6, 1)$$

↑

MID

## Bruhat interval $f$ -vectors

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∃ analogy

Intervals  $[e, w]$  in Bruhat order

↔

Face lattices of convex polytopes

Weyl group

↔

rational polytope

Schubert variety

↔

toric variety

Kazhdan-Lusztig polynomial

↔

$g$ -polynomial

Also: Both determine regular CW decompositions of a sphere  
Intersection cohomology lurks in the background

Remark:

For **all** polytopes: ∃ combinatorial intersection cohomology theory satisfying hard Lefschetz (recent work of K. Karu and others)

Question: ∃ ??? combinatorial intersection cohomology theory for **all** Coxeter groups ("virtual Schubert varieties")?

## Bruhat interval $f$ -vectors

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Note: Analogy with  $f$ -vector of convex polytope **Compare:** Known for  $f$ -vector of **simplicial**  $(d + 1)$ -dimensional polytope:

(1)  $f_i \leq f_j$  if  $i < j \leq d - i$ . In particular,

- $f_0 \leq f_1 \leq \cdots \leq f_{d/2}$  and  $f_i \leq f_{d-i}$

(2)  $f_{3d/4} \geq f_{(3d/4)-1} \geq \cdots \geq f_d$

(3) The bounds  $d/2$  and  $3d/4$  are best possible.

**Conjecture:** (2) is true for **all** polytopes.

## Bruhat interval $f$ -vectors

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Does it make sense to ask such questions for  $f^w$ -vectors of Bruhat intervals  $[e, w]$ ?

Perhaps ... — consider this:

**THM** (Carrell-Peterson 1994)

The Schubert variety  $X_w$  is rationally smooth

$$\iff f_i^w = f_{\ell(w)-i}^w, \forall i$$

**THM** (Brion 2000)

$$\sum_{0 \leq i \leq k} f_i^w \leq \sum_{0 \leq i \leq k} f_{\ell(w)-i}^w$$

## Bruhat interval $f$ -vectors

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**Theorem 1.** *The  $f^{w,J}$ -vector  $f^{w,J} = \{f_0, f_1, \dots, f_{\ell(w)}\}$  of an interval  $[e, w]^J$  in a crystallographic Coxeter group satisfies:*

$$f_i \leq f_j \quad , \quad \text{if } 0 \leq i < j \leq \ell(w) - i.$$

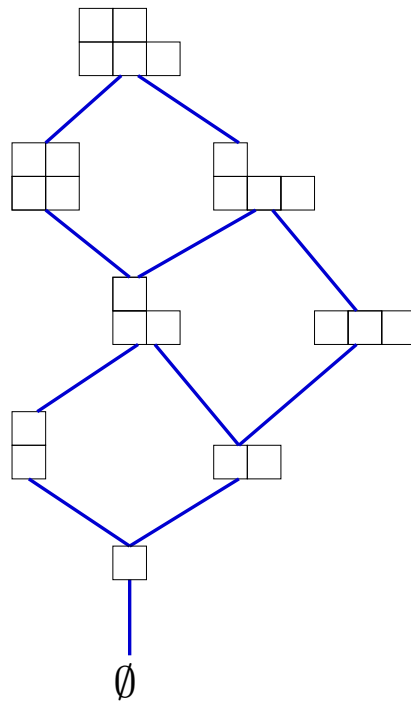
Equivalently,

- $f_i \leq f_{\ell(w)-i}$ , for all  $i < \ell(w)/2$
- $f_0 \leq f_1 \leq \dots \leq f_{\lceil \ell(w)/2 \rceil}$

## Bruhat interval $f$ -vectors

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Gives new inequalities already for the special case of Young's lattice:



Lower intervals  $[\emptyset, \lambda]$ : Ferrers' diagrams contained in shape  $\lambda$ , and ordered by containment

## Bruhat interval $f$ -vectors

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Recall definition:  $(W, S)$  is *crystallographic* if  $m(s, s') \in \{2, 3, 4, 6, \infty\}$  for all distinct generators  $s$  and  $s'$ .

**Fact:** Crystallographic  $\Leftrightarrow$  appears as Weyl group of a Kac-Moody algebra

**Fact:** Crystallographic  $\Rightarrow \exists$  Schubert varieties

## Bruhat interval $f$ -vectors

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Let  $(W, S)$  be crystallographic,  $J \subseteq S$ .

