Rook placements in Young diagrams

Matthieu Josuat-Vergès

Université Paris-sud

Séminaire Lotharingien de Combinatoire ’08

Partially supported by a Junior Research Fellowship from the Erwin Schrödinger Institute, Vienna
Introduction

Context: The PASEP, Partially Assymmetric Self-Exclusion Process, is a 1D-model of particles in $n$ sites, hopping from each site to its neighbours.

This model is solved by a matrix ansatz (cf. Derrida & al). If:

$$DE - qED = D + E,$$

we can write $(D + E)^n$ in normal form:

$$(D + E)^n = \sum_{i,j \geq 0} c_{ij} E^i D^j,$$

Then the partition function is $Z = \langle (D + E)^n \rangle = \sum c_{ij}$. 
Introduction

If we define:

\[ \hat{D} = \frac{q^{-1}}{q} D + \frac{1}{q}, \quad \hat{E} = \frac{q^{-1}}{q} E + \frac{1}{q}. \]

Then we have inversion formulas:

\[ (1 - q)^n (D + E)^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} (-1)^k q^k (\hat{D} + \hat{E})^k, \quad \text{and} \]

\[ q^n (\hat{D} + \hat{E})^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} (-1)^k (1 - q)^k (D + E)^k. \]

And the commutation relation is (cf. Uchiyama-Sasamoto, Evans):

\[ \hat{D} \hat{E} - q \hat{E} \hat{D} = \frac{1-q}{q^2} \]
The rewriting of \((D + E)^n\) in normal form is combinatorially described by alternative tableaux (cf. Viennot).

This explains the link between the PASEP and the combinatorics of permutations (cf. Corteel-Williams).

The rewriting of \((\hat{D} + \hat{E})^n\) in normal form is combinatorially described by rook placements in Young diagrams.
Rewriting rules for $\hat{D}$ and $\hat{E}$

**Definition**
A rook placement is a filling of the cells of a Young diagram with $\circ$, with at most one $\circ$ per line (resp. column).

We distinguish by a $\times$ the cells that are not directly below or to the left of a $\circ$ (cf. Garsia-Remmel).

Each $\circ$ has a weight $p$.
Each $\times$ has a weight $q$.

**Theorem**
Suppose more generally that $\hat{D}\hat{E} - q\hat{E}\hat{D} = p$, then $< (\hat{D} + \hat{E})^n >$ is the sum of weight of rook placements of half-perimeter $n$. 
Rewriting rules for $\hat{D}$ and $\hat{E}$

Definition
A rook placement is a filling of the cells of a Young diagram with $\circ$, with at most one $\circ$ per line (resp. column).

We distinguish by a $\times$ the cells that are not directly below or to the left of a $\circ$ (cf. Garsia-Remmel).

Each $\circ$ has a weight $p$.
Each $\times$ has a weight $q$.

Theorem
Suppose more generally that $\hat{D}\hat{E} - q\hat{E}\hat{D} = p$, then $<(\hat{D} + \hat{E})^n>$ is the sum of weight of rook placements of half-perimeter $n$. 
Rewriting rules for $\hat{D}$ and $\hat{E}$

Since $(\hat{D} + \hat{E})^n$ expands into the sum of all words of length $n$ in $\hat{D}$ and $\hat{E}$, it is consequence of:

**Proposition**

*Let $w$ be a word in $\hat{D}$ and $\hat{E}$. Then $\langle w \rangle$ is the sum of weights of rook placements of shape $\lambda(w)$.*

\[ w = \hat{D}\hat{E}\hat{E}\hat{D}... \quad \lambda(w) = \]

\[ \hat{D} \quad \hat{E} \quad \hat{E} \quad \hat{D} \]

\[ \vdots \]

\[ \hat{E} \quad \hat{D} \quad \vdots \]
Rewriting rules: Sketch of proof

Operator point of view:

\[ \hat{D}\hat{E}\hat{D}(\hat{D}\hat{E})\hat{D}\hat{E}\hat{E} = \hat{D}\hat{E}\hat{D}(q\hat{E}\hat{D})\hat{D}\hat{E}\hat{E} + \hat{D}\hat{E}\hat{D}(p)\hat{D}\hat{E}\hat{E} \]

Combinatorial point of view:
Rewriting rules: Sketch of proof

Operator point of view:

\[ \hat{D} \hat{E} \hat{D}(\hat{D} \hat{E}) \hat{D} \hat{E} \hat{E} = \hat{D} \hat{E} \hat{D}(q \hat{E} \hat{D}) \hat{D} \hat{E} \hat{E} + \hat{D} \hat{E} \hat{D}(p) \hat{D} \hat{E} \hat{E} \]

Combinatorial point of view:

\[ \begin{array}{c c c c c c c c c c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
\end{array} =
\begin{array}{c c c c c c c c c c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
\end{array} +
\begin{array}{c c c c c c c c c c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
\end{array} = q
\begin{array}{c c c c c c c c c c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
\end{array} + p
\begin{array}{c c c c c c c c c c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
\end{array} \]
Rewriting rules: Sketch of proof

Operator point of view:

\[ \hat{D} \hat{E} \hat{D}(\hat{D} \hat{E}) \hat{D} \hat{E} \hat{E} = \hat{D} \hat{E} \hat{D}(q \hat{E} \hat{D}) \hat{D} \hat{E} \hat{E} + \hat{D} \hat{E} \hat{D}(p) \hat{D} \hat{E} \hat{E} \]

Combinatorial point of view:
Rewriting rules: Sketch of proof

Operator point of view:

\[
\hat{D}\hat{E}\hat{D}(\hat{D}\hat{E})\hat{D}\hat{E}\hat{E} = \hat{D}\hat{E}\hat{D}(q\hat{E}\hat{D})\hat{D}\hat{E}\hat{E} + \hat{D}\hat{E}\hat{D}(p)\hat{D}\hat{E}\hat{E}
\]

Combinatorial point of view:

\[
\begin{align*}
\begin{array}{ccc}
\infty & \infty & \infty \\
\infty & \infty & \infty \\
\infty & \infty & \infty
\end{array} & = & \begin{array}{ccc}
\times & \infty & \infty \\
\infty & \infty & \infty \\
\infty & \infty & \infty
\end{array} & + & \begin{array}{ccc}
\odot & \infty & \infty \\
\infty & \infty & \infty \\
\infty & \infty & \infty
\end{array} & = & q & \begin{array}{ccc}
\infty & \infty & \infty \\
\infty & \infty & \infty \\
\infty & \infty & \infty
\end{array} & + & p & \begin{array}{ccc}
\infty & \infty & \infty \\
\infty & \infty & \infty \\
\infty & \infty & \infty
\end{array}
\end{align*}
\]
Rewriting rules: Sketch of proof

Operator point of view:

\[ \hat{D}\hat{E}\hat{D}(\hat{D}\hat{E})\hat{D}\hat{E}\hat{E} = \hat{D}\hat{E}\hat{D}(q\hat{E}\hat{D})\hat{D}\hat{E}\hat{E} + \hat{D}\hat{E}\hat{D}(p)\hat{D}\hat{E}\hat{E} \]

Combinatorial point of view:

These are identical recurrence relations.
Enumeration of rook placements: Examples

Let $T_{j,k,n}$ be the sum of weights of rook placements of half-perimeter $n$, with $k$ lines and $j$ lines without rook. We have:

**Proposition**

$$T_{k,k,n} = \begin{bmatrix} n \\ k \end{bmatrix}_q.$$  

**Proposition**

*When $p = 1$ and $q = 0$, $T_{0,k,n}$ is the number of (left factor of) Dyck paths of $n$ steps ending at height $n - 2k$. Hence:*

$$T_{0,k,n} = \binom{n}{k} - \binom{n}{k-1}.$$
This is a consequence of:

**Proposition**

For any $\lambda$ there is at most one rook placement of shape $\lambda$ with no $\times$ and one rook per line, with equality in the case where the NE boundary of $\lambda$ is a Dyck path.

If the path goes below the diagonal, it is impossible to place one rook per line.

If it is a Dyck path there is only one way to place the rooks:
This is a consequence of:

**Proposition**

*For any* \( \lambda \) *there is at most one rook placement of shape* \( \lambda \) *with no \( \times \) and one rook per line, with equality in the case where the NE boundary of* \( \lambda \) *is a Dyck path.*

If the path goes below the diagonal, it is impossible to place one rook per line.

If it is a Dyck path there is only one way to place the rooks:

- There is one in each corner,
This is a consequence of:

**Proposition**

For any $\lambda$ there is at most one rook placement of shape $\lambda$ with no $\times$ and one rook per line, with equality in the case where the NE boundary of $\lambda$ is a Dyck path.

If the path goes below the diagonal, it is impossible to place one rook per line.

If it is a Dyck path there is only one way to place the rooks:

- There is one in each corner,
- One in each corner of the remaining shape, and so on.
This is a consequence of:

**Proposition**

*For any $\lambda$ there is at most one rook placement of shape $\lambda$ with no $\times$ and one rook per line, with equality in the case where the NE boundary of $\lambda$ is a Dyck path.*

If the path goes below the diagonal, it is impossible to place one rook per line.

If it is a Dyck path there is only one way to place the rooks:

- There is one in each corner,
- One in each corner of the remaining shape, and so on.
This is a consequence of:

**Proposition**

*For any $\lambda$ there is at most one rook placement of shape $\lambda$ with no $\times$ and one rook per line, with equality in the case where the NE boundary of $\lambda$ is a Dyck path.*

If the path goes below the diagonal, it is impossible to place one rook per line.

If it is a Dyck path there is only one way to place the rooks:

- There is one in each corner,
- One in each corner of the remaining shape, and so on.
This is a consequence of:

**Proposition**

*For any* \( \lambda \) *there is at most one rook placement of shape* \( \lambda \) *with no* \( \times \) *and one rook per line, with equality in the case where the NE boundary of* \( \lambda \) *is a Dyck path.*

If the path goes below the diagonal, it is impossible to place one rook per line.

If it is a Dyck path there is only one way to place the rooks:

- There is one in each corner,
- One in each corner of the remaining shape, and so on.
This is a consequence of:

**Proposition**

For any $\lambda$ there is at most one rook placement of shape $\lambda$ with no $\times$ and one rook per line, with equality in the case where the NE boundary of $\lambda$ is a Dyck path.

If the path goes below the diagonal, it is impossible to place one rook per line.

If it is a Dyck path there is only one way to place the rooks:

- There is one in each corner,
- One in each corner of the remaining shape, and so on.
Enumeration: The bijective part

For each rook placement we define an involution (cf. Kerov):
For each rook placement we define an involution (cf. Kerov):
Enumeration: The bijective part

For each rook placement we define an involution (cf. Kerov):

\[ I = (\cdots) \]

\[ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \]
Enumeration: The bijective part

For each rook placement we define an involution (cf. Kerov):

\[ I = \begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\circ & \times & \times & \times & \circ & \times & \times & \times & \circ & \circ & \circ
\end{array} \]
Enumeration: The bijective part

For each rook placement we define an involution (cf. Kerov):

\[ I = \begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\end{array} \]
Enumeration: The bijective part

For each rook placement we define an involution (cf. Kerov):
Enumeration: The bijective part

For each rook placement we define an involution (cf. Kerov):
Enumeration: The bijective part

For each rook placement we define an involution (cf. Kerov):

\[ I = \begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\end{array} \]
Enumeration: The bijective part

For each rook placement we define an involution (cf. Kerov):

This is not a bijection because fixed points may correspond either to empty lines or empty columns.
Enumeration: The bijective part

For each rook placement we define an involution (cf. Kerov):

\[
I = \begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
\odot & \times & \times & 5 & \odot & \times & \times & 6 & \odot & \times & \times \\
\times & \times & \times & \times & \odot & \times & \times & \times & \times & \times & \times \\
\end{array}
\]

This is not a bijection because fixed points may correspond either to empty lines or empty columns.

To keep track of empty lines or columns, we also define:

\[
\lambda = \begin{array}{c}
\end{array}
\]
We have a bijection between rook placements of half-perimeter $n$, and couples $(I, \lambda)$ where:

- $I$ is an involution on $\{1, \ldots, n\}$,
- $\lambda$ is a Young diagram of half-perimeter $\#\text{Fix}(I)$.

**Proposition**

*With respect to this decomposition $R \mapsto (I, \lambda)$, the parameter ”number of crosses” is additive:*

$$\#\text{crosses}(R) = |\lambda| + \mu(I)$$
We have a bijection between rook placements of half-perimeter \( n \), and couples \((I, \lambda)\) where:

- \( I \) is an involution on \( \{1, \ldots, n\} \),
- \( \lambda \) is a Young diagram of half-perimeter \( \#\text{Fix}(I) \).

**Proposition**

*With respect to this decomposition \( R \mapsto (I, \lambda) \), the parameter ”number of crosses” is additive:*

\[
\#\text{crosses}(R) = |\lambda| + \mu(I)
\]

It is possible to describe \( \mu \) precisely:

\[
\mu(I) = \#\text{crossings}(I) + \sum_{x \in \text{Fix}(I)} \text{height}(x)
\]
\[ R = \begin{array}{ccccccc}
\times & \times & \times & \circ & \times & \circ & \times \\
\circ & \times & \times & \times & \circ & \times & \\
\circ & \circ & \times & \times & \circ & \times & \\
\times & \times & \times & \times & \times & \times & \\
\end{array} \]

- $|\lambda|$ counts the number of $\times$ with no rook in the same line, no rook in the same column.
- $\#\text{crossings}$ counts the number of $\times$ with one rook in the same line, one rook in the same column.
- $\sum \text{height}(x)$ counts all remaining $\times$.

$|\lambda| = 3$, $\#\text{crossings} = 2$, $\sum \text{height}(x) = 1 + 1 + 2 + 0 = 4$
| $\lambda$ | counts the number of $\times$ with no rook in the same line, no rook in the same column. |
| $\#\text{crossings}$ | counts the number of $\times$ with one rook in the same line, one rook in the same column. |
| $\sum \text{height}(x)$ | counts all remaining $\times$. |

$|\lambda| = 3, \quad \#\text{crossings} = 2,$

$\sum \text{height}(x) = 1 + 1 + 2 + 0 = 4$
\[ R = \]

\[
\lambda = \]

\[
= \]

- \(|\lambda|\) counts the number of \(\times\) with no rook in the same line, no rook in the same column.

- \(#\text{crossings}\) counts the number of \(\times\) with one rook in the same line, one rook in the same column.

- \(\sum \text{height}(x)\) counts all remaining \(\times\).

\(|\lambda| = 3, \quad #\text{crossings} = 2, \quad \sum \text{height}(x) = 1 + 1 + 2 + 0 = 4\)
Rewriting rules for $\hat{D}$ and $\hat{E}$

Enumeration of rook placements

- $|\lambda|$ counts the number of $\times$ with no rook in the same line, no rook in the same column.

- $\#\text{crossings}$ counts the number of $\times$ with one rook in the same line, one rook in the same column.

- $\sum \text{height}(x)$ counts all remaining $\times$.

\[
|\lambda| = 3, \quad \#\text{crossings} = 2, \quad \sum \text{height}(x) = 1 + 1 + 2 + 0 = 4
\]
Rewriting rules for $\hat{D}$ and $\hat{E}$

Enumeration of rook placements

Conclusion

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>counts the number of $\times$ with no rook in the same line, no rook in the same column.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$#\text{crossings}$</td>
<td>counts the number of $\times$ with one rook in the same line, one rook in the same column.</td>
</tr>
<tr>
<td>$\sum \text{height}(x)$</td>
<td>counts all remaining $\times$.</td>
</tr>
</tbody>
</table>

$|\lambda| = 3, \quad \#\text{crossings} = 2, \quad \sum \text{height}(x) = 1 + 1 + 2 + 0 = 4$
\begin{itemize}
  \item $|\lambda|$ counts the number of $\times$ with no rook in the same line, no rook in the same column.
  \item $\#\text{crossings}$ counts the number of $\times$ with one rook in the same line, one rook in the same column.
  \item $\sum \text{height}(x)$ counts all remaining $\times$.
\end{itemize}

$|\lambda| = 3$, $\#\text{crossings} = 2$, $\sum \text{height}(x) = 1 + 1 + 2 + 0 = 4$
$R = \begin{array}{cccccc}
\times & \times & \circ & \circ & \circ & \times \\
\circ & \times & \times & \times & \times & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ \\
\times & \times & \times & \times & \times & \times \\
\end{array}$

- $|\lambda|$ counts the number of $\times$ with no rook in the same line, no rook in the same column.
- $\#\text{crossings}$ counts the number of $\times$ with one rook in the same line, one rook in the same column.
- $\sum\text{height}(x)$ counts all remaining $\times$.

$|\lambda| = 3, \quad \#\text{crossings} = 2, \quad \sum\text{height}(x) = 1 + 1 + 2 + 0 = 4$
Rewriting rules for $\hat{D}$ and $\hat{E}$

Enumeration of rook placements

Introduction

Conclusion

$R = \begin{array}{cccc}
\times & \times & \times & \times \\
\circ & \times & \times & \times \\
\circ & \times & \circ & \times \\
\times & \times & \times & \times \\
\end{array}$

- $|\lambda|$ counts the number of $\times$ with no rook in the same line, no rook in the same column.

- $\#\text{crossings}$ counts the number of $\times$ with one rook in the same line, one rook in the same column.

- $\sum \text{height}(x)$ counts all remaining $\times$.

$|\lambda| = 3, \quad \#\text{crossings} = 2,$

$\sum \text{height}(x) = 1 + 1 + 2 + 0 = 4$
Consequence: Remember that $T_{j,k,n}$ is the sum of weights of rook placements of half-perimeter $n$, with $k$ lines, $j$ lines without rook.

Then we have a factorization:

$$T_{j,k,n} = \left[ n - 2k + 2j \right]_q T_{0,k-j,n}.$$
Besides this factorization property, we have a recurrence relation:

\[ T_{0,k,n} = T_{0,k,n-1} + p T_{1,k,n-1}. \]

**Case 1:** The first column contains no rook.

**Case 2:** The first column contains a rook.

Hence:

\[ T_{0,k,n} = T_{0,k,n-1} + p[n + 1 - 2k]q T_{0,k-1,n-1}. \]
Proposition

This recurrence is solved by:

\[
T_{0,k,n} = \left( \frac{p}{1-q} \right)^k \sum_{i=0}^{k} (-1)^i q^{i(i+1)/2} \binom{n-2k+i}{i} q \left( \binom{n}{k-i} - \binom{n}{k-i-1} \right).
\]

It remains to compute:

\[
< (\hat{D} + \hat{E})^n > = \sum_{j,k} T_{j,k,n} = \sum_{j,k} \left[ \binom{n-2k+2j}{j} \right] q T_{0,k-j,n}.
\]
In the PASEP case, i.e. $p = \frac{1-q}{q^2}$, we can simplify this sum with q-binomial identities. We obtain:

**Proposition**

$$< (\hat{D} + \hat{E})^n > = \frac{2F(n) - F(n+1)}{q^n(1-q)},$$

where

$$F(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left( \binom{n}{k} - \binom{n}{k-1} \right) \sum_{j=0}^{n-2k} q^{j(n+1-2k-j)}.$$
Remember that \((\hat{D} + \hat{E})^n\) and \((D + E)^n\) are linked by inversion formulas. We get a new proof of:

**Theorem**

\[
< (D + E)^{n-1} > = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left( \binom{2n}{n-k} - \binom{2n}{n-k-1} \right) \\
\times \left( \sum_{j=0}^{k} q^j(k+1-j) - \sum_{j=0}^{k-1} q^j(k-j) \right).
\]

Conclusion

\[ < (D + E)^n > \] is the one-parameter function partition of the PASEP, but also:

- The \(q\)-enumeration of permutations wrt the number of 13-2 patterns (or equivalently, the number of crossings)
- The \(q\)-enumeration of permutation tableaux wrt the number of non-topmost 1’s.
- The momentum of simple \(q\)-Laguerre polynomials.

These results also give an expression for the 3-parameter partition function of the PASEP, although it seems there is no nice simplification.

A generalization to \((\alpha D + E)^n\) and \((\alpha \hat{D} + \hat{E})^n\) would give the momentum of (non-simple) \(q\)-Laguerre polynomials.