Schur functions and characters of Lie algebras and superalgebras

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Inspired by a conjecture by Joris Van der Jeugt (University of Gent, Belgium) including ongoing joint work with Angèle Hamel (Wilfred Laurier University, Canada)

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Motivation

- Received query from Joris Van der Jeugt (working with Stijn Lievens and Neli Stoilova)
- Studying representations of the orthosymplectic Lie superalgebra $osp(1|2n)$ built using parabosons
- Identified Fock space modules $\overline{V}(p)$ for any $p \in \mathbb{N}$
- Constructed unitary irreducible infinite-dimensional representations $V(p) = \overline{V}(p)/M(p)$ where $M(p)$ is the maximal submodule of $\overline{V}(p)$, and found that
  - for $p \geq n$ irrep $V(p) = \overline{V}(p)$
  - for $p < n$ irrep $V(p) = \overline{V}(p)/M(p)$
- Also calculated the characters of both $\overline{V}(p)$ and $V(p)$

Van der Jeugt's conjecture

Proposition [Van der Jeugt, Lievens and Stoilova, 2007]

Let $x = (x_1, x_2, \ldots, x_n)$, then

$$\mathrm{ch}\, V(p) = (x_1 x_2 \cdots x_n)^{p/2} \sum_{\lambda : \ell(\lambda) \leq p} s_\lambda(x)$$

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Conjecture [Van der Jeugt, Lievens and Stoilova, 2007]

$$\sum_{\lambda : \ell(\lambda) \leq p} s_\lambda(x) = \frac{\sum_\eta (-1)^{c_\eta} \, s_\eta(x)}{\prod_{1 \leq i \leq n} (1 - x_i) \, \prod_{1 \leq j < k \leq n} (1 - x_i x_j)}$$

with the sum over all partitions η which in Frobenius notation take the form $\eta = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ a_1 + p & a_2 + p & \cdots & a_r + p \end{pmatrix}$

with $c_\eta = (|\eta| - rp + r)/2$

Macdonald's Theorem

- Joris Van der Jeugt asked if the result was known
- If so where could it be found, if not could I supply a proof?
- Angèle Hamel reminded me of:

  Theorem [Macdonald 79]

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x) = \frac{\left|x_i^{n-j} - x_i^{n+p+j-1}\right|}{\prod_{1\leq i\leq n}(1-x_i)\ \prod_{1\leq j<k\leq n}(x_j-x_k)(1-x_jx_k)}$$

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- Need to compare this with an immediate Corollary to Van der Jeugt's Conjecture

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x) = \frac{\sum_\eta (-1)^{c_\eta}\ s_{\eta'}(x)}{\prod_{1\leq i\leq n}(1-x_i)\ \prod_{1\leq j<k\leq n}\ (1-x_ix_j)}$$

Strategy

- Try to recast the numerator of Macdonald's formula as a signed sum of Schur functions
- Use conjugacy to recover Van der Jeugt's formula
- Try to identify the origin of the row length restriction $\ell(\lambda') \leq p$ in Macdonald's formula
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- First some preliminaries on
  - Schur functions and Schur functions series
  - Partitions, Young diagrams, Frobenius notation
  - Determinantal identities and modifications

Schur functions

- Let n be a fixed positive integer
- Let $x = (x_1, x_2, \ldots, x_n)$ be a sequence of indeterminates
- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ be a partition of weight $|\lambda|$ and length $\ell(\lambda) \leq n$
- Then the Schur function $s_\lambda(x)$ is defined by:

$$s_\lambda(x) = \frac{\left| x_i^{\lambda_j+n-j} \right|_{1\leq i,j\leq n}}{\left| x_i^{n-j} \right|_{1\leq i,j\leq n}}$$

- where $\left| x_i^{n-j} \right|_{1\leq i,j\leq n} = \prod_{1\leq i<j\leq n} (x_i - x_j)$

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$$s_\lambda(x) = \frac{\left| x_i^{\lambda_j + n - j} \right|_{1 \leq i,j \leq n}}{\left| x_i^{n-j} \right|_{1 \leq i,j \leq n}}$$

- where $\left| x_i^{n-j} \right|_{1 \leq i,j \leq n} = \prod_{1 \leq i < j \leq n} (x_i - x_j)$
- These Schur functions form a $\mathbb{Z}$-basis of Λ_n, the ring of polynomial symmetric functions of $x_1, \dots, x_n$.

Partitions and Young diagrams

- Young diagrams F^λ consists of $|\lambda|$ boxes arranged in $\ell(\lambda)$ rows of lengths λ_i for $i = 1, 2, \ldots \ell(\lambda)$
- Conjugate partition λ' is the partition defined by the $\ell(\lambda')$ columns of F^λ of lengths λ'_j for $j = 1, 2 \ldots, \ell(\lambda')$
- Frobenius notation If F^λ has r boxes on the main diagonal, with arm and leg lengths a_k and b_k for $k = 1, 2, \ldots, r$, then $\lambda = \begin{pmatrix} a_1 \; a_2 \cdots a_r \\ b_1 \; b_2 \cdots b_r \end{pmatrix}$ has rank $r(\lambda) = r$ with $a_1 > a_2 > \cdots > a_r \geq 0$ and $b_1 > b_2 > \cdots > b_r \geq 0$

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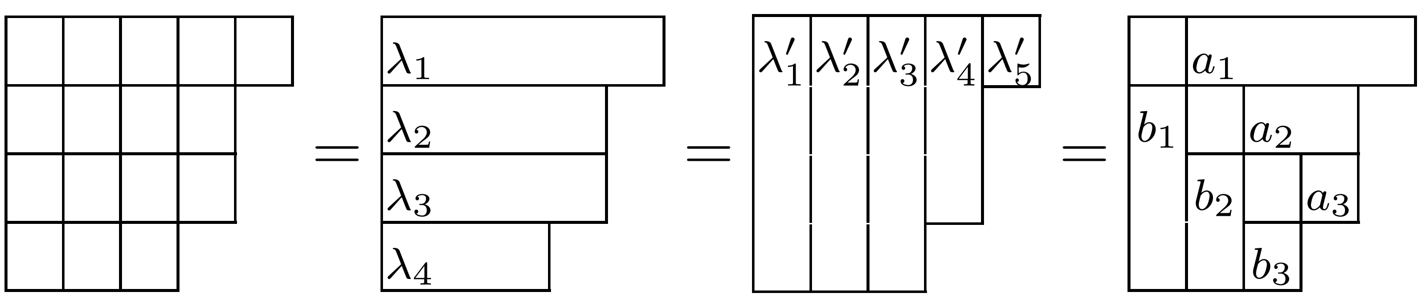

Special families of partitions

- Let $\mathcal{P}$ be the set of all partitions, including the zero partition $\lambda = 0 = (0, 0, \ldots, 0)$.
- The zero partition is the unique partition of weight, length and rank zero, ie. $|0| = \ell(0) = r(0) = 0$
- Then for any integer t let

$$\mathcal{P}_t = \left\{ \lambda = \begin{pmatrix} a_1 \; a_2 \cdots a_r \\ b_1 \; b_2 \cdots b_r \end{pmatrix} \in \mathcal{P} \;\middle|\; a_k - b_k = t \quad \begin{array}{l} \text{for} \quad k = 1, 2, \ldots, r \\ \text{and} \quad r = 0, 1, \ldots \end{array} \right\}$$

- Note: The zero partition belongs to $\mathcal{P}_t$ for all integer t

Modification rules

- For $n \in \mathbb{N}$ let $x = (x_1, x_2, \ldots, x_n)$ and $\mathbf{x} = x_1 x_2 \cdots x_n$
- Let $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ with $\kappa_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, n$
- Let $s_\kappa(x) = \dfrac{\left| x_i^{\kappa_j + n - j} \right|_{1 \le i,j \le n}}{\left| x_i^{n-j} \right|_{1 \le i,j \le n}}$
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- Either $s_\kappa(x) = 0$ or $s_\kappa(x) = \pm\, \mathbf{x}^k\, s_\lambda(x)$ for some partition λ and some integer k
- Permuting columns leads to various identities, such as
  - $s_\kappa(x) = -s_\mu(x)$ and $s_\kappa(x) = (-1)^{j-1} s_\nu(x)$ with
  - $\mu = (\kappa_1, \ldots, \kappa_{j+1} - 1, \kappa_j + 1, \ldots, \kappa_n)$
  - $\nu = (\kappa_{j+1} - j, \kappa_1 + 1, \ldots, \kappa_j + 1, \kappa_{j+2} \ldots, \kappa_n)$

Example

- If $n = 4$ and $\kappa = (0,4,0,9)$ then $s_\kappa(x) = (-1)^{3+1}\, s_\lambda(x)$ with $\lambda = (6,4,2,1)$ since

$$\frac{\begin{vmatrix} x_i^3 & x_i^6 & x_i & x_i^9 \end{vmatrix}}{\begin{vmatrix} x_i^3 & x_i^2 & x_i & 1 \end{vmatrix}} = \frac{\begin{vmatrix} x_i^9 & x_i^6 & x_i^3 & x_i \end{vmatrix}}{\begin{vmatrix} x_i^3 & x_i^2 & x_i & 1 \end{vmatrix}}$$

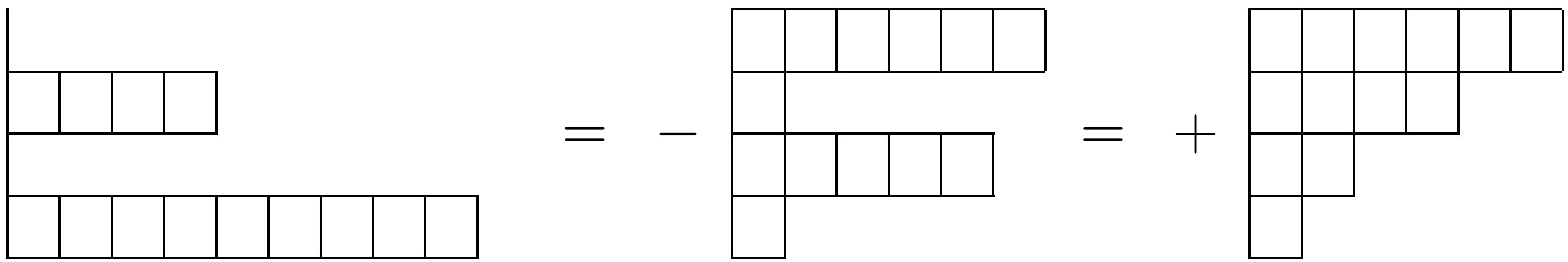
  where just the ith row of each determinant has been displayed

- Alternatively, one can proceed iteratively using the previous identities

$$s_{0409}(x) \;=\; -\; s_{6151}(x) \;=\; +\; s_{6421}(x)$$

Diagrammatically

Ex: $s_\kappa(x) = s_{0409}(x) = -s_{6151}(x) = +s_{6421}(x) = +s_\lambda(x)$



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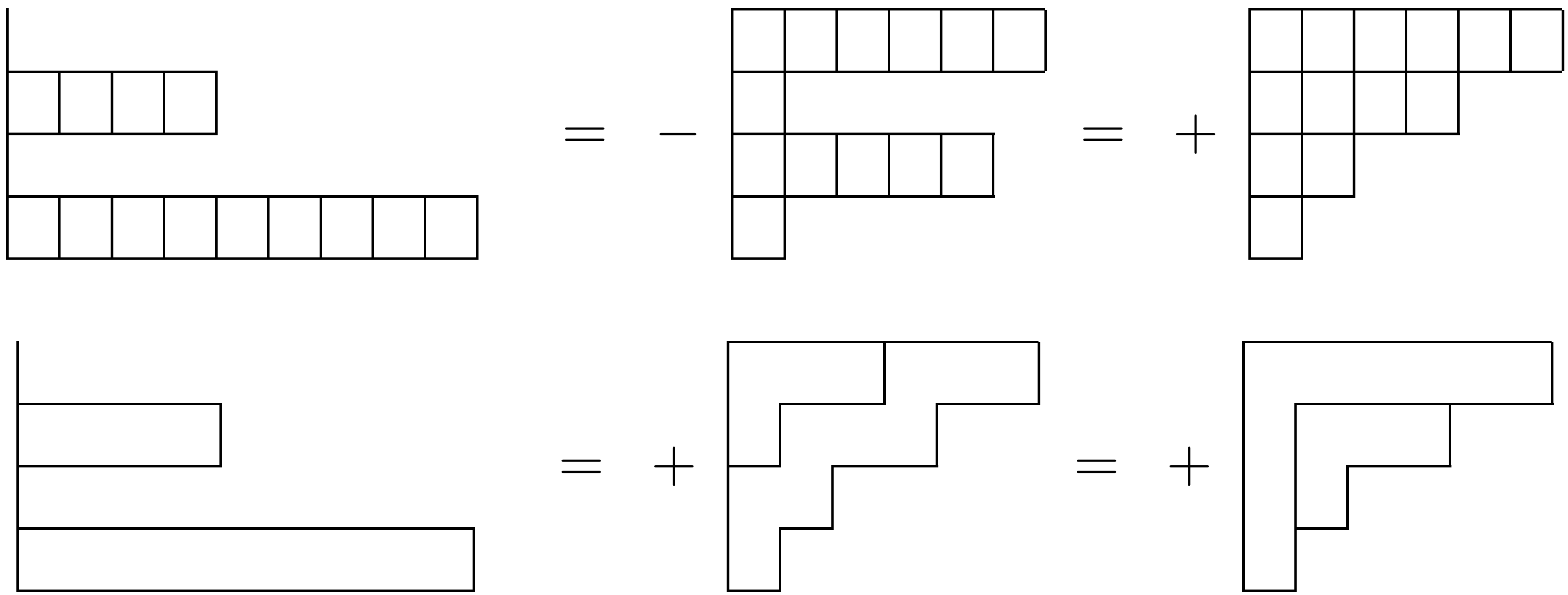