For each  $w \in W^J$  there exists a complex projective variety (called *Schubert variety*)  $\overline{X}_w$  containing closed subvarieties  $\overline{X}_u$  for all  $u \in [e, w]^J$ , which are disjoint unions

$$\overline{X}_u = \bigsqcup_z X_z,$$

where  $z \in [e, u]^J$ .

Furthermore,  $X_u$  is a subvariety of  $\overline{X}_w$  isomorphic to affine space  $\mathbf{A}^{\ell(u)}$ .

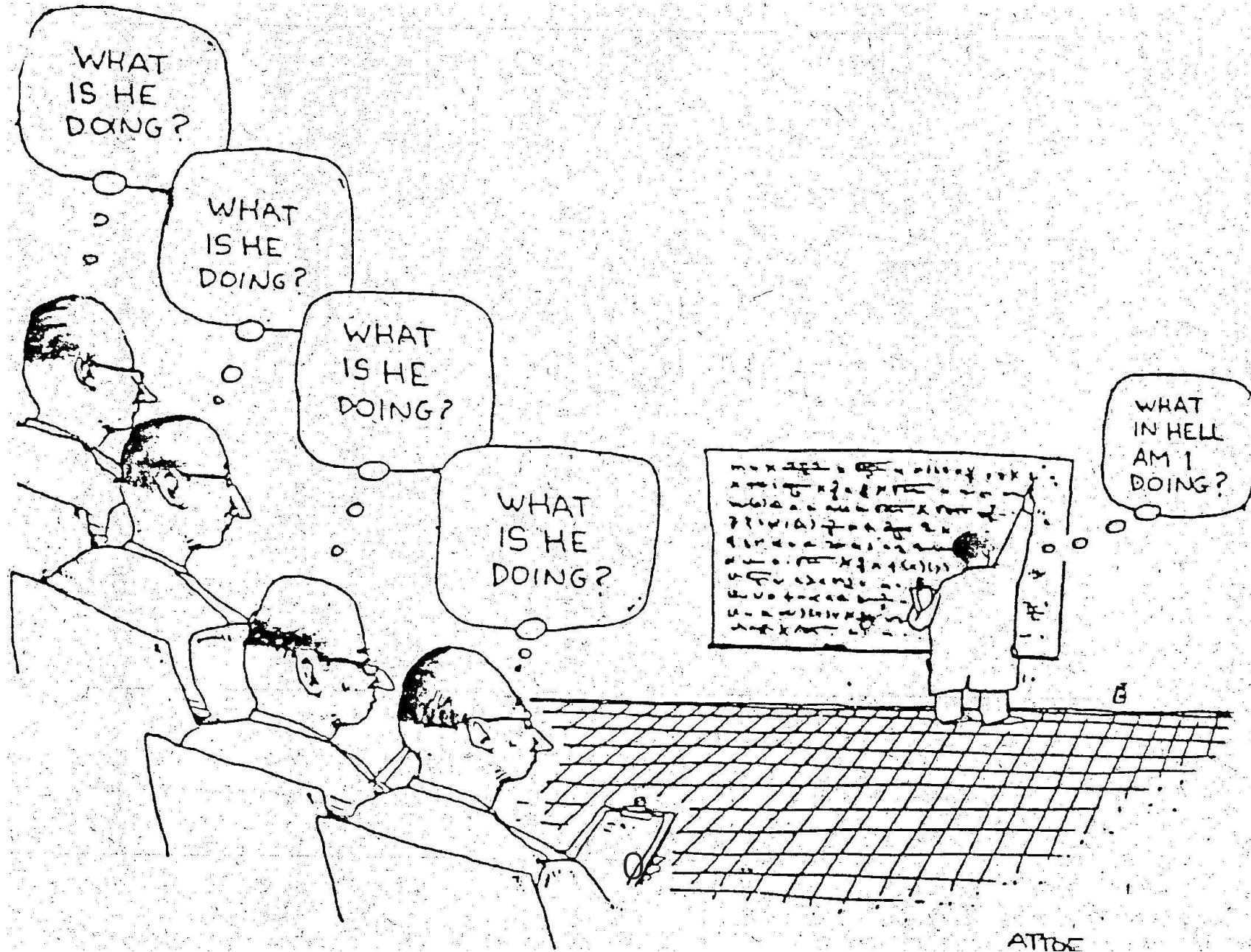
## Bruhat interval $f$ -vectors

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Idea of proof of Thm:

