Dyck partitions, quasi-minuscule quotiens and Kazhdan-Lusztig polynomials

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partly based on a joint work with Francesco Brenti and Mario Marietti
1. Background

1.1 The symmetric group

\[ P = \{1, 2, 3, \ldots\}, \quad [n] = \{1, 2, \ldots, n\} \quad (n \in P), \]

*Symmetric group*: \( S_n = \{v : [n] \to [n] \text{ bijection}\} \).

We denote \( v \in S_n \) by the word \( v(1)v(2)\ldots v(n) \) and by its *diagram*.

**Example.** \( v = 61523748 \in S_8 \) has diagram
$S_n$ is a Coxeter group, with generators the simple transpositions:

$$S = \{(1, 2), (2, 3), \ldots, (n - 1, n)\}.$$ 

When we refer to these generators, the transposition $(i, i+1)$ is simply denoted by $i$. With this convention, the set of generators of $S_n$ is

$$S = [n-1].$$

Let $J \subseteq [n-1]$. The quotient of $S_n$ by $J$ is

$$(S_n)^J = \{v \in S_n : v^{-1}(r) < v^{-1}(r+1) \text{ for all } r \in J\}.$$
The maximal quotients of $S_n$ are obtained by taking

$$J = [n-1] \setminus \{i\} \quad (i \in [n-1]).$$

The quasi-minuscule quotients of $S_n$ are obtained by taking

$$J = [n-1] \setminus \{i-1, i\} \quad (2 \leq i \leq n-1)$$
or

$$J = [n-1] \setminus \{1, n-1\}.$$

In this talk we study the quasi-minuscule quotients of $S_n$. 
1.2 Partitions and lattice paths

We identify a partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \subseteq (n^m) \) with its diagram:

\[ \{(i, j) \in P^2 : 1 \leq i \leq k \text{ and } 1 \leq j \leq \lambda_i \} . \]

Example. \( \lambda = (3, 2, 2, 1, 1) \subseteq (4^5) . \)
Given a partition $\lambda \subseteq (n^m)$, the path associated with $\lambda$ is the lattice path from $(0, m)$ to $(n + m, n)$, with steps $(1, 1)$ (up steps) and $(1, -1)$ (down steps) which is the upper border of the diagram of $\lambda$:

$$\text{path}(\lambda) = x_1x_2\ldots x_{n+m}, \quad x_k \in \{U, D\},$$

Note that $\text{path}(\lambda)$ has exactly $n$ U’s and $m$ D’s.

**Example.** $\lambda = (3, 2, 2, 1, 1) \subseteq (4^5)$. 

\[
\begin{align*}
\lambda &= \begin{array}{c}
\text{Diagram of } \lambda \\
\end{array} \quad \text{path}(\lambda) &= \begin{array}{c}
\text{Lattice path} \\
= \text{UDDUDDUDUDU}
\end{array}
\end{align*}
\]
Let $\lambda, \mu \in \mathcal{P}$, with $\mu \subseteq \lambda$. Then we call $\lambda \setminus \mu$ a **skew partition**.

A skew partition is a **border strip** (also called a **ribbon**) if it contains no $2 \times 2$ square of cells. For brevity, we call a connected (by which we mean “rookwise connected”) border strip a **cbs**.

The **outer border strip** $\theta$ of $\lambda \setminus \mu$ is the set of cells of $\lambda \setminus \mu$ such that the cell directly above it is not in $\lambda \setminus \mu$. 

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**Diagram:**

- **Skew partition** $\lambda \setminus \mu$
- **Outer border strip** of $\lambda \setminus \mu$
A cbs $\theta \subset P^2$ is called a *Dyck cbs* if it is a “Dyck path”, which means that no cell of $\theta$ has level strictly less than that of either the leftmost or the rightmost of its cells. (In particular, in a Dyck cbs the leftmost and rightmost cells have the same level.)

Dyck  non-Dyck  non-Dyck
Let $\lambda \setminus \mu \subset \mathbb{P}^2$ be a skew partition.

Recall that $\lambda \setminus \mu$ is defined to be \textit{Dyck} in the following inductive way:

(1) the empty partition is \textit{Dyck},

(2) if $\lambda \setminus \mu$ is connected, then $\lambda \setminus \mu$ is \textit{Dyck} if and only if
   
   (a) its outer border strip $\theta$ is a Dyck cbs,
   (b) $(\lambda \setminus \mu) \setminus \theta$ is Dyck,

(3) if $\lambda \setminus \mu$ is not connected, then $\lambda \setminus \mu$ is \textit{Dyck} if and only if all of its
    connected components are Dyck.
Let $\lambda \setminus \mu \subset P^2$ be a skew partition (not necessarily Dyck).