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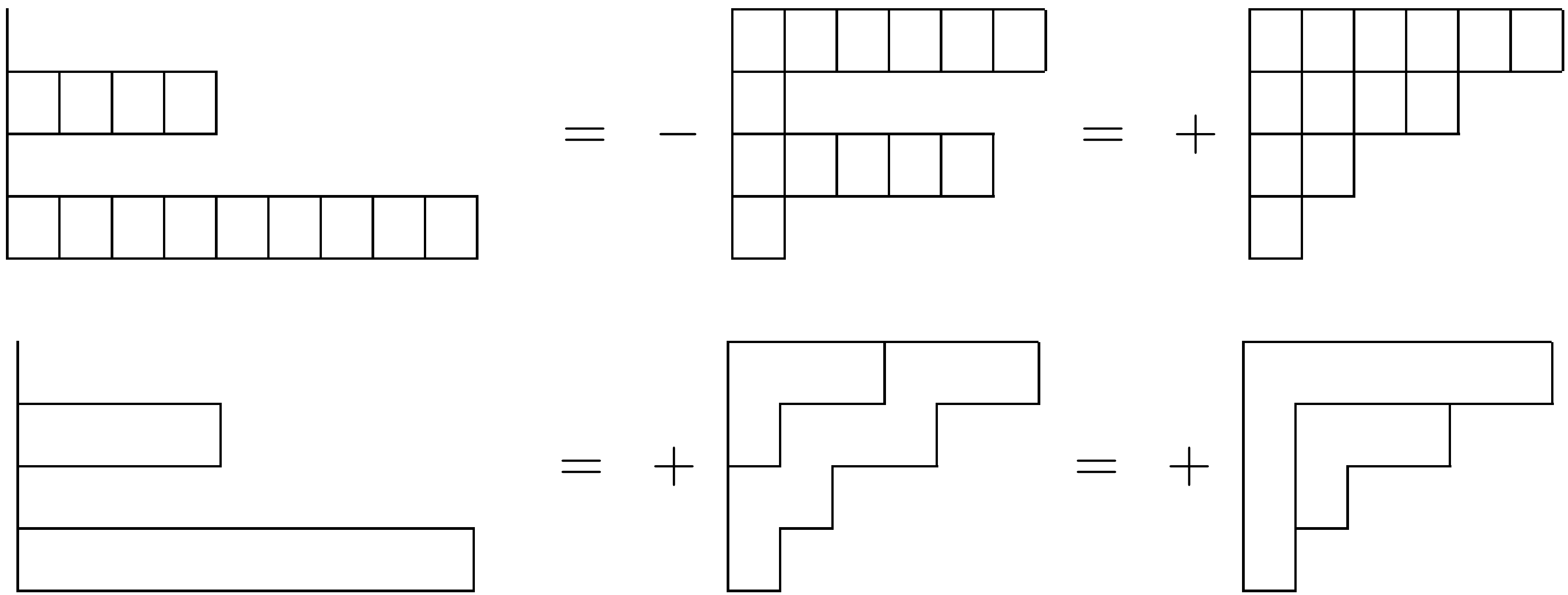

- Note $\lambda = (6421) = \begin{pmatrix} 5\,2 \\ 3\,1 \end{pmatrix}$

Frobenius notation and modifications

- Let $\kappa_j = 0$ unless $j \in \{b_1{+}1, b_2{+}1, \ldots, b_r{+}1\}$
- Let $b_1 > b_2 > \cdots > b_r \geq 0$ without loss of generality
- Let $\kappa(j) = a_k + b_k + 1$ if $j = b_k{+}1$ so that

$$\kappa = (0^{b_r}, a_r{+}b_r{+}1, 0^{b_{r-1}-b_r-1}, \ldots, a_2{+}b_2{+}1, 0^{b_1-b_2-1}, a_1{+}b_1{+}1)$$

- Then, if $a_1 > a_2 > \cdots > a_r \geq 0$,

$$s_\kappa(x) \;=\; (-1)^{b_1+b_2+\cdots+b_r}\, s_\lambda(x)$$

with

$$\lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

and $r = r(\lambda)$

Example

- For $\kappa = (0,4,0,9)$ we have $\kappa = 0$ unless $j \in \{2,4\}$
- Hence $r = 2$, $b_1 = 3$, $b_2 = 1$, with $b_1 > b_2 \geq 0$
- Since $\kappa_4 = a_1 + b_1 + 1 = 9$ and $\kappa_2 = a_2 + b_2 + 1 = 4$ we have $a_1 = 5$, $a_2 = 2$ with $a_1 > a_2 \geq 0$
- Hence we have $s_\kappa(x) = s_{0409}(x) = (-1)^{3+1}\, s_{6421}(x)$
- In Frobenius notation $\lambda = (6,4,2,1) = \begin{pmatrix} 5\,2 \\ 3\,1 \end{pmatrix}$

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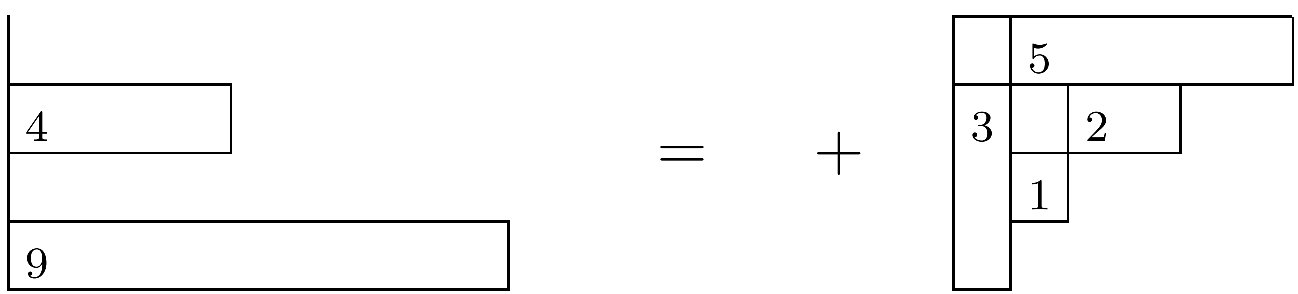

Schur function series

- Littlewood [1940] For all $n \geq 1$ and $x = (x_1, x_2, \dots, x_n)$:

$$\sum_{\lambda} s_\lambda(x) = \prod_{1 \le i \le n} (1 - x_i)^{-1} \prod_{1 \le j < k \le n} (1 - x_j x_k)^{-1}$$

$$\sum_{\lambda \text{ even}} s_\lambda(x) = \prod_{1 \le j \le k \le n} (1 - x_j x_k)^{-1}$$

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- A partition is even if all its non-zero parts are even
- The infinite sums over λ involve no restriction on either $\ell(\lambda)$ or $\ell(\lambda')$, but $s_\lambda(x) = 0$ if $\ell(\lambda) > n$.

Inverse Schur function series

- Littlewood [1940] For all $n \geq 1$ and $x = (x_1, x_2, \ldots, x_n)$

$$\sum_{\lambda \in \mathcal{P}_0} (-1)^{(|\lambda|+r(\lambda))/2} \, s_\lambda(x) \;=\; \prod_{1 \leq i \leq n} (1 - x_i) \prod_{1 \leq j < k \leq n} (1 - x_j x_k)$$

$$\sum_{\lambda \in \mathcal{P}_1} (-1)^{|\lambda|/2} \, s_\lambda(x) \;=\; \prod_{1 \leq j \leq k \leq n} (1 - x_j x_k)$$

$$\sum_{\lambda \in \mathcal{P}_{-1}} (-1)^{|\lambda|/2} \, s_\lambda(x) \;=\; \prod_{1 \leq j < k \leq n} (1 - x_j x_k)$$

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- These series are finite for all finite n
- For finite n both $\ell(\lambda)$ and $\ell(\lambda')$ are restricted, since for $\lambda \in \mathcal{P}_t$ these differ by t

Determinantal identities

- Littlewood [1940] For all $n \geq 1$ and $x = (x_1, x_2, \ldots, x_n)$

$$\frac{\left|x_i^{n-j} - x_i^{n+j-1}\right|}{\left|x_i^{n-j}\right|} = \sum_{\lambda \in \mathcal{P}_0} (-1)^{[|\lambda|+r(\lambda)]/2} \, s_\lambda(x)$$

$$\frac{\left|x_i^{n-j} - x_i^{n+j}\right|}{\left|x_i^{n-j}\right|} = \sum_{\lambda \in \mathcal{P}_1} (-1)^{|\lambda|/2} \, s_\lambda(x)$$

$$\frac{\left|x_i^{n-j} + \chi_{\,j>1}\, x_i^{n+j-2}\right|}{\left|x_i^{n-j}\right|} = \sum_{\lambda \in \mathcal{P}_{-1}} (-1)^{|\lambda|/2} \, s_\lambda(x)$$

- the determinants are all $n \times n$ with $i, j = 1, 2, \ldots, n$
- and, for any proposition P, $\chi_P = \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{if } P \text{ is false} \end{cases}$

General determinantal identity

- Lemma K [2008] For all $n \geq 1$ and $x = (x_1, x_2, \ldots, x_n)$

$$\frac{\left| x_i^{n-j} + q\, \chi_{j>-t}\; x_i^{n+t+j-1} \right|}{\left| x_i^{n-j} \right|} = \sum_{\lambda \in \mathcal{P}_t} (-1)^{[|\lambda| - r(\lambda)(t+1)]/2}\; q^{r(\lambda)}\; s_\lambda(x)$$

  - where t is any integer, and q is arbitrary
  - and the determinants are all $n \times n$
  - so that $i, j = 1, 2, \ldots, n$

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  - and the determinants are all $n \times n$
  - so that $i, j = 1, 2, \ldots, n$

- The special cases:

$$q = -1, t = 0; \qquad q = -1, t = 1; \qquad q = 1, t = -1,$$

correspond to Littlewood's previous formulae

Algebraic proof

$$\frac{\left| x_i^{n-j} + q\ \chi_{\,j>-t}\ x_i^{n+t+j-1} \right|}{\left| x_i^{n-j} \right|} = \frac{\left| x_i^{n-j} + q\ \chi_{\,j>-t}\ x_i^{2j-1+t+n-j} \right|}{\left| x_i^{n-j} \right|}$$

$$= \sum_{r=0}^{n} \sum_{\kappa}\ q^r\ s_\kappa(x) = \sum_{\lambda \in \mathcal{P}_t} (-1)^{(j_r-1)+\cdots+(j_2-1)+(j_1-1)}\ q^r\ s_\lambda(x)$$

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- $\kappa_j = 2j-1+t$ for $j \in \{j_1, j_2, \ldots, j_r\}$ and $\kappa_j = 0$ otherwise
- with $n \geq j_1 > j_2 > \cdots > j_r \geq 1 - \chi_{t<0} t$
- $\lambda = \begin{pmatrix} j_1 - 1 + t & j_2 - 1 + t & \cdots & j_r - 1 + t \\ j_1 - 1 & j_2 - 1 & \cdots & j_r - 1 \end{pmatrix} \in \mathcal{P}_t$
- $r = r(\lambda)$
- $|\lambda| = 2((j_1-1) + (j_2-1) + \cdots + (j_r-1)) + r(t+1)$

Example with $n = 4$ and $t = 2$

$$\frac{\left| x_i^{4-j} + q\,\chi_{j>-2}\; x_i^{5+j} \right|}{\left| x_i^{4-j} \right|}$$

$$= \frac{\begin{vmatrix} x_i^3 + q\,x_i^6 & x_i^2 + q\,x_i^7 & x_i + q\,x_i^8 & 1 + q\,x_i^9 \end{vmatrix}}{\begin{vmatrix} x_i^3 & x_i^2 & x_i & 1 \end{vmatrix}}$$

$$\begin{aligned}
= \quad & s_{0000} + q\left(s_{3000} + s_{0500} + s_{0070} + s_{0009}\right) \\
& + q^2\left(s_{3500} + s_{3070} + s_{0570} + s_{3009} + s_{0509} + s_{0079}\right) \\
& + q^3\left(s_{3570} + s_{3509} + s_{3079} + s_{0579}\right) + q^4\, s_{3579} \\
= \quad & 1 + q\left(s_3 - s_{41} + s_{511} - s_{6111}\right) \\
& + q^2\left(-s_{44} + s_{541} - s_{552} - s_{6411} + s_{6521} - s_{6622}\right) \\
& + q^3\left(-s_{555} + s_{6551} - s_{6652} + s_{6663}\right) + q^4\, s_{6666}
\end{aligned}$$

Example with $n=4$ and $t=2$ contd.