- \* Use  $\ell$ -adic étale cohomology  $H^*(X, \mathbb{Q}_\ell)$  and intersection cohomology  $IH^*(X, \mathbb{Q}_\ell)$ .
- \* There is a  $H^*(X, \mathbb{Q}_\ell)$ -module map  $\varphi : H^*(X, \mathbb{Q}_\ell) \rightarrow IH^*(X, \mathbb{Q}_\ell)$
- \* For Schubert varieties  $X = X_w$  this map  $\varphi$  is injective.
- \*  $f_i^w = \dim_{\mathbb{Q}_\ell} H^{2i}(X_w, \mathbb{Q}_\ell)$

# Bruhat interval $f$ -vectors



## Bruhat interval $f$ -vectors

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### Idea of proof of Thm (cont'd)

Let  $X = X_w$ . The map  $\varphi$  is an  $H^*(X, \mathbb{Q}_\ell)$ -module map

$\Rightarrow$  for  $i \leq j \leq m - i$  it commutes with multiplication by  $c_1(\mathcal{L})^{j-i}$

$\Rightarrow$  commutative diagram

$$\begin{array}{ccc} H^{2i}(X, \mathbb{Q}_\ell) & \longrightarrow & IH^{2i}(X, \mathbb{Q}_\ell) \\ \downarrow \cap c_1(\mathcal{L})^{j-i} & & \downarrow \cap c_1(\mathcal{L})^{j-i} \\ H^{2j}(X, \mathbb{Q}_\ell) & \longrightarrow & IH^{2j}(X, \mathbb{Q}_\ell). \end{array}$$

The horizontal maps  $\varphi$  are injective, and the right vertical map is an injection by hard Lefschetz.

## Bruhat interval $f$ -vectors

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### Idea of proof of Thm (cont'd)

For  $i \leq j \leq m - i$ , we have a commutative diagram

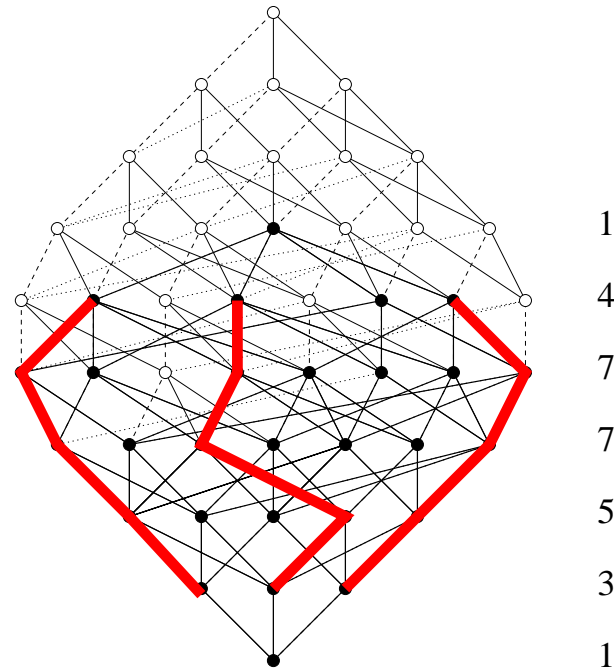
$$\begin{array}{ccc} H^{2i}(X, \mathbb{Q}_\ell) & \longrightarrow & IH^{2i}(X, \mathbb{Q}_\ell) \\ \downarrow \cap c_1(\mathcal{L})^{j-i} & & \downarrow \cap c_1(\mathcal{L})^{j-i} \\ H^{2j}(X, \mathbb{Q}_\ell) & \longrightarrow & IH^{2j}(X, \mathbb{Q}_\ell). \end{array}$$

The horizontal maps  $\varphi$  are injective, and the right vertical map is an injection by hard Lefschetz. **Hence the left vertical map is injective**, giving

$$f_i^w = \dim_{\mathbb{Q}_\ell} H^{2i}(X, \mathbb{Q}_\ell) \leq \dim_{\mathbb{Q}_\ell} H^{2j}(X, \mathbb{Q}_\ell) = f_j^w.$$

## Bruhat interval $f$ -vectors

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**Theorem 2.** Let  $(W, S)$  be crystallographic,  $J \subseteq S$ . Fix  $w \in W^J$  and  $i$  such that  $0 \leq i < \ell(w)/2$ . Then, in  $[e, w]^J$  there exist  $f_i^{w, J}$  pairwise disjoint chains

$$u_i < u_{i+1} < \cdots < u_{\ell(w)-i}$$

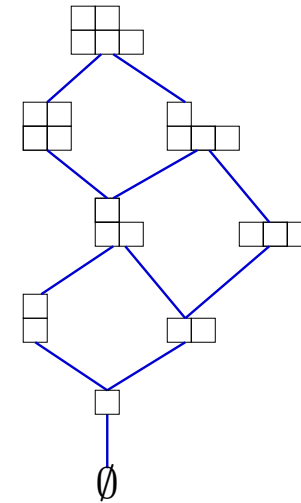
such that  $\ell(u_j) = j$ .

## Bruhat interval $f$ -vectors

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References:

Stanley (1980) did the  $J = S$ ,  $w = w_0$  case for finite groups



Interpretation of Theorem 2 for Young's lattice:

Given a partition  $\lambda$  of  $n$  and  $k \leq n/2$ . Suppose that there are  $b = b(\lambda, k)$  partitions of  $k$  below  $\lambda$ . Then there exist  $b$  standard Young tableaux of shape  $\lambda$ ,  $T^1, \dots, T^b$ , such that

$$\text{shape}(T_p^i) \neq \text{shape}(T_p^j)$$

for all  $i \neq j$  and all  $p = k, k + 1, \dots, n - k$ .

Here  $T_p$  is the subtableau gotten by erasing the boxes with numbers  $> p$ .

## Bruhat interval $f$ -vectors

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**Question:** what about the case of equality in some of the relations  $f_i \leq f_{\ell(w)-i}$ ?

**From now on:** Only the  $J = \emptyset$  case.

Then  $W^J = W$ , so we drop "  $J$  " from the notation.

## Bruhat interval $f$ -vectors

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Fix  $w \in W$ , and let  $m := \lfloor (\ell(w) - 1)/2 \rfloor$ . Let

$$P_{e,w}(q) = 1 + \beta_1 q + \cdots + \beta_m q^m$$

be the Kazhdan-Lusztig polynomial of the interval  $[e, w]$ .

Known:

\*  $\beta_i \geq 0$  if  $W$  is crystallographic,

\*  $P_{e,w}(q) = 1 \iff X_w$  is rationally smooth

\*  $X_w$  is rationally smooth  $\iff f_i^w = f_{\ell(w)-i}^w, \forall i$   
(Carrell-Peterson '94)

\* For simply-laced  $W$ : smooth  $\iff$  rationally smooth

## Bruhat interval $f$ -vectors

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**Theorem 3.** *Suppose that  $W$  is crystallographic,  $w \in W$  and  $1 \leq k \leq m$ . Then the following conditions are equivalent:*

(a)  $f_i^w = f_{\ell(w)-i}^w$ , for  $i = 1, \dots, k$ ,

(b)  $\beta_i = 0$ , for  $i = 1, \dots, k$ .

Remark: The equivalence of (a) and (b) in the case  $k = m$  gives the Carrell-Peterson criterion for rational smoothness of the Schubert variety  $X_w$ .

## Bruhat interval $f$ -vectors

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**Theorem 4.** *Suppose that  $W$  is crystallographic,  $w \in W$  and  $1 \leq k \leq m$ . Then the following conditions are equivalent:*

(a)  $f_i^w = f_{\ell(w)-i}^w$ , for  $i = 1, \dots, k$ ,

(b)  $\beta_i = 0$ , for  $i = 1, \dots, k$ .

Furthermore, if  $k < m$  then (a) and (b) imply

(c)  $\beta_{k+1} = f_{\ell(w)-k-1}^w - f_{k+1}^w$ .

## Bruhat interval $f$ -vectors

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Idea of proof: Based on

\* Monotonicity theorem for K-L polynomials  
(extending Braden-MacPherson '01).

\* Polynomial  $F_w(q) = \sum_{x \leq w} q^{\ell(x)} P_{x,w}(q)$  is palindromic  
(Kazhdan-Lusztig '79).

## Bruhat interval $f$ -vectors

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More can be said about the increasing inequalities

$$f_0 \leq f_1 \leq \cdots \leq f_{\lceil \ell(w)/2 \rceil},$$

namely, **the sequence cannot grow too fast.**

## Bruhat interval $f$ -vectors

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Recall:

For  $n, k \geq 1$  there is a unique expansion

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i},$$

with  $a_k > a_{k-1} > \cdots > a_i \geq i \geq 1$ . Let

$$\partial^k(n) = \binom{a_k - 1}{k - 1} + \binom{a_{k-1} - 1}{k - 2} + \cdots + \binom{a_i - 1}{i - 1},$$

$$\partial^k(0) = 0.$$

## Bruhat interval $f$ -vectors

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### **Theorem** (Macaulay-Stanley)

For an integer sequence  $(1, m_1, m_2, \dots)$  the following are equivalent (and this defines an  $M$ -sequence):

- (1)  $\partial^k(m_k) \leq m_{k-1}$ , for all  $k \geq 1$ ,
- (2) some order ideal of monomials contains exactly  $m_k$  monomials of degree  $k$ ,
- (3)  $\dim(A_k) = m_k$  for some graded commutative algebra  $A = \bigoplus_{k \geq 0} A_k$  which is generated by  $A_1$ .

## Bruhat interval $f$ -vectors

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More can be said about the increasing inequalities

$$f_0 \leq f_1 \leq \cdots \leq f_{\lceil \ell(w)/2 \rceil},$$

namely, the sequence cannot grow too fast.

**Theorem 5.** *In the case of finite Weyl groups every  $f$ -vector  $f^w = \{f_0, f_1, \dots, f_{\ell(w)}\}$  is an  $M$ -sequence.*

## Bruhat interval $f$ -vectors

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Idea of proof of Thm: Based on

\* The  $f$ -vector  $f^w = \{f_0, f_1, \dots, f_{\ell(w)}\}$  is coeff-sequence of Poincaré polynomial of  $H^*(X_w)$ , the cohomology algebra of the Schubert variety  $X_w$  (over  $\mathbb{C}$ ).

\* So, we only need that  $H^*(X_w)$  is generated in degree one (dim = 2).

\* For  $w = w_0$  this is classical:  $H^*(X_{w_0}) \cong$  coinvariant algebra of  $W$ .

\* For  $w \neq w_0$  there is algebra surjection  $H^*(X_{w_0}) \rightarrow H^*(X_w)$

## Bruhat interval $f$ -vectors

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Remarks:

\*\*  $M$ -sequence property fails for the affine group  $\tilde{C}_2$ :

$$\sum q^{\ell(w)} = 1 + 3q + 5q^2 + 8q^3 + \dots$$

Consequence:  $H^*(X_w)$  not necessarily generated in degree one for affine Schubert varieties  $X_w$ .

\*\*  $M$ -sequence property fails for general intervals in finite  $B_4$ :

$$\sum_{w \leq x \leq w_0} q^{\ell(x) - \ell(w)} = 1 + 4q + 11q^2 + \dots$$

for certain  $w \in B_4$ .

## Bruhat interval $f$ -vectors

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The increasing inequalities  $f_0 \leq f_1 \leq \cdots \leq f_{\lceil \ell(w)/2 \rceil}$  have decreasing counterparts at the upper end of the Bruhat interval — but the information we are able to give about this is much weaker.

## Bruhat interval $f$ -vectors

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**Theorem 6.** *For all  $k \geq 1$  there exists a number  $N_k$ , such that for every finite Coxeter group  $(W, S)$  and every  $w \in W$  such that  $\ell(w) \geq N_k$  we have that*

$$f_{\ell(w)-k}^w \geq f_{\ell(w)-k+1}^w \geq \cdots \geq f_{\ell(w)}^w.$$

## Bruhat interval $f$ -vectors

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### Questions

**1.** Do  $f^w$ -vectors satisfy more inequalities?

(Noticed by D. Stanton: unimodality fails on some intervals in Young's lattice. Unimodality might be true for full intervals, i.e.  $J = \emptyset$ .)

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2. Are the theorems true for general (non-crystallographic) Coxeter groups?

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2. Are the theorems true for general (non-crystallographic) Coxeter groups?
3. Does there exist some  $\alpha < 1$  such that

$$f_{\lfloor \alpha \cdot \ell(w) \rfloor}^w \geq f_{\lfloor \alpha \cdot \ell(w) \rfloor + 1}^w \geq \cdots \geq f_{\ell(w)}^w.$$

Will  $\alpha = \frac{3}{4}$  do?