The *depth* of $\lambda \setminus \mu$ is defined inductively by

$$dp(\lambda \setminus \mu) = \begin{cases} 0, & \text{if } \lambda = \mu, \\ c(\theta) + dp((\lambda \setminus \mu) \setminus \theta), & \text{otherwise}, \end{cases}$$

where $\theta$ is the outer border strip of $\lambda \setminus \mu$ and

$c(\theta) = \# \text{ connected components of } \theta$. 
Example. Dyck skew partition:

\[ \lambda \setminus \mu \]

\[ dp(\lambda \setminus \mu) = 8. \]
2. Parabolic Kazhdan-Lusztig polynomials

**Theorem.** (Deodhar, 1987) Let \((W, S)\) be any Coxeter system and let \(J \subseteq S\). Then, there is a unique family of polynomials

\[
\{P^J_{u,v}(q)\}_{u,v \in W^J} \subseteq \mathbb{Z}[q]
\]

such that, for all \(u, v \in W^J\), with \(u \leq v\), and fixed \(s \in D(v)\), one has

\[
P^J_{u,v}(q) = \tilde{P}(q) - \sum_{\{u \leq w \leq vs : ws < w\}} \mu(w, vs) q^{\frac{\ell(w, v)}{2}} P^J_{u,w}(q),
\]

where

\[
\begin{align*}
\tilde{P}(q) &= \begin{cases} 
P^J_{us,vs}(q) + qP^J_{u,vs}(q), & \text{if } us < u, \\
qP^J_{us,vs}(q) + P^J_{u,vs}(q), & \text{if } u < us \in W^J, \\
0, & \text{if } u < us \notin W^J.
\end{cases}
\end{align*}
\]

and

\[
\mu(u, v) = \left[ q^{\frac{\ell(u,v)-1}{2}} \right](P^J_{u,v}).
\]
The $P_{u,v}^J(q)$ are the \textit{parabolic Kazhdan-Lusztig polynomials} of $W^J$.

For $J = \emptyset$, we get the (\textit{ordinary}) \textit{Kazhdan-Lusztig polynomials} of $W$:

$$P_{u,v}(q) = P_{u,v}^\emptyset(q).$$

Conversely, parabolic Kazhdan-Lusztig polynomials can be expressed in terms their ordinary counterparts.

\textbf{Proposition.} Let $J \subseteq S$, and $u, v \in W^J$. Then

$$P_{u,v}^J(q) = \sum_{w \in W_J} (-1)^{\ell(w)} P_{wu,v}(q).$$
3. Quasi-minuscule quotiens

We will now give a combinatorial description of the quasi-minuscule quotients in $S_n$. We start with the following simple observation.

A permutation $v \in S_n$ belongs to $S_n^{(n-1)\setminus\{i-1,i\}}$ if and only if

$$v^{-1}(1) < \cdots < v^{-1}(i-1) \quad \text{and} \quad v^{-1}(i) < \cdots < v^{-1}(n).$$

Example. $v = 61523748 \in S_8^{[7]\setminus\{4,5\}}$
Let $\lambda \subseteq (n^m)$ be a partition and let

$$\text{path}(\lambda) = x_1 \ldots x_{n+m}, \quad x_k \in \{U, D\}.$$

We say that an index $k \in [n + m - 1]$ is a

$$\begin{cases} 
\text{valley of } \lambda, & \text{if } (x_k, x_{k+1}) = (D, U), \\
\text{peak of } \lambda, & \text{if } (x_k, x_{k+1}) = (U, D).
\end{cases}$$

**Definition.** A **rooted partition** is a pair $(\lambda, r)$, where $\lambda$ is a partition with at least one valley and $r$ is one of its valleys.

We think of a rooted partition as a lattice path with a ball in one of its valleys. If $\lambda \subseteq (n^m)$ and $\text{path}(\lambda) = x_1 \ldots, x_{n+m}$, then

$$\text{path}(\lambda, r) = x_1 \ldots x_r \bullet x_{r+1} \ldots x_{n+m}$$
Let $v \in S_n^{[n-1]\{i-1,i\}}$. The partition associated with $v$, denoted by $\Lambda(v)$, is the non-increasing rearrangement of the inversion table of $v$.