In Frobenius notation $s_\lambda(x) = \begin{pmatrix} a_1\, a_2 \cdots a_r \\ b_1\, b_2 \cdots b_r \end{pmatrix}$, we have

$$1 + q\left[\begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \end{pmatrix}\right]$$

$$+q^2\left[-\begin{pmatrix} 32 \\ 10 \end{pmatrix} + \begin{pmatrix} 42 \\ 20 \end{pmatrix} - \begin{pmatrix} 43 \\ 21 \end{pmatrix} - \begin{pmatrix} 52 \\ 30 \end{pmatrix} + \begin{pmatrix} 53 \\ 31 \end{pmatrix} - \begin{pmatrix} 54 \\ 32 \end{pmatrix}\right]$$

$$+q^3\left[-\begin{pmatrix} 432 \\ 210 \end{pmatrix} + \begin{pmatrix} 532 \\ 310 \end{pmatrix} - \begin{pmatrix} 542 \\ 320 \end{pmatrix} + \begin{pmatrix} 543 \\ 321 \end{pmatrix}\right]$$

$$+q^4\begin{pmatrix} 5432 \\ 3210 \end{pmatrix}$$

Example with $n = 4$ and $t = -2$

$$\frac{\left| x_i^{4-j} + q\, \chi_{j>2}\, x_i^{1+j} \right|}{\left| x_i^{4-j} \right|}$$

$$= \frac{\begin{vmatrix} x_i^3 & x_i^2 & x_i + q\, x_i^4 & 1 + q\, x_i^5 \end{vmatrix}}{\begin{vmatrix} x_i^3 & x_i^2 & x_i & 1 \end{vmatrix}}$$

$$= s_{0000} + q\,(s_{0030} + s_{0005}) + q^2\, s_{0035}$$

$$= 1 + q\,(s_{111} - s_{2111}) - q^2\, s_{2222}$$

$$= 1 + q \begin{pmatrix} 0 \\ 2 \end{pmatrix} - q \begin{pmatrix} 1 \\ 3 \end{pmatrix} - q^2 \begin{pmatrix} 1\ 0 \\ 3\ 2 \end{pmatrix}$$

Row length restricted Schur function series

$$\sum_{\lambda:\ell(\lambda')\le p} s_\lambda(x) \quad \text{with} \quad x=(x_1,x_2,\ldots,x_n),\, n\ge 1,\, p\ge 0$$

$$= \frac{\left|x_i^{n-j}-x_i^{n+p+j-1}\right|}{\prod_{1\le i\le n}(1-x_i)\prod_{1\le j<k\le n}(x_j-x_k)(1-x_jx_k)} \quad \text{Macdonald}$$

$$= \frac{\left|x_i^{n-j}-x_i^{n+p+j-1}\right| / \left|x_i^{n-j}\right|}{\prod_{1\le i\le n}(1-x_i)\prod_{1\le j<k\le n}(1-x_jx_k)} \quad \text{Vandermonde}$$

$$= \frac{\sum_{\mu\in\mathcal{P}_p}(-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x)}{\prod_{1\le i\le n}(1-x_i)\prod_{1\le j<k\le n}(1-x_jx_k)} \quad \text{Lemma: q=-1,t=p}$$

$$= \frac{\sum_{\mu\in\mathcal{P}_p}(-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x)}{\sum_{\nu\in\mathcal{P}_0}(-1)^{[|\nu|+r(\nu)]/2}\, s_\nu(x)} \quad \text{Littlewood}$$

Column length restricted Schur function series

- Using the conjugacy involution $\omega : s_\lambda(x) \mapsto s_{\lambda'}(x)$ for all λ
- and noting that $\lambda \in \mathcal{P}_t \implies \lambda' \in \mathcal{P}_{-t}$ for all t, we have

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$$= \frac{\sum_{\mu\in\mathcal{P}_{-p}} (-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x)}{\sum_{\nu\in\mathcal{P}_0} (-1)^{[|\nu|+r(\nu)]/2}\, s_\nu(x)} \quad \text{Conjugacy}$$

$$= \frac{\sum_{\mu\in\mathcal{P}_{-p}} (-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x)}{\prod_{1\le i\le n}(1-x_i)\prod_{1\le j<k\le n}(1-x_jx_k)} \quad \text{Van der Jeugt}$$

$$= \frac{\left|x_i^{n-j} - (-1)^p \chi_{\,j>p}\, x_i^{n-p+j-1}\right| / \left|x_i^{n-j}\right|}{\prod_{1\le i\le n}(1-x_i)\prod_{1\le j<k\le n}(1-x_jx_k)} \quad \text{Lemma} \quad \begin{array}{c} q=-(-1)^p \\ t=-p \end{array}$$

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- We have recast the numerator of Macdonald’s formula as a signed sum of Schur functions
- We have then used conjugacy to prove Van der Jeugt’s conjecture

So far

- We have recast the numerator of Macdonald's formula as a signed sum of Schur functions
- We have then used conjugacy to prove Van der Jeugt's conjecture
- We have not exploited all of Littlewood's series
- We have only used two special cases of the Lemma: $q = -1, \;\; t = p$ and $q = -(-1)^p, \;\; t = -p$
- But there exist further row (and as we shall see column) restricted Schur function series

Row length restricted Schur function series

Theorem [Macdonald 79; Désarménien 87, Stembridge 90, Proctor 90; Bressoud 98, Okada 98]

For all $n \geq 1$, $x = (x_1, x_2, \ldots, x_n)$ and $p \geq 0$:

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x) = \frac{\left|x_i^{n-j} - x_i^{n+p+j-1}\right|}{\left|x_i^{n-j}\right| \prod_{1\leq i\leq n}(1-x_i) \prod_{1\leq j<k\leq n}(1-x_jx_k)}$$

$$\sum_{\lambda \,\mathrm{even}\,:\ell(\lambda')\leq 2p} s_\lambda(x) = \frac{\left|x_i^{n-j} - x_i^{n+2p+j}\right|}{\left|x_i^{n-j}\right| \prod_{1\leq j\leq k\leq n}(1-x_jx_k)}$$

$$\sum_{\lambda' \,\mathrm{even}\,:\ell(\lambda')\leq p} s_\lambda(x) = \frac{\frac{1}{2}\left|x_i^{n-j} - x_i^{n+p+j-2}\right| + \frac{1}{2}\left|x_i^{n-j} + x_i^{n+p+j-2}\right|}{\left|x_i^{n-j}\right| \prod_{1\leq j<k\leq n}(1-x_jx_k)}$$

Row length restricted Schur function series

Using the Lemma for given q and t as indicated, we find

Corollary For all $x=(x_1,x_2,\ldots)$

$q=-1, t=p$

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_p} (-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x)}{\prod_{1\leq i\leq n}(1-x_i)\prod_{1\leq j<k\leq n}(1-x_jx_k)}$$

$q=-1, t=2p+1$

$$\sum_{\lambda\ \text{even}\ :\ell(\lambda')\leq 2p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{2p+1}} (-1)^{[|\mu|-r(\mu)(2p)]/2}\, s_\mu(x)}{\prod_{1\leq j\leq k\leq n}(1-x_jx_k)}$$

$q=\pm 1, t=p-1$

$$\sum_{\lambda'\ \text{even}\ :\ell(\lambda')\leq p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{p-1}:r(\mu)\ \text{even}} (-1)^{[|\mu|-r(\mu)p]/2}\, s_\mu(x)}{\prod_{1\leq j<k\leq n}(1-x_jx_k)}$$

Row length restricted Schur function series

Littlewood's inverse Schur function series formulae then give:

Corollary For all $x = (x_1, x_2, \ldots)$

$q = -1, t = p$

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_p} (-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x)}{\sum_{\nu\in\mathcal{P}_0} (-1)^{[|\nu|+r(\nu)]/2}\, s_\nu(x)}$$

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$$\sum_{\lambda'\text{ even }:\ell(\lambda)\leq 2p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{-2p-1}} (-1)^{[|\mu|-r(\mu)(2p)]/2}\, s_\mu(x)}{\sum_{\nu\in\mathcal{P}_{-1}} (-1)^{|\nu|/2}\, s_\nu(x)}$$

$$\sum_{\lambda\text{ even }:\ell(\lambda)\leq p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{-p+1}:r(\mu)\text{ even}} (-1)^{[|\mu|-r(\mu)p]/2}\, s_\mu(x)}{\sum_{\nu\in\mathcal{P}_1} (-1)^{|\nu|/2}\, s_\nu(x)}$$

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$$\sum_{\lambda'\text{ even }:\ell(\lambda)\le 2p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{-2p-1}} (-1)^{[|\mu|-r(\mu)(2p)]/2}\, s_\mu(x)}{\prod_{1\le j<k\le n}(1-x_jx_k)}$$

$$\sum_{\lambda\text{ even }:\ell(\lambda)\le p} s_\lambda(x) = \frac{\sum_{\mu\in\mathcal{P}_{-p+1}:r(\mu)\text{ even}} (-1)^{[|\mu|-r(\mu)p]/2}\, s_\mu(x)}{\prod_{1\le j\le k\le n}(1-x_jx_k)}$$

- Note: The first of these was Van der Jeugt's Conjecture

Column length restricted Schur function series

Using $(q,t)=(-(-1)^p,-p)$, $(\pm 1,-p+1)$ and $(1,-2p-1)$ in our Lemma, we find

Theorem For all $n\geq 1$, $x=(x_1,x_2,\ldots,x_n)$ and $p\geq 0$:

$$\sum_{\lambda:\ell(\lambda)\leq p} s_\lambda(x) \quad = \quad \frac{\left|x_i^{n-j}-(-1)^p\chi_{\,j>p}\; x_i^{n-p+j-1}\right|}{\left|x_i^{n-j}\right|\,\prod_{1\leq i\leq n}(1-x_i)\prod_{1\leq j<k\leq n}(1-x_jx_k)}$$

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Row length restricted Schur function series

Alternative universal expressions giving each restricted series as a product of an unrestricted series and a correction factor for all $x = (x_1, x_2, \dots)$ take the form

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x) = \sum_{\lambda} s_\lambda(x) \cdot \sum_{\mu\in\mathcal{P}_p} (-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x)$$

$$\sum_{\lambda\,\mathrm{even}\,:\ell(\lambda')\leq 2p} s_\lambda(x) = \sum_{\lambda\,\mathrm{even}} s_\lambda(x) \cdot \sum_{\mu\in\mathcal{P}_{2p+1}} (-1)^{[|\mu|-r(\mu)(2p)]/2}\, s_\mu(x)$$

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Rank restricted Schur function series

- The row length restricted series takes the form

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x) = \sum_{\lambda} s_\lambda(x) \cdot \sum_{\mu\in\mathcal{P}_p} (-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x)$$

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- Conjecture The rank restricted series takes the form

$$\sum_{\lambda:r(\lambda)\leq p} s_\lambda(x) = \sum_\lambda s_\lambda(x) \cdot \sum_{\mu\in\mathcal{P}_0:r(\mu)=p+1} (-1)^{[|\mu|+r(\mu)]/2}\, s_\mu(x)$$

So far

- We have obtained three determinantal formulae for column length restricted partitions analogous to those for row length restricted partitions

So far

- We have obtained three determinantal formulae for column length restricted partitions analogous to those for row length restricted partitions
- We have not explained why the various determinants lead to row or column length restrictions
- To do this we need to exploit the fact that they define characters of particular representations of classical groups as emphasized by Okada
- Then we may look for an alternative way of evalauting these characters through the use of Howe dual pairs of groups

Classical groups and their characters

Let $x=(x_1,x_2,\ldots,x_n)$ and $\overline{x}=(\overline{x}_1,\overline{x}_2,\ldots,\overline{x}_n)$
with $x_i=e^{\epsilon_i}$ and $\overline{x}_i=x_i^{-1}=e^{-\epsilon_i}$ for $i=1,2,\ldots,n$

$$\mathrm{ch}\,V^{\lambda}_{GL(n)} = \frac{\left|x_i^{\lambda_j+n-j}\right|}{\left|x_i^{n-j}\right|}$$

$$\mathrm{ch}\,V^{\lambda}_{SO(2n+1)} = \frac{\left|x_i^{\lambda_j+n-j+\frac12}-\overline{x}_i^{\lambda_j+n-j+\frac12}\right|}{\left|x_i^{n-j+\frac12}-\overline{x}_i^{n-j+\frac12}\right|}$$

$$\mathrm{ch}\,V^{\lambda}_{Sp(2n)} = \frac{\left|x_i^{\lambda_j+n-j+1}-\overline{x}_i^{\lambda_j+n-j+1}\right|}{\left|x_i^{n-j+1}-\overline{x}_i^{n-j+1}\right|}$$

$$\mathrm{ch}\,V^{\lambda}_{SO(2n)} = \frac{\left|x_i^{\lambda_j+n-j}+\overline{x}_i^{\lambda_j+n-j}\right|+\left|x_i^{\lambda_j+n-j}-\overline{x}_i^{\lambda_j+n-j}\right|}{\left|x_i^{n-j}+\overline{x}_i^{n-j}\right|}$$

Characters expressed in terms of Schur functions

$$\begin{aligned}
\mathrm{ch}\, V^{\lambda}_{GL(n)} &= s_\lambda(x) \\
\mathrm{ch}\, V^{\lambda}_{SO(2n+1)} &= \sum_{\mu\in\mathcal{P}_0} (-1)^{(|\mu|-r(\mu))/2}\, s_{\lambda/\mu}(x,\overline{x}) \\
\mathrm{ch}\, V^{\lambda+\frac{1}{2}^p}_{SO(2n+1)} &= \mathrm{ch}\, V^{\Delta}_{SO(2n)} \sum_{\mu\in\mathcal{P}_{-1}} (-1)^{|\mu|/2}\, s_{\lambda/\mu}(x,\overline{x}) \\
\mathrm{ch}\, V^{\lambda}_{Sp(2n)} &= \sum_{\mu\in\mathcal{P}_{-1}} (-1)^{|\mu|/2}\, s_{\lambda/\mu}(x,\overline{x}) \\
\mathrm{ch}\, V^{\lambda}_{SO(2n)} &= \sum_{\mu\in\mathcal{P}_{1}} (-1)^{|\mu|/2}\, s_{\lambda/\mu}(x,\overline{x}) \\
\mathrm{ch}\, V^{\lambda+\frac{1}{2}^p}_{SO(2n)} &= \mathrm{ch}\, V^{\Delta}_{SO(2n)} \sum_{\mu\in\mathcal{P}_0} (-1)^{(|\mu|+r(\mu))/2}\, s_{\lambda/\mu}(x,\overline{x})
\end{aligned}$$

Row length restricted series and characters

Theorem [Macdonald, Désarménien, Stembridge, Proctor, Bressoud, Okada]

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x) = \frac{\left|x_i^{n-j} - x_i^{n+p+j-1}\right|}{\left|x_i^{n-j} - x_i^{n+j-1}\right|} = \mathbf{x}^{p/2}\ \mathrm{ch}\, V_{SO(2n+1)}^{(p/2)^n}(x,\overline{x},1)$$

$$\sum_{\lambda\ \mathrm{even}\,:\ell(\lambda')\leq 2p} s_\lambda(x) = \frac{\left|x_i^{n-j} - x_i^{n+2p+j}\right|}{\left|x_i^{n-j} - x_i^{n+j}\right|} = \mathbf{x}^{p}\ \mathrm{ch}\, V_{Sp(2n)}^{p^n}(x,\overline{x})$$

$$\sum_{\lambda'\ \mathrm{even}\,:\ell(\lambda')\leq p} s_\lambda(x) = \frac{\left|x_i^{n-j} - x_i^{n+p+j-2}\right| + \left|x_i^{n-j} + x_i^{n+p+j-2}\right|}{\left|x_i^{n-j} + x_i^{n+j-2}\right|}$$

$$= \mathbf{x}^{p/2}\ \mathrm{ch}\, V_{SO(2n)}^{(p/2)^{n-1},(-)^n(p/2)}(x,\overline{x})$$

where $\mathbf{x} = x_1x_2\cdots x_n = \mathrm{ch}\, V_{GL(n}^{1^n}(x)$

Proof of formulae in terms of characters

- Start from the original determinantal formulae
- In each determinant permute columns under

$$j \to n-j+1$$

- Extract factors $(-1)^n$ by changing signs of all terms of the form $\quad x_i^a - x_i^b$
- Extract factors
  - $x_i^{n-\frac{1}{2}+\frac{p}{2}}$ and $x_i^{n-\frac{1}{2}}$
  - x_i^{n+p} and x_i^n
  - $x_i^{n-1+\frac{p}{2}}$ and x_i^{n-1}

  from each row of numerator and denominator determinants

Howe dual pairs of groups

Definition [Howe 85]

- Let groups G and H act on a linear vector space V
- Let their actions mutually commute
- As a representation of $G \times H$, let

$$V = \oplus_{k \in K} V_G^{\lambda(k)} \otimes V_H^{\mu(k)}$$

  - k varies over some index set K
  - $V_G^{\lambda(k)}$ and $V_H^{\mu(k)}$ are irreps of G and H
  - $V_G^{\lambda(k)}$ and $V_H^{\mu(k)}$ vary without repetition
- In such a case we say that G and H form a (Howe) dual pair with respect to V.

Howe dual pairs of classical groups

- In some cases V is an irrep of a group $F \supseteq G \times H$
- On restriction to the subgroup $G \times H$

$$\mathrm{ch}\, V^F_{G\times H} = \sum_{k\in K} \mathrm{ch}\, V_G^{\lambda(k)}\ \mathrm{ch}\, V_H^{\mu(k)}$$

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Ex: [Howe 89, Hasegawa 89] For V the spin irrep of an othogonal group with character $\mathrm{ch}\, V^\Delta$, dual pairs are defined through each of the following restrictions:

$$\begin{array}{rcl} O(4np) & \supseteq & SO(2n) \times O(2p) \\ O(4np+2p) & \supseteq & SO(2n+1) \times O(2p) \\ O(4np+2n) & \supseteq & SO(2n) \times O(2p+1) \\ O(4np+2n+2p+1) & \supseteq & SO(2n+1) \times O(2p+1) \\ O(4np) & \supseteq & Sp(2n) \times Sp(2p) \end{array}$$

Notation for p^n-complements

- For any partition $\lambda \subseteq n^p$ we have $\lambda' \subseteq p^n$
- In such a case, let $\lambda^\dagger = (p - \lambda'_n, \ldots, p - \lambda'_2, p - \lambda'_1)$
- Then $\lambda^\dagger$ is also a partition

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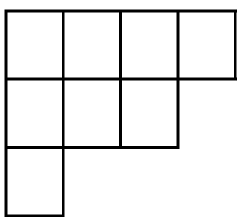
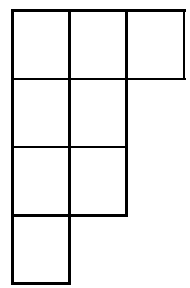
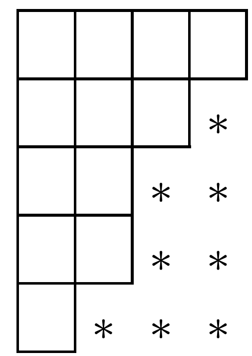
- Then $\lambda^\dagger$ is also a partition

  Ex: If $p = 4$, $n = 5$ and $\lambda = (4,3,1)$

  then $\lambda' = (3,2,2,1)$ and $\lambda^\dagger = (4,3,2,2,1)$

$F^{\lambda} =$  $F^{\lambda'} =$  $F^{\lambda^\dagger} =$ 

- Note: $0^\dagger = p^n = (p, p, \ldots, p)$

The spin module and Howe dual pairs

Theorem [Morris 58,60; Hasegawa 89; Terada 93; Bump and Gamburd 05] On restriction to the appropriate subgroup:

$$
\begin{aligned}
\mathrm{ch}\, V^{\Delta}_{O(4np)} &= \sum_{\lambda \subseteq n^p} \mathrm{ch}\, V^{\lambda^\dagger}_{SO(2n)}\, \mathrm{ch}\, V^{\lambda}_{O(2p)} \\
\mathrm{ch}\, V^{\Delta}_{O(4np+2p)} &= \sum_{\lambda \subseteq n^p} \mathrm{ch}\, V^{\lambda^\dagger}_{SO(2n+1)}\, \mathrm{ch}\, V^{\Delta;\lambda}_{O(2p)} \\
\mathrm{ch}\, V^{\Delta}_{O(4np+2n)} &= \sum_{\lambda \subseteq n^p} \mathrm{ch}\, V^{\Delta;\lambda^\dagger}_{SO(2n)}\, \mathrm{ch}\, V^{\lambda}_{O(2p+1)} \\
\mathrm{ch}\, V^{\Delta}_{O(4np+2n+2p+1)} &= \sum_{\lambda \subseteq n^p} \mathrm{ch}\, V^{\Delta;\lambda^\dagger}_{SO(2n+1)}\, \mathrm{ch}\, V^{\Delta;\lambda}_{O(2p+1)} \\
\mathrm{ch}\, V^{\Delta}_{O(4np)} &= \sum_{\lambda \subseteq n^p} \mathrm{ch}\, V^{\lambda^\dagger}_{Sp(2n)}\, \mathrm{ch}\, V^{\lambda}_{Sp(2p)}
\end{aligned}
$$

Exploitation of Howe duality

- Let (G, H) be a Howe dual pair with $F \supseteq G \times H$ such that $\mathrm{ch}\, V^F_{G\times H} = \sum_{k\in K} \mathrm{ch}\, V_G^{\lambda(k)}\, \mathrm{ch}\, V_H^{\mu(k)}$
- The character $\mathrm{ch}\, V_G^{\lambda(k)}$ is just the coefficient of $\mathrm{ch}\, V_H^{\mu(k)}$ in any formula we can devise for $\mathrm{ch}\, V^F_{G\times H}$

Exploitation of Howe duality

- Let (G,H) be a Howe dual pair with $F \supseteq G \times H$ such that $\mathrm{ch}\, V^F_{G\times H} = \sum_{k\in K} \mathrm{ch}\, V_G^{\lambda(k)}\, \mathrm{ch}\, V_H^{\mu(k)}$
- The character $\mathrm{ch}\, V_G^{\lambda(k)}$ is just the coefficient of $\mathrm{ch}\, V_H^{\mu(k)}$ in any formula we can devise for $\mathrm{ch}\, V^F_{G\times H}$
- In the case of the spin character identities all that is needed are:
  - dual Cauchy formula
  - expressions for classical group characters in terms of Schur functions [Littlewood 1940]
  - some modification rules [Newell 1951]

Spin characters and their decomposition

In terms of appropriate parameters

$$\mathrm{ch}\, V^{\Delta}_{O(2n)}(x,\overline{x}) = \prod_{i=1}^{n} (x_i^{\frac{1}{2}} + x_i^{-\frac{1}{2}}) = \mathbf{x}^{-\frac{1}{2}} \prod_{i=1}^{n} (1 + x_i)$$

$$\mathrm{ch}\, V^{\Delta}_{O(4np)}(xy, x\overline{y}, \overline{x}y, \overline{xy})$$

$$= \prod_{i=1}^{n}\prod_{j=1}^{p} (x_i^{\frac{1}{2}} y_j^{\frac{1}{2}} + x_i^{-\frac{1}{2}} y_j^{-\frac{1}{2}})(x_i^{\frac{1}{2}} y_j^{-\frac{1}{2}} + x_i^{-\frac{1}{2}} y_j^{\frac{1}{2}})$$

$$= \prod_{i=1}^{n}\prod_{j=1}^{p} (x_i + \overline{x}_i + y_j + \overline{y}_j)$$

$$= \mathbf{x}^{-p} \prod_{i=1}^{n}\prod_{j=1}^{p} (1 + x_i y_j)(1 + x_i \overline{y}_j) = \mathbf{x}^{-p} \sum_{\zeta \subseteq n^{2p}} s_{\zeta'}(x)\; s_{\zeta}(y,\overline{y})$$

Application to Howe dual pair contd.

$$= \mathbf{x}^{-p} \sum_{\zeta \subseteq n^{2p}} s_{\zeta'}(x)\, s_\zeta(y,\overline{y}) = \mathbf{x}^{-p} \sum_{\zeta \subseteq n^{2p}} s_{\zeta'}(x)\, \mathrm{ch}\, V^{\zeta}_{GL(2p)}(y,\overline{y})$$

$$= \mathbf{x}^{-p} \sum_{\zeta \subseteq n^{2p}} s_{\zeta'}(x) \sum_{\beta:\beta'\, even} \mathrm{ch}\, V^{\zeta/\beta}_{Sp(2p)}(y,\overline{y})$$

$$= \mathbf{x}^{-p} \sum_{\eta \subseteq n^{2p}} \mathcal{W}_{2p}\left(\sum_{\beta:\beta'\, even} s_{\eta'}(x)\, s_{\beta'}(x) \right) \mathrm{ch}\, V^{\eta}_{Sp(2p)}(y,\overline{y})$$

$$= \mathbf{x}^{-p} \sum_{\eta \subseteq n^{2p}} \mathcal{W}_{2p}\left(\sum_{\delta\, even} s_{\eta'}(x)\, s_{\delta}(x) \right) \mathrm{ch}\, V^{\eta}_{Sp(2p)}(y,\overline{y})$$

$$= \sum_{\lambda \subseteq n^p} \mathrm{ch}\, V^{\lambda^\dagger}_{Sp(2n)}(x,\overline{x})\ \ \mathrm{ch}\, V^{\lambda}_{Sp(2p)}(y,\overline{y})$$ dual pair Theorem

where $\mathcal{W}_{2p}$ restricts any sum of Schur functions $s_\nu(x)$ to those having $\nu_1 = \ell(\nu') \leq 2p$

Character formula

It follows that

$$\mathrm{ch}\, V^{\lambda^\dagger}_{Sp(2n)}(x,\overline{x}) = \mathbf{x}^{-p} \sum_{\eta\subseteq n^{2p}} \varepsilon_{\eta,\lambda}\, \mathcal{W}_{2p}\Bigg(\sum_{\delta\; even} s_{\eta'}(x)\, s_\delta(x)\Bigg)$$

where the modification rules for $Sp(2p)$ characters are such that

$$\varepsilon_{\eta,\lambda} = \begin{cases} \pm 1 & \text{if } \mathrm{ch}\, V^{\eta}_{Sp(2p)}(y,\overline{y}) = \pm\,\mathrm{ch}\, V^{\lambda}_{Sp(2p)}(y,\overline{y}) \\ 0 & \text{otherwise} \end{cases}$$

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To be more precise [K and Wybourne 00]

$$\mathrm{ch}\, V^{\lambda^\dagger}_{Sp(2n)}(x,\overline{x}) = \mathbf{x}^{-p} \sum_{\alpha\in\mathcal{P}_{-1}} \sum_{\delta\, even} (-1)^{|\alpha|/2}\, \mathcal{W}_{2p}\Big(s_{(\lambda,\alpha)'}(x)\, s_\delta(x)\Big)$$

Character formula

- Here $(\lambda, \alpha) = (\lambda_1, \ldots, \lambda_p, \alpha_1 \ldots, \alpha_p)$
- Standardisation is necessary if $\lambda_p < \alpha_1$
- For given λ only a finite number of terms $\alpha \in \mathcal{P}_{-1}$ give non-zero contributions

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Example

- If $\lambda = 0$ then only the case $\alpha = 0$ survives. In this case $\lambda^\dagger = (p^n)$ and

$$\begin{aligned} \operatorname{ch} V^{p^n}_{Sp(2n)}(x, \overline{x}) &= \mathbf{x}^{-p}\, \mathcal{W}_{2p}\Bigg(\sum_{\delta\, even} s_\delta(x) \Bigg) \\ &= \mathbf{x}^{-p} \sum_{\delta\, even : \ell(\delta') \leq 2p} s_\delta(x) \qquad \text{as before} \end{aligned}$$

Character formula

- Howe duality thus leads directly to a formula for one of the row length restricted Schur function series
- It involves a character of rectangular shape, since $\lambda^\dagger = p^n$

Character formula

- Howe duality thus leads directly to a formula for one of the row length restricted Schur function series
- It involves a character of rectangular shape, since $\lambda^\dagger = p^n$
- If $\lambda = m$ then only the case $\alpha = 0$ survives. In this case $\lambda^\dagger = p^n/1^m = (p^{n-m}, (p-1)^m)$ and

$$
\begin{aligned}
\mathrm{ch}\, V_{Sp(2n)}^{p^{n-m},(p-1)^m}(x,\overline{x}) &= \mathbf{x}^{-p}\, \mathcal{W}_{2p}\left(\sum_{\delta\ even} s_{1^m}(x)\ s_\delta(x) \right) \\
&= \mathbf{x}^{-p} \sum_{\mu\in(2p)^n:oddrows(\mu)=p} s_\mu(x)
\end{aligned}
$$

- This is a formula for a character of near rectangular shape, previously derived by Krattenthaler [98]

Character formula

- If $\lambda = 1^m$ then two terms survive. In this case $\lambda^\dagger = p^n/m = (p^{n-1}, p-m)$ and

$$\mathrm{ch}\, V_{Sp(2n)}^{p^{n-1},p-m}(x,\overline{x})$$
$$= \mathbf{x}^{-p}\, \mathcal{W}_{2p}\left(\sum_{\delta\, even} \left(s_m(x) - s_{2p+2-m}(x)\right)\, s_\delta(x)\right)$$

- This gives another character of near rectangular shape
- Some care is required to effect the cancellations necessary to express the character as a sum of wholly positive terms, see [Krattenthaler 98]
- Further examples can easily be generated, but they involve more complicated cancellations

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- Further examples can easily be generated, but they involve more complicated cancellations

The spin module and Howe dual pairs

- Thus we have recovered the formula for the symplectic group characters as a sum of row length restricted Schur functions specified by even partitions
- Similar formulae for orthogonal group characters may be recovered in the same way using Howe dual pairs
- In each case the row length restriction owes its origin to the bijective correspondence between irreps of the dual groups specified by $\lambda^\dagger$ and λ

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- Thus we have recovered the formula for the symplectic group characters as a sum of row length restricted Schur functions specified by even partitions
- Similar formulae for orthogonal group characters may be recovered in the same way using Howe dual pairs
- In each case the row length restriction owes its origin to the bijective correspondence between irreps of the dual groups specified by $\lambda^\dagger$ and λ
- We would like to identify other Howe dual pairs that might lead to characters expressible as our sums of column length restricted Schur functions
- Such characters are necessarily infinite dimensional

The metaplectic module and Howe dual pairs

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Ex: [Howe 89] For V the metaplectic irrep of a symplectic group with character $\mathrm{ch}\, V^{\tilde{\Delta}}$, dual pairs are defined through each of the following restrictions:

$$\begin{aligned} Sp(4np) &\supseteq Sp(2n) \times O(2p) \\ Sp(4np+2p) &\supseteq Sp(2n) \times O(2p+1) \\ Sp(4np) &\supseteq SO(2n) \times Sp(2p) \end{aligned}$$

Metaplectic dual pair character formula

Theorem [Moshinsky and Quesne 71, Kashiwara and Vergne 78, Howe 85, K and Wybourne 85]

On restriction to the appropriate subgroup:

$$\mathrm{ch}\, V^{\tilde{\Delta}}_{Sp(4np)} = \sum_{\lambda:\lambda'_1+\lambda'_2\leq 2p,\, \lambda'_1\leq n} \mathrm{ch}\, V^{p(\lambda)}_{Sp(2n)}\, \mathrm{ch}\, V^{\lambda}_{O(2p)}$$

$$\mathrm{ch}\, V^{\tilde{\Delta}}_{Sp(4np+2n)} = \sum_{\lambda:\lambda'_1+\lambda'_2\leq 2p+1,\, \lambda'_1\leq n} \mathrm{ch}\, V^{p+\frac{1}{2}(\lambda)}_{Sp(2n)}\, \mathrm{ch}\, V^{\lambda}_{O(2p+1)}$$

$$\mathrm{ch}\, V^{\tilde{\Delta}}_{Sp(4np)} = \sum_{\lambda:\lambda'_1\leq \min(p,n)} \mathrm{ch}\, V^{p(\lambda)}_{SO(2n)}\, \mathrm{ch}\, V^{\lambda}_{Sp(2p)}$$

Metaplectic characters and their decomposition

In terms of appropriate parameters

$$\mathrm{ch}\, V^{\tilde{\Delta}}_{Sp(2n)}(x,\overline{x}) = \prod_{i=1}^{n} (x_i^{-\frac{1}{2}} - x_i^{\frac{1}{2}})^{-1} = \mathbf{x}^{\frac{1}{2}} \prod_{i=1}^{n} (1-x_i)^{-1}$$

$$\mathrm{ch}\, V^{\tilde{\Delta}}_{Sp(4np)}(xy, x\overline{y}, \overline{x}y, \overline{xy})$$

$$= \prod_{i=1}^{n}\prod_{j=1}^{p} (x_i^{-\frac{1}{2}} y_j^{-\frac{1}{2}} - x_i^{\frac{1}{2}} y_j^{\frac{1}{2}})^{-1} (x_i^{-\frac{1}{2}} y_j^{\frac{1}{2}} - x_i^{\frac{1}{2}} y_j^{-\frac{1}{2}})^{-1}$$

$$= \mathbf{x}^{p} \prod_{i=1}^{n}\prod_{j=1}^{p} (1 - x_i y_j)^{-1} (1 - x_i \overline{y}_j)^{-1}$$

$$= \mathbf{x}^{p} \sum_{\zeta:\ell(\zeta)\le \min(n,2p)} s_\zeta(x)\; s_\zeta(y,\overline{y})$$

Application to Howe dual pair contd.

$$= \mathbf{x}^p \sum_{\zeta:\ell(\zeta)\le\min(n,2p)} s_\zeta(x)\; s_\zeta(y,\overline{y})$$

$$= \mathbf{x}^p \sum_{\zeta:\ell(\zeta)\le\min(n,2p)} s_\zeta(x)\; \mathrm{ch}\, V^{\zeta}_{GL(2p)}(y,\overline{y})$$

$$= \mathbf{x}^p \sum_{\zeta:\ell(\zeta)\le\min(n,2p)} s_\zeta(x)\; \sum_{\delta\, even} \mathrm{ch}\, V^{\zeta/\delta}_{O(2p)}(y,\overline{y})$$

$$= \mathbf{x}^p \sum_{\eta:\ell(\eta)\le\min(n,2p)} \mathcal{L}_{2p}\left(\sum_{\delta\, even} s_\eta(x)\, s_\delta(x)\right) \mathrm{ch}\, V^{\eta}_{O(2p)}(y,\overline{y})$$

$$= \sum_{\lambda:\lambda'_1+\lambda'_2\le 2p,\, \lambda'_1\le n} \mathrm{ch}\, V^{p(\lambda)}_{Sp(2n)}(x,\overline{x})\;\; \mathrm{ch}\, V^{\lambda}_{O(2p)}(y,\overline{y}) \quad \text{dual pair}$$

where $\mathcal{L}_{2p}$ restricts any sum of Schur functions $s_\nu(x)$ to those having $\nu'_1 = \ell(\nu) \le 2p$

Character formula

It follows that

$$\mathrm{ch}\, V^{p(\lambda)}_{Sp(2n)}(x,\overline{x}) = \ \mathbf{x}^p \sum_{\eta:\ell(\zeta)\leq\min(n,2p)} \varepsilon_{\eta,\lambda}\, \mathcal{L}_{2p}\Bigg(\sum_{\delta\; even} s_\eta(x)\, s_\delta(x)\Bigg)$$

where the modification rules for $O(2p)$ characters are such that

$$\varepsilon_{\eta,\lambda} = \begin{cases} \pm 1 & \text{if } \ \mathrm{ch}\, V^{\eta}_{O(2p)}(y,\overline{y}) = \pm\,\mathrm{ch}\, V^{\lambda}_{O(2p)}(y,\overline{y}) \\ 0 & \text{otherwise} \end{cases}$$

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In the special case $\lambda = 0$ this gives

$$\mathrm{ch}\, V^{p(0)}_{Sp(2n)}(x,\overline{x}) = \ \mathbf{x}^p\, \mathcal{L}_{2p}\Big(\sum_{\delta\ even} s_\delta(x)\Big) = \ \mathbf{x}^p \sum_{\delta\ even:\ell(\delta)\leq 2p} s_\delta(x)$$

The metaplectic module and Howe dual pairs

- Thus we have obtained a formula for a particular symplectic group character as a sum of column length restricted Schur functions specified by even partitions
- Our other column length restricted Schur function formula may be also be identifed with characters in the same way
- In each case the column length restriction owes its origin to the bijective correspondence between irreps of the dual groups specified by $p(\lambda)$ and λ

Dual pairs in spin modules

- The spin modules Δ of $O(N)$ give rise to the following dual pairs of subgroups $G \times H$:

$$\begin{aligned}
O(4np) &\supseteq SO(2n) \times O(2p) \\
O(4np+2p) &\supseteq SO(2n+1) \times O(2p) \\
O(4np+2n) &\supseteq SO(2n) \times O(2p+1) \\
O(4np+2n+2p+1) &\supseteq SO(2n+1) \times O(2p+1) \\
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- The dual pairs may be found by
  - verifying that the actions of G and H mutually centralize one another
  - determining multiplicity free common highest weight vectors of G and H [Hasegawa 89]

Dual pairs in spin modules

- Each dual pair gives rise to an identity of characters of the form $\operatorname{ch} V^{\Delta}_{O(N)} = \sum_{k\in K} \operatorname{ch} V^{\lambda(k)}_{G} \operatorname{ch} V^{\mu(k)}_{H}$
- Such identities have been derived by
  - Using the Laplace expansion of $\operatorname{ch} V^{\Delta}_{O(N)}$
    - orthogonal subgroup case [Morris 58, 61]
    - symplectic subgroup case [Bump and Gamburd 05]
  - Using a Robinson-Schensted-Knuth-Berele procedure in the symplectic subgroup case [Terada 91]

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- Here, in the symplectic subgroup case, we offer an alternative derivation based on a jeu-de-taquin procedure

Semistandard Young tableaux

- Let $\mathcal{T}^{\lambda}(n)$ be the set of $gl(n)$-tableaux T obtained by filling the boxes of F^{λ} with entries from $\{1 < 2 < \ldots < n\}$ such that they
  - T1 weakly increase across each row from left to right;
  - T2 strictly increase down each column from top to bottom;

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- Ex: For $n = 6$, $\lambda = (3,3,2)$ we have

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 4 & 4 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \quad \in \mathcal{T}^{332}(6)$$

Schur functions and tableaux

- For $x = (x_1, x_2, \ldots, x_n)$ and any $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ let $x^\kappa = x_1^{\kappa_1}\, x_2^{\kappa_2}\, \cdots\, x_n^{\kappa_n}$
- Then

$$\mathrm{ch}\, V^\lambda_{GL(n)} = s_\lambda(x) = \sum_{T \in \mathcal{T}^\lambda(n)} x^{\mathrm{wgt}\,(T)}$$

where $\mathrm{wgt}\,(T)_k \;=\; \#k \in T$ for $k = 1, 2, \ldots, n$

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Symplectic tableaux

- Let $\mathcal{S}pT^{\lambda}(n)$ be the set of $sp(2n)$-tableaux T obtained by filling the boxes of F^{λ} with entries from $\{\overline{1} < 1 < \overline{2} < 2 < \cdots < \overline{n} < n\}$ such that they

  S1 weakly increase across each row from left to right;

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  S3 k and $\overline{k}$ appear no lower than the kth row.

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- Ex: For $n = 4$, $\lambda = (3,3,2,1)$

$$T = \begin{array}{|c|c|c|}\hline \overline{1} & \overline{2} & \overline{3} \\ \hline 2 & 3 & 3 \\ \hline \overline{3} & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \quad \in \mathcal{S}pT^{3321}(4)$$

Symplectic characters and tableaux

- Let $x = (x_1, x_2, \ldots, x_n)$ and $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ with $\overline{x}_k = x_k^{-1}$ for $k = 1, 2, \ldots, n$
- Then

$$\mathrm{ch}\, V^{\lambda}_{Sp(2n)} = sp_{\lambda}(x, \overline{x}) = \sum_{T \in \mathcal{S}pT^{\lambda}(n)} x^{\mathrm{wgt}\,(T)}$$

where $\mathrm{wgt}\,(T)_k \; = \; \#k \in T - \#\overline{k} \in T$ for $k = 1, 2, \ldots, n$

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Dual pair identity

- The identity to be proved takes the form

$$\mathrm{ch}\, V^{\Delta}_{O(4np)} = \sum_{\lambda \subseteq n^p} \mathrm{ch}\, V^{\lambda^\dagger}_{Sp(2n)}\, \mathrm{ch}\, V^{\lambda}_{Sp(2p)} = \sum_{\lambda \subseteq p^n} \mathrm{ch}\, V^{\lambda}_{Sp(2n)}\, \mathrm{ch}\, V^{\lambda^\dagger}_{Sp(2p)}$$

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- where $\quad \mathrm{ch}\, V^{\lambda}_{Sp(2n)}\, \mathrm{ch}\, V^{\lambda^\dagger}_{Sp(2p)} = sp_\lambda(x,\overline{x})\; sp_{\lambda^\dagger}(y,\overline{y})$

- and

$$\begin{aligned}
&\mathrm{ch}\, V^{\Delta}_{O(4np)}(xy, x\overline{y}, \overline{x}y, \overline{xy}) \\
&= \prod_{i=1}^{n}\prod_{j=1}^{p} (x_i^{\frac12} y_j^{\frac12} + x_i^{-\frac12} y_j^{-\frac12})(x_i^{\frac12} y_j^{-\frac12} + x_i^{-\frac12} y_j^{\frac12}) \\
&= \prod_{i=1}^{n}\prod_{j=1}^{p} (x_i + \overline{x}_i + y_j + \overline{y}_j)
\end{aligned}$$

Pairs of symplectic tableau

- Let $\mathcal{R}(n,p)$ be the set of tableaux $R=(TS^\dagger)$ composed, for some $\lambda\subseteq(p^n)$, of $T\in\mathcal{S}pT^\lambda(n)$ and $S\in\mathcal{S}pT^{\lambda^\dagger}(p)$ reoriented so as to constitute a rectangular tableaux of shape $F^{(p^n)}$

- Ex: $n=4$, $p=5$, $\lambda=(3,3,2,1)$, $\lambda^\dagger=(4,4,2,1,0)$

$$T=\begin{array}{|c|c|c|}\hline \overline{1} & \overline{2} & \overline{3} \\ \hline 2 & 3 & 3 \\ \hline \overline{3} & 4 & \multicolumn{1}{|c}{} \\ \cline{1-2} 4 & \multicolumn{1}{|c}{} & \\ \cline{1-1} \end{array} \qquad S=\begin{array}{|c|c|c|c|}\hline \overline{1}' & 1' & 1' & 2' \\ \hline \overline{2}' & \overline{4}' & \overline{4}' & 4' \\ \hline \overline{4}' & 4' & \multicolumn{1}{|c}{} & \\ \cline{1-2} 5' & \multicolumn{1}{|c}{} & & \\ \cline{1-1} \end{array} \qquad R=\begin{array}{|c|c|c|c|c|}\hline \overline{1} & \overline{2} & \overline{3} & 4' & 2' \\ \hline 2 & 3 & 3 & \overline{4}' & 1' \\ \hline \overline{3} & 4 & 4' & \overline{4}' & 1' \\ \hline 4 & 5' & \overline{4}' & \overline{2}' & \overline{1}' \\ \hline \end{array}$$

Observation

$$\sum_{\lambda \subseteq p^n} sp_\lambda(x, \overline{x}) \; sp_{\lambda^\dagger}(y, \overline{y})$$

$$= \sum_{\lambda \subseteq p^n} \sum_{T \in \mathcal{S}pT^{\lambda}(n)} x^{\mathrm{wgt}\,(T)} \sum_{S \in \mathcal{S}pT^{\lambda^\dagger}(n)} y^{\mathrm{wgt}\,(S)}$$

$$= \sum_{R \in \mathcal{R}(n,p)} \big(x\, y\big)^{\mathrm{wgt}\,(R)}$$

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$$= \sum_{\lambda\subseteq p^n}\ \sum_{T\in\mathcal{S}pT^\lambda(n)} x^{\mathrm{wgt}\,(T)} \sum_{S\in\mathcal{S}pT^{\lambda^\dagger}(n)} y^{\mathrm{wgt}\,(S)}$$

$$= \sum_{R\in\mathcal{R}(n,p)} (x\,y)^{\mathrm{wgt}\,(R)}$$

- Ex: $n=4,\ p=5,\ \lambda=(3,3,2,1),\ \lambda^\dagger=(4,4,2,1,0)$

$$R = \begin{array}{|c|c|c|c|c|}\hline \overline{1} & \overline{2} & \overline{3} & 4' & 2' \\ \hline 2 & 3 & 3 & \overline{4}' & 1' \\ \hline \overline{3} & 4 & 4' & \overline{4}' & 1' \\ \hline 4 & 5' & \overline{4}' & \overline{2}' & \overline{1}' \\ \hline \end{array} \qquad (x\,y)^{\mathrm{wgt}\,(R)} = x_1^{-1}\,x_4^2\,y_1\,y_4^{-1}\,y_5$$

New rectangular tableaux

- Let $\mathcal{D}(n,p)$ be the set of tableaux D obtained by filling the boxes of $F^{(p^n)}$ with entries from $\{\overline{1} < 1 < \overline{2} < \cdots < \overline{n} < n < \overline{1}' < 1' < \overline{2}' < \cdots < \overline{p}' < p'\}$ in such a way that:

  D1 each unprimed entry k or $\overline{k}$ lies in the kth row counted from top to bottom;

  D2 each primed entry k' or $\overline{k}'$ lies in the kth column counted from right to left.

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  D1 each unprimed entry k or $\overline{k}$ lies in the kth row counted from top to bottom;

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- Typically $D = \begin{array}{|c|c|c|c|c|} \hline \overline{1} & 1 & \overline{1} & 2' & 1' \\ \hline 5' & 4' & 2 & \overline{2}' & \overline{2} \\ \hline \overline{3} & \overline{4}' & 3 & 2' & 1' \\ \hline 4 & \overline{4}' & 4 & \overline{2}' & \overline{1}' \\ \hline \end{array} \in \mathcal{D}(4,5)$

Metaplectic character

$$\prod_{i=1}^{n} \prod_{j=1}^{p} (x_i + \overline{x}_i + y_j + \overline{y}_j) = \sum_{D \in \mathcal{D}(n,p)} (x\, y)^{\mathrm{wgt}\,(D)}$$

- $(x, y) = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_p)$
- $\mathrm{wgt}\,(D)_i = \#k - \#\overline{k}$ for $i = k$ with $k = 1, 2, \ldots, n$
- $\mathrm{wgt}\,(D)_i = \#k' - \#\overline{k}'$ for $i = n + k$ with $k = 1, 2, \ldots, p$

Metaplectic character

$$\prod_{i=1}^{n}\prod_{j=1}^{p}(x_i+\overline{x}_i+y_j+\overline{y}_j)=\sum_{D\in\mathcal{D}(n,p)}(x\,y)^{\mathrm{wgt}\,(D)}$$

- $(x,y)=(x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_p)$
- $\mathrm{wgt}\,(D)_i=\#k-\#\overline{k}$ for $i=k$ with $k=1,2,\ldots,n$
- $\mathrm{wgt}\,(D)_i=\#k'-\#\overline{k}'$ for $i=n+k$ with $k=1,2,\ldots,p$
- Ex: $D=\begin{array}{|c|c|c|c|c|c}\cline{1-5} \overline{1} & 1 & \overline{1} & 2' & 1' & -1 \\ \cline{1-5} 5' & 4' & 2 & \overline{2}' & \overline{2} & 0 \\ \cline{1-5} \overline{3} & \overline{4}' & 3 & 2' & 1' & 0 \\ \cline{1-5} 4 & \overline{4}' & 4 & \overline{2}' & \overline{1}' & 2 \\ \cline{1-5} \multicolumn{1}{c}{1} & \multicolumn{1}{c}{-1} & \multicolumn{1}{c}{0} & \multicolumn{1}{c}{0} & \multicolumn{1}{c}{1} & \end{array}\Rightarrow\begin{array}{l}(x,y)^{\mathrm{wgt}\,(D)}\\ =x_1^{-1}\,x_4^{2}\,y_1\,y_4^{-1}\,y_5\end{array}$
- Note: Entry in the (i,j)th box is any one of $\{i,\overline{i}',j',\overline{j}'\}$

Lemma

Lemma For all $n, p \in \mathbb{N}$

$$\sum_{R \in \mathcal{R}(n,p)} (x\, y)^{\mathrm{wgt}\,(R)} = \sum_{D \in \mathcal{D}(n,p)} (x\, y)^{\mathrm{wgt}\,(D)}$$

Lemma

Lemma For all $n, p \in \mathbb{N}$

$$\sum_{R \in \mathcal{R}(n,p)} (x\, y)^{\mathrm{wgt}\,(R)} = \sum_{D \in \mathcal{D}(n,p)} (x\, y)^{\mathrm{wgt}\,(D)}$$

- Construct a weight preserving bijection between $\mathcal{R}(n,p)$ and $\mathcal{D}(n,p)$
- Use jeu de taquin to map each $R \in \mathcal{R}(n,p)$ to corresponding $D \in \mathcal{D}(n,p)$
- Move each primed entry k' or $\overline{k'}$ north-west to its own column, the kth, and then north while moving each unprimed entry i or $\overline{i}$ to its own row, the ith.
- To right of kth column maintain S1-S3 and S1†-S3†

Legitimate moves for k'

- k' in position (i, j) with $i > 1$ and $j < k$

$$\begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{b} \\ \hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{k'} \\ \hline \end{array} \quad \Leftrightarrow \quad \left\{ \begin{array}{ll} \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{k'} \\ \hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{b} \\ \hline \end{array} & \text{if} \quad a \leq b \\ \\ \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{b} \\ \hline \multicolumn{1}{|c|}{k'} & \multicolumn{1}{c|}{a} \\ \hline \end{array} & \text{if} \quad a > b \end{array} \right.$$

Legitimate moves for k'

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$$\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & b \\ \hline a & k' \\ \hline\end{array} \quad \Leftrightarrow \quad \begin{cases} \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & k' \\ \hline a & b \\ \hline\end{array} & \text{if} \quad a \leq b \\[2em] \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & b \\ \hline k' & a \\ \hline\end{array} & \text{if} \quad a > b \end{cases}$$

- k' in position $(1,j)$ with $j < k$

$$\begin{array}{|c|c|}\hline a & k' \\ \hline\end{array} \quad \Leftrightarrow \quad \begin{array}{|c|c|}\hline k' & a \\ \hline\end{array}$$

Legitimate moves for k'

- k' in position (i,k) with $i>1$

$$\begin{array}{|c|}\hline b \\ \hline k' \\ \hline\end{array} \quad \Leftrightarrow \quad \begin{array}{|c|}\hline k' \\ \hline b \\ \hline\end{array} \quad \text{if} \quad b \leq i$$

Legitimate moves for k'

- k' in position (i,k) with $i>1$

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- Note

$\begin{array}{|c|}\hline k' \\ \hline k' \\ \hline\end{array}$ allowed by S1† $\quad$ $\begin{array}{|c|c|}\hline k' & k' \\ \hline\end{array}$ forbidden by S2†

Legitimate moves for $\overline{k}'$

- $\overline{k}'$ in position (i,j) with $i>1$ and $j<k$

$$\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & b \\ \hline a & \overline{k}' \\ \hline\end{array} \quad \Leftrightarrow \quad \begin{cases} \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & \overline{k}' \\ \hline a & b \\ \hline\end{array} & \text{if} \quad a \le b \\[2em] \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & b \\ \hline \overline{k}' & a \\ \hline\end{array} & \text{if} \quad a > b \end{cases}$$

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- $\overline{k}'$ in position $(1,j)$ with $j<k$

$$\begin{array}{|c|c|}\hline a & \overline{k}' \\ \hline\end{array} \quad \Leftrightarrow \quad \begin{array}{|c|c|}\hline \overline{k}' & a \\ \hline\end{array}$$

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$$\begin{array}{|c|}\hline b \\ \hline \overline{k}' \\ \hline \end{array} \quad \Leftrightarrow \quad \begin{array}{|c|}\hline \overline{k}' \\ \hline b \\ \hline \end{array} \quad \text{if} \quad b \le i$$

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$$\begin{array}{|c|}\hline \overline{k}' \\ \hline \overline{k}' \\ \hline \end{array}$$ allowed by S1† $$\begin{array}{|c|c|}\hline \overline{k}' & \overline{k}' \\ \hline \end{array}$$ forbidden by S2†

No transformations necessary

- Note

$$\boxed{i \mid i} \quad \text{and} \quad \boxed{\bar{i} \mid \bar{i}} \quad \text{allowed by S1}$$

No transformations necessary

- Note

$$\begin{array}{|c|c|}\hline i & i \\ \hline\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline \bar{i} & \bar{i} \\ \hline\end{array} \quad \text{allowed by S1}$$

- Note

$$\begin{array}{|c|}\hline i \\ \hline i \\ \hline\end{array} \quad \text{and} \quad \begin{array}{|c|}\hline \bar{i} \\ \hline \bar{i} \\ \hline\end{array} \quad \text{forbidden by S2}$$

Weight preserving transformations

- k' in position (i,k) so that k' is in kth column, but blocks $\overline{k}'$ from moving to kth column

$$\boxed{k'}\boxed{\overline{k}'} \quad \Leftrightarrow \quad \boxed{i}\boxed{\overline{i}}$$

Weight preserving transformations

- k' in position (i,k) so that k' is in kth column, but blocks $\overline{k}'$ from moving to kth column

$$\begin{array}{|c|c|}\hline k' & \overline{k}' \\ \hline \end{array} \quad \Leftrightarrow \quad \begin{array}{|c|c|}\hline i & \overline{i} \\ \hline \end{array}$$

- i in position (i,k) so that i is in ith row, but blocks $\overline{i}$ from moving to ith row

$$\begin{array}{|c|}\hline \overline{i} \\ \hline i \\ \hline \end{array} \quad \Leftrightarrow \quad \begin{array}{|c|}\hline \overline{k}' \\ \hline k' \\ \hline \end{array}$$

Map from $R \in \mathcal{R}(n,p)$ to $D \in \mathcal{D}(n,p)$

- Identify largest primed entries. Move topmost such entry, k' or $\overline{k}'$, North-West by a sequence of interchanges with nearest neighbours until it reaches kth column and then North as far as possible in this column, while moving unprimed entries, i or $\overline{i}$, South to the ith row and changing any vertical pair $\overline{i}\,i$ to $\overline{k}'\,k'$.

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$\overline{1}$	$\overline{2}$	$\overline{3}$	$4'$	$2'$
2	3	3	$\overline{4}'$	$1'$
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$\overline{1}$	1	$\overline{1}$	$\overline{2}$	$2'$
$5'$	$4'$	2	$\overline{3}$	$1'$
$\overline{3}$	3	3	$\overline{4}'$	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

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$$\begin{array}{|c|c|c|c|c|}
\hline
\overline{1} & 1 & \overline{1} & \overline{2} & 2' \\
\hline
5' & 4' & 2 & \overline{3} & 1' \\
\hline
\overline{3} & 3 & \overline{4}' & 3 & 1' \\
\hline
4 & 4 & \overline{4}' & \overline{2}' & \overline{1}' \\
\hline
\end{array}$$

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- Identify largest primed entries. Move topmost such entry, k' or $\overline{k}'$, North-West by a sequence of interchanges with nearest neighbours until it reaches kth column and then North as far as possible in this column, while moving unprimed entries, i or $\overline{i}$, South to the ith row and changing any vertical pair $\overline{i}\,i$ to $\overline{k}'\,k'$.

$\overline{1}$	1	$\overline{1}$	$\overline{2}$	$2'$
$5'$	$4'$	2	$\overline{3}$	$1'$
$\overline{3}$	$\overline{4}'$	3	3	$1'$
4	4	$\overline{4}'$	$\overline{2}'$	$\overline{1}'$

Map from $R \in \mathcal{R}(n,p)$ to $D \in \mathcal{D}(n,p)$

- Identify largest primed entries. Move topmost such entry, k' or $\overline{k}'$, North-West by a sequence of interchanges with nearest neighbours until it reaches kth column and then North as far as possible in this column, while moving unprimed entries, i or $\overline{i}$, South to the ith row and changing any vertical pair $\overline{i}\,i$ to $\overline{k}'\,k'$.

$$\begin{array}{|c|c|c|c|c|}
\hline
\overline{1} & 1 & \overline{1} & \overline{2} & 2' \\
\hline
5' & 4' & 2 & \overline{3} & 1' \\
\hline
\overline{3} & \overline{4}' & 3 & 3 & 1' \\
\hline
4 & 4 & \overline{4}' & \overline{2}' & \overline{1}' \\
\hline
\end{array}$$

Map from $R \in \mathcal{R}(n,p)$ to $D \in \mathcal{D}(n,p)$

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Bijection

- Thus we have a map from $R \in \mathcal{R}(n,p)$ to $D \in \mathcal{D}(n,p)$ illustrated by:

$$R = \begin{array}{|c|c|c|c|c|}\hline \overline{1} & \overline{2} & \overline{3} & 4' & 2' \\ \hline 2 & 3 & 3 & \overline{4}' & 1' \\ \hline \overline{3} & 4 & 4' & \overline{4}' & 1' \\ \hline 4 & 5' & \overline{4}' & \overline{2}' & \overline{1}' \\ \hline \end{array} \quad \Leftrightarrow \quad \begin{array}{|c|c|c|c|c|}\hline \overline{1} & 1 & \overline{1} & 2' & 1' \\ \hline 5' & 4' & 2 & \overline{2}' & \overline{2} \\ \hline \overline{3} & \overline{4}' & 3 & 2' & 1' \\ \hline 4 & \overline{4}' & 4 & \overline{2}' & \overline{1}' \\ \hline \end{array} = D$$

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- Every step is reversible - the map is bijective
- The map is weight preserving
- Hence our dual pair character identity is proven

Skew Young diagrams

- Given partitions λ and μ such that all boxes of F^μ are contained in F^λ we write $\mu \subseteq \lambda$.
- Removing the boxes of F^μ from F^λ leaves the skew Young diagram $F^{\lambda/\mu}$
- Ex: $\lambda = (5,4,2)$, $\mu = (3,1)$, $\quad F^{\lambda/\mu} = \begin{array}{ccccc} * & * & * & \square & \square \\ * & \square & \square & \square & \\ \square & \square & & & \end{array}$

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- Let $\mathcal{T}^{\lambda/\mu}(n)$ be the set of $gl(n)$-tableaux T obtained by filling the boxes of $F^{\lambda/\mu}$ with entries from $\{1 < 2 < \ldots < n\}$ such that they

  T1 weakly increase across each row from left to right

  T2 strictly increase down each column from top to bottom

Skew Schur function

- For $x = (x_1, x_2, \ldots, x_n)$ with $n \in \mathbb{N}$

$$s_{\lambda/\mu}(x) = \sum_{T \in \mathcal{T}^{\lambda/\mu}(n)} x^{\mathrm{wgt}\,(T)}$$

where $\mathrm{wgt}\,(T)_k \;=\; \#k \in T$ for $k = 1, 2, \ldots, n$

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- Ex: $n = 6$, $\lambda = (5,4,2)$, $\mu = (3,1)$

$$T^{\lambda/\mu} = \begin{array}{ccccc} * & * & * & \boxed{2} & \boxed{3} \\ * & \boxed{1} & \boxed{4} & \boxed{4} & \\ \boxed{1} & \boxed{5} & & & \end{array} \qquad x^{\mathrm{wgt}\,(T)} = x_1^2\, x_2\, x_3\, x_4^2\, x_5$$

Schur function expansion

- For $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_n)$ with $m, n \in \mathbb{N}$

$$s_\lambda(x, y) = \sum_\mu s_\mu(x)\ s_{\lambda/\mu}(y)$$

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$$s_\lambda(x, y) = \sum_\mu s_\mu(x)\; s_{\lambda/\mu}(y)$$

- Ex: $m = 4$, $n = 6$, $\lambda = (5, 4, 2)$, $\mu = (3, 1)$

1	3	3	2	3
4	1	4	4	
1	5			

$$(x\,y)^{\mathrm{wgt}\,(T)} = x_1\, x_3^2\, x_4\; y_1^2\, y_2\, y_3\, y_4^2\, y_5$$

Cauchy formula and its inverse

- Let m, n be positive integers
- Then for all $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$

$$\sum_{\lambda} s_\lambda(x)\, s_\lambda(y) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 - x_i\, y_j)^{-1}$$

$$\sum_{\lambda} (-1)^{|\lambda|}\, s_\lambda(x)\, s_{\lambda'}(y) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 - x_i\, y_j)$$

Cauchy formula and its inverse

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$$\sum_{\lambda} (-1)^{|\lambda|}\, s_\lambda(x)\, s_{\lambda'}(y) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 - x_i\, y_j)$$

- The first sum over λ is infinite with non-zero terms arising for all $\ell(\lambda) \leq \min\{m, n\}$, and no restriction on $\ell(\lambda')$
- The second sum over λ is finite, with $\lambda \subseteq n^m$, since $s_\lambda(x) = 0$ if $\ell(\lambda) > m$ and $s_{\lambda'}(y) = 0$ if $\ell(\lambda') > n$

Determinantal identity

- For $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_n)$ with $m, n \in \mathbb{N}$

$$\sum_{\lambda} (-1)^{|\lambda|} \, s_\lambda(x) \, s_{\lambda'}(y) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 - x_i \, y_j)$$

$$= \frac{1}{\left|x_i^{m-j}\right| \; \left|y_a^{n-b}\right|} \cdot \begin{vmatrix} y_{n+1-i}^{j-1} \\ \cdots \\ x_{i-n}^{m+n-j} \end{vmatrix}$$

- The $(m+n) \times (m+n)$ determinant is partitioned after the nth row

Determinantal identity

- For $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_n)$ with $m, n \in \mathbb{N}$

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- The $(m+n) \times (m+n)$ determinant is partitioned after the nth row

- Proof Use either Laplace expansion to obtain Schur functions directly, or three Vandermonde identities to obtain product form

Row length restricted Cauchy formula

Theorem [Kwon 08, Hamel and K. 08]

Let $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$ with $m, n \geq 1$.

Then for all $p \geq 0$ we have

$$\sum_{\lambda : \ell(\lambda') \leq p} s_\lambda(x)\, s_\lambda(y) = (y_1\, y_2\, \cdots\, y_n)^p\, s_{p^n}(x, \overline{y})$$

$$= \frac{1}{\left|x_i^{m-j}\right|\ \left|y_a^{n-b}\right|\ \prod_{i=1}^m \prod_{a=1}^n (1 - x_i y_a)} \cdot \begin{vmatrix} y_{n+1-i}^{\,j-1+\chi_{j>n}p} \\ \cdots \\ x_{i-n}^{\,m+n-j+\chi_{j\leq n}p} \end{vmatrix}$$

$$= \frac{1}{\prod_{i=1}^m \prod_{a=1}^n (1 - x_i y_a)} \cdot \sum_{\zeta \subseteq n^m} (-1)^{|\zeta|}\, s_\sigma(x)\, s_\tau(y)$$

where $\sigma = (\zeta + p^r)$ and $\tau = (\zeta' + p^r)$ with $r = r(\zeta)$

Proof

- For $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_n)$ and $p \in \mathbb{N}$

$$s_{p^n}(x, \overline{y}) = \sum_{\zeta \subseteq p^n} s_\zeta(x)\, s_{p^n/\zeta}(y) = \sum_{T \in \mathcal{T}^{p^n}(m+n)} (x\,\overline{y})^{\mathrm{wgt}\,(T)}$$

Proof

- For $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_n)$ and $p \in \mathbb{N}$

$$s_{p^n}(x, \overline{y}) = \sum_{\zeta \subseteq p^n} s_\zeta(x)\, s_{p^n/\zeta}(y) = \sum_{T \in \mathcal{T}^{p^n}(m+n)} (x\,\overline{y})^{\mathrm{wgt}\,(T)}$$

- Ex: Typically, for $m = 6$, $n = 4$, $p = 5$, and order $1 < 2 < 3 < 4 < 5 < 6 < \overline{4} < \overline{3} < \overline{2} < \overline{1}$ we have

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & \overline{4} \\ \hline 2 & 4 & 4 & \overline{4} & \overline{3} \\ \hline 5 & \overline{4} & \overline{4} & \overline{2} & \overline{2} \\ \hline \overline{3} & \overline{2} & \overline{1} & \overline{1} & \overline{1} \\ \hline \end{array}$$

Proof

- However, separating the blue entries from the red entries and taking the complement of each column of the latter with respect to $1\,2\,3\,4$ gives

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 5 & \bar{4} \\ \hline 2 & 4 & 4 & \bar{4} & \bar{3} \\ \hline 5 & \bar{4} & \bar{4} & \bar{2} & \bar{2} \\ \hline \bar{3} & \bar{2} & \bar{1} & \bar{1} & \bar{1} \\ \hline \end{array} \Leftrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 2 & 4 & 4 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array} \cdot \begin{array}{|c|c|c|c|c|}\hline \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} \\ \hline \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} \\ \hline \bar{3} & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\ \hline \bar{4} & \bar{4} & \bar{4} & \bar{4} & \bar{4} \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 \\ \cline{1-1} \end{array}$$

Proof

- However, separating the blue entries from the red entries and taking the complement of each column of the latter with respect to $1\,2\,3\,4$ gives

$$
\begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 5 & \overline{4} \\ \hline
2 & 4 & 4 & \overline{4} & \overline{3} \\ \hline
5 & \overline{4} & \overline{4} & \overline{2} & \overline{2} \\ \hline
\overline{3} & \overline{2} & \overline{1} & \overline{1} & \overline{1} \\ \hline
\end{array}
\;\Leftrightarrow\;
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 5 \\ \hline
2 & 4 & 4 \\ \cline{1-3}
5 \\ \cline{1-1}
\end{array}
\cdot
\begin{array}{|c|c|c|c|c|}
\hline
\overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} \\ \hline
\overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} \\ \hline
\overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{3} \\ \hline
\overline{4} & \overline{4} & \overline{4} & \overline{4} & \overline{4} \\ \hline
\end{array}
\cdot
\begin{array}{|c|c|c|c|}
\hline
1 & 1 & 2 & 3 \\ \hline
2 & 3 & 3 \\ \cline{1-3}
4 \\ \cline{1-1}
\end{array}
$$

- Hence, setting $\mathbf{y} = y_1 y_2 \cdots y_n$ we have

$$
s_{p^n}(x, \overline{y}) = \mathbf{y}^{-p} \sum_{\zeta \subseteq p^n} s_\zeta(x)\; s_\zeta(y)
$$

Proof

It follows that $\displaystyle\sum_{\zeta:\ell(\zeta')\le p} s_\zeta(x)\, s_\zeta(y) = \mathbf{y}^p\, s_{p^n}(x,\overline{y})$

$$= \mathbf{y}^p \frac{\begin{vmatrix} \overline{y}_{n+1-i}^{m+n-j+p} & \vdots & \overline{y}_{n+1-i}^{m+n-j} \\ \cdots & \cdots & \cdots \\ x_{i-n}^{m+n-j+p} & \vdots & x_{i-n}^{m+n-j} \end{vmatrix}}{\begin{vmatrix} \overline{y}_{n+1-i}^{m+n-j} & \vdots & \overline{y}_{n+1-i}^{m+n-j} \\ \cdots & \cdots & \cdots \\ x_{i-n}^{m+n-j} & \vdots & x_{i-n}^{m+n-j} \end{vmatrix}} = \frac{\begin{vmatrix} y_{n+1-i}^{j-1} & \vdots & y_{n+1-i}^{j-1+p} \\ \cdots & \cdots & \cdots \\ x_{i-n}^{m+n-j+p} & \vdots & x_{i-n}^{m+n-j} \end{vmatrix}}{\begin{vmatrix} y_{n+1-i}^{j-1} \\ \cdots \\ x_{i-n}^{m+n-j} \end{vmatrix}}$$

$$= \frac{1}{\left|x_i^{m-j}\right| \; \left|y_a^{n-b}\right| \; \prod_{i=1}^{m}\prod_{a=1}^{n}(1-x_iy_a)} \cdot \begin{vmatrix} y_{n+1-i}^{j-1+\chi_{j>n}p} \\ \cdots \\ x_{i-n}^{m+n-j+\chi_{j\le n}p} \end{vmatrix}$$

Lemma [K. 2008]

- Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$
- Then for each pair of integers p and q we have

$$\frac{1}{\left|x_i^{m-j}\right| \; \left|y_i^{n-j}\right|} \cdot \left| \begin{array}{c:c} y_{n+1-i}^{j-1} & \chi_{\,j>n-q}\, y_{n+1-i}^{j-1+q} \\ \cdots & \cdots \\ \chi_{\,j\le n+p}\, x_{i-n}^{m+n-j+p} & x_{i-n}^{m+n-j} \end{array} \right|$$

$$= \sum_{\zeta \subseteq n^m} (-1)^{|\zeta|}\, s_{\zeta + p^{r(\zeta)}}(x)\, s_{\zeta' + q^{r(\zeta)}}(y)$$

- where the large determinant is $(m+n) \times (m+n)$, and is partitioned after the nth row and nth column

Lemma contd.

- If $\zeta = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \in (n^m)$
- with $a_1 < n$, $b_1 < m$ and $r = r(\zeta)$, then
  - $\zeta + p^r = \begin{pmatrix} a_1+p & a_2+p & \cdots & a_r+p \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$
  - $\zeta' + p^r = \begin{pmatrix} b_1+q & b_2+q & \cdots & b_r+q \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$
  - with $a_r \geq \max\{0, -p\}$ and $b_r \geq \max\{0, -q\}$

Lemma contd.

- If $\zeta = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \in (n^m)$
- with $a_1 < n$, $b_1 < m$ and $r = r(\zeta)$, then
  - $\zeta + p^r = \begin{pmatrix} a_1+p & a_2+p & \cdots & a_r+p \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$
  - $\zeta' + p^r = \begin{pmatrix} b_1+q & b_2+q & \cdots & b_r+q \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$
  - with $a_r \geq \max\{0, -p\}$ and $b_r \geq \max\{0, -q\}$

Proof: By Laplace expansion

Examples of key determinant

- $m = 3, \ n = 4, \ p = 2, \ q = 1$

Examples of key determinant

- $m = 3, \; n = 4, \; p = 2, \; q = 1$

$$\begin{vmatrix} 1 & y_4 & y_4^2 & y_4^3 & \vdots & y_4^5 & y_4^6 & y_4^7 \\ 1 & y_3 & y_3^2 & y_3^3 & \vdots & y_3^5 & y_3^6 & y_3^7 \\ 1 & y_2 & y_2^2 & y_2^3 & \vdots & y_2^5 & y_2^6 & y_2^7 \\ 1 & y_1 & y_1^2 & y_1^3 & \vdots & y_1^5 & y_1^6 & y_1^7 \\ \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ x_1^8 & x_1^7 & x_1^6 & x_1^5 & \vdots & x_1^2 & x_1 & 1 \\ x_2^8 & x_2^7 & x_2^6 & x_2^5 & \vdots & x_2^2 & x_2 & 1 \\ x_3^8 & x_3^7 & x_3^6 & x_3^5 & \vdots & x_3^2 & x_3 & 1 \end{vmatrix}$$

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Ex 1: $m = 3$**,** $n = 4$**,** $p = 2$**,** $q = 1$

Ex. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 5 & 7 & 1 & 4 & 6 \end{pmatrix}$

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$$= - \quad s_{4311}(y) \left| y_a^{4-b} \right| \cdot s_{641}(x) \left| x_i^{3-j} \right|$$

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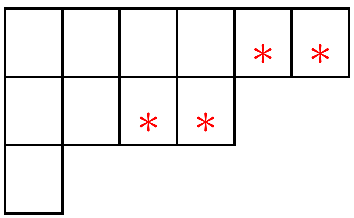
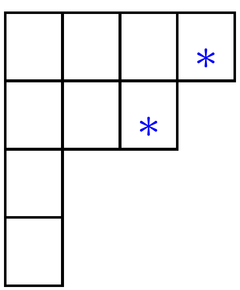
Ex. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 5 & 7 & 1 & 4 & 6 \end{pmatrix}$ $\quad (-1)^{\pi} = (-1)^{|\zeta|} = -1$

$$\zeta = (4,2,1) = \begin{pmatrix} 3 & 0 \\ 2 & 0 \end{pmatrix}$$



$$\sigma = \begin{pmatrix} 3+2 & 0+2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 0 \end{pmatrix} = (6,4,1)$$

$$\zeta' = (3,2,1,1) = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}$$



$$\tau = \begin{pmatrix} 2+1 & 0+1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 0 \end{pmatrix} = (4,3,1,1)$$

Ex 2: $m = 3$**,** $n = 4$**,** $p = -2$**,** $q = -1$

Ex. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 6 & 7 & 1 & 2 & 5 \end{pmatrix}$

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$$\begin{vmatrix} 1 & y_4 & y_4^2 & y_4^3 & \vdots & - & y_4^4 & y_4^5 \\ 1 & y_3 & y_3^2 & y_3^3 & \vdots & - & y_3^4 & y_3^5 \\ 1 & y_2 & y_2^2 & y_2^3 & \vdots & - & y_2^4 & y_2^5 \\ 1 & y_1 & y_1^2 & y_1^3 & \vdots & - & y_1^4 & y_1^5 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ x_1^4 & x_1^3 & - & - & \vdots & x_1^2 & x_1 & 1 \\ x_2^4 & x_2^3 & - & - & \vdots & x_2^2 & x_2 & 1 \\ x_3^4 & x_3^3 & - & - & \vdots & x_3^2 & x_3 & 1 \end{vmatrix} \sim - \begin{vmatrix} {y_4}^2 & y_4^3 & y_4^4 & y_4^5 \\ y_3^2 & y_3^3 & y_3^4 & y_3^5 \\ y_2^2 & y_2^3 & y_2^4 & y_2^5 \\ y_1^2 & y_1^3 & y_1^4 & y_1^5 \end{vmatrix} \cdot \begin{vmatrix} x_1^4 & x_1^3 & x_1^2 \\ x_2^4 & x_2^3 & x_2^2 \\ x_3^4 & x_3^3 & x_3^2 \end{vmatrix}$$

$$= - \quad s_{2222}(y) \ \left|y_a^{4-b}\right| \cdot s_{222}(x) \ \left|x_i^{3-j}\right|$$

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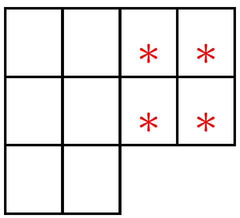