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2. Are Theorems 1–2 true for general (non-crystallographic) Coxeter groups?
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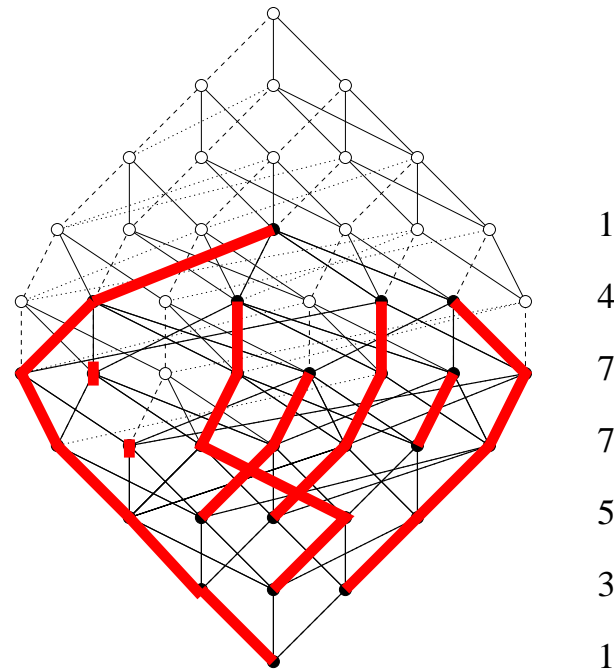
$$f_{\lfloor \alpha \cdot \ell(w) \rfloor}^w \geq f_{\lfloor \alpha \cdot \ell(w) \rfloor + 1}^w \geq \cdots \geq f_{\ell(w)}^w.$$

Will  $\alpha = \frac{3}{4}$  do?

4. What can be said about the shape of general intervals  $[u, w]^J$ ?  
(I.e., for  $u \neq e$ )

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Def: An **upper chain decomposition** is a partition of  $[e, w]^J$  into pairwise disjoint saturated chains

$$u_i < u_{i+1} < \cdots < u_k$$

such that  $\ell(j) = j$  for all  $j = i, \dots, k$ , and  $k \geq \ell(w) - i$ .

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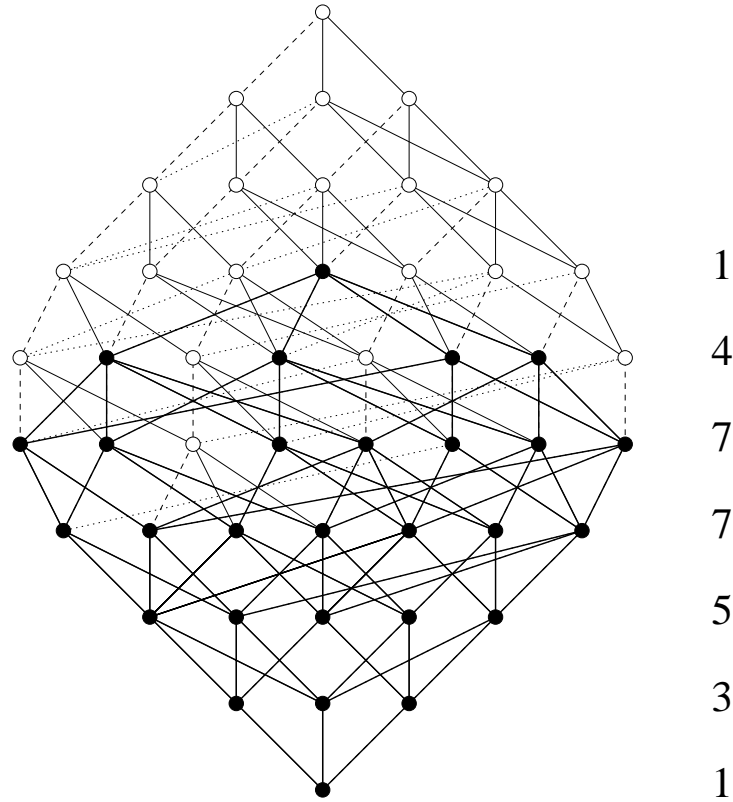
5. Do the intervals  $[e, w]^J$  admit upper chain decompositions?

### Note:

1. This would imply Thms 6 and 7.
2. Specializes to symmetric chain decomposition, if  $f^w$ -vector is symmetric.
3. Symmetric chain decomposition question still open for intervals  $[\emptyset, \lambda]$  in Young's lattice,  $\lambda$  of rectangular shape.

# Shape of Bruhat intervals

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THE END