**Example.** $v = 61523748 \in S_8^{[7]\{4,5\}}$. Then

$$\Lambda(v) = (3, 2, 2, 1, 1) = \text{Diagram}$$

**Remark.** $\Lambda(v) \subseteq ((n - i + 1)^i)$ and $v^{-1}(i)$ is a valley of $\Lambda(v)$. 
The \textit{rooted partition} associated with $v$ is

$$\Lambda^\bullet(v) = (\Lambda(v), v^{-1}(i)).$$

\textbf{Example.} $v = 61523748 \in S_8^{[7]\{4,5\}}$. Then

$$\Lambda^\bullet(v) = ((3, 2, 2, 1, 1), 3) = \phantom{x}$$

\textbf{Proposition.} The map $v \mapsto \Lambda^\bullet(v)$ is a bijection

$$S_n^{[n-1]\{i-1,i\}} \longleftrightarrow \{\text{rooted partitions } \subseteq ((n - i + 1)^i)\}.$$

Furthermore, $\ell(v) = |\Lambda(v)|$. 
4. \(-\text{Dyck partitions}\)

This is the main new combinatorial concept arising from this work.

If \((\lambda, r)\) and \((\mu, t)\) are two rooted partitions such that \(\mu \subseteq \lambda\), then we call \((\lambda, r) \setminus (\mu, t)\) a skew rooted partition.
Definition. A skew rooted partition \((\lambda, r) \setminus (\mu, t)\) is \(-\text{Dyck}\) if

1. there are no peaks of \(\lambda\) strictly between the two roots,
2. at least one of \(\lambda \setminus \mu\) and \(\lambda \setminus \mu^t\) is Dyck.

Let \((\lambda, r) \setminus (\mu, t)\) be \(-\text{Dyck}\). The depth of \((\lambda, r) \setminus (\mu, t)\) is

\[
\text{dp}((\lambda, r) \setminus (\mu, t)) = \begin{cases} 
\text{dp}(\lambda \setminus \mu), & \text{if } \lambda \setminus \mu \text{ is Dyck,} \\
\text{dp}(\lambda \setminus \mu^t) + 1, & \text{if } \lambda \setminus \mu^t \text{ is Dyck,}
\end{cases}
\]

Proposition. Let \(\lambda \setminus \mu\) be skew partition and let \(t\) be a valley of \(\mu\). Suppose that at least one of \(\lambda \setminus \mu\) and \(\lambda \setminus \mu^t\) is Dyck. Then \(\lambda \setminus \mu\) and \(\lambda \setminus \mu^t\) are both Dyck if and only if \(t\) is a peak of \(\lambda\). In this case,

\[
\text{dp}(\lambda \setminus \mu) = \text{dp}(\lambda \setminus \mu^t) + 1.
\]
Four \(\bullet\)-Dyck skew rooted partitions:

For all of them,

\[ |\lambda \setminus \mu| = 98 \quad \text{and} \quad dp((\lambda, r) \setminus (\mu, t)) = 8. \]
5. Main result

Theorem. (Brenti, I., Marietti, 2008) Let \( u, v \in S^{[n-1]\{i-1,i\}} \), with \( \Lambda \circ \circ(v) = (\lambda, r) \) and \( \Lambda \circ \circ(u) = (\mu, t) \).

Then

\[
P_{u,v}(q) = \begin{cases} 
q^{\frac{|\lambda \backslash \mu| - dp((\lambda, r) \backslash (\mu, t))}{2}}, & \text{if } (\lambda, r) \backslash (\mu, t) \text{ is } \bullet \text{-Dyck}, \\
0, & \text{otherwise}.
\end{cases}
\]

Example. If \((\lambda, r) \backslash (\mu, t)\) is one of the previous four, then

\[
P_{u,v}(q) = \frac{98 - 8}{2} = q^{45}.
\]
Our main result implies the analog result for *maximal quotients*, found by Brenti in [*Pacific Journal of Mathematics* 207 (2002), 257–286].

**Corollary.** (Brenti, 2002) Let $u, v \in S^{[n-1]\{i\}}$, with

$$
\Lambda(v) = \lambda \quad \text{and} \quad \Lambda(u) = \mu.
$$

Then

$$
P^J_{u,v}(q) = \begin{cases} 
\frac{|\lambda \setminus \mu| - dp(\lambda \setminus \mu)}{2}, & \text{if } \lambda \setminus \mu \text{ is Dyck,} \\
0, & \text{otherwise.}
\end{cases}
$$
6. Enumerative results

6.1 Enumeration of Dyck partitions

Let $\lambda \subseteq (n^m)$ be a partition and consider the associated path

$$\text{path}(\lambda) = x_1 \ldots x_{n+m}, \quad x_k \in \{U, D\}.$$  

We make the substitution $U \leftrightarrow ( \quad D \leftrightarrow )$.

We define the *matching set* and the *matching number* of $\lambda$ by

$$M(\lambda) = \{k \in [n+m] : \text{parenthesis } x_k \text{ is matched}\},$$

$$\text{mtc}(\lambda) = \frac{|M(\lambda)|}{2} = \# \text{ pairs of matched parentheses in } \text{path}(\lambda).$$
Example. $\lambda = (4, 3, 3, 2, 2, 2) \subseteq (5^6)$. 

$\lambda = \begin{array}{c}
\begin{array}{c}
\lambda \\
\end{array}
\end{array}
$

\text{path}(\lambda) = \left( \begin{array}{c}
\begin{array}{c}
( )
\end{array}
\end{array} \right)

\text{M}(\lambda) = \{1, 2, 3, 4, 6, 7, 10, 11\}

\text{mtc}(\lambda) = 4
In 2002, Brenti enumerated the partitions $\mu$ contained in a given partition $\lambda$ such that $\lambda \setminus \mu$ is Dyck and found a $q$-analog formula. This is a reformulation of his result.

**Theorem.** (Brenti, 2002) Let $\lambda \subseteq (n^m)$. Then

$$|\{\mu \subseteq \lambda : \lambda \setminus \mu \text{ is Dyck}\}| = 2^{\text{mtc}(\lambda)}.$$

More generally, the following $q$-analog holds:

$$\sum_{\substack{\mu \subseteq \lambda \\ \lambda \setminus \mu \text{ is Dyck}}} q^{dp(\lambda \setminus \mu)} = (q + 1)^{\text{mtc}(\lambda)}.$$
Recently, *all* the Dyck skew partition contained in a given rectangle have been enumerated and a $q$-analog has been found.

**Theorem.** (I., August 2008)

$$|\{\lambda \setminus \mu \subseteq \binom{n^m}{m} \text{ Dyck}\}| = \min\{n,m\} \sum_{k=0}^{\min\{n,m\}} \frac{n + m - 2k + 1}{n + m - k + 1} \binom{n + m}{k} 2^k.$$

More generally, the following $q$-analog holds:

$$\sum_{\substack{\lambda \setminus \mu \subseteq \binom{n^m}{m} \\ \lambda \setminus \mu \text{ is Dyck}}} q^{dp(\lambda \setminus \mu)} = \min\{n,m\} \sum_{k=0}^{\min\{n,m\}} \frac{n + m - 2k + 1}{n + m - k + 1} \binom{n + m}{k} (q + 1)^k.$$
6.2 Connection with paths on regular trees

For any integer \( d \geq 2 \), we denote by \( T_d \) the \( d \)-regular tree, that is the (infinite) tree where all the vertices have degree \( d \).
Given two vertices $x$ and $y$ in a graph $G$, we denote by $\text{Paths}_{G,\ell}(x, y)$ the set of all paths in $G$ of length $\ell$ from $x$ to $y$.

**Theorem.** (I., August 2008) Let $n, m \in \mathbb{P}$.

Let $x, y$ be two vertices of $T_3$ at distance $|n - m|$. Then

$$|\{\lambda \setminus \mu \subseteq (n^m) : \lambda \setminus \mu \text{ is Dyck}\}| = |\text{Paths}_{T_3,n+m}(x, y)|.$$

More generally, we have the following $q$-analog.

Let $q \in \mathbb{Z}_{\geq 0}$ and $x, y$ be two vertices of $T_{q+2}$ at distance $|n - m|$. Then

$$\sum_{\substack{\lambda \setminus \mu \subseteq (n^m) \\ \lambda \setminus \mu \text{ is Dyck}}} q^{\text{dp}(\lambda \setminus \mu)} = |\text{Paths}_{T_{q+2},n+m}(x, y)|.$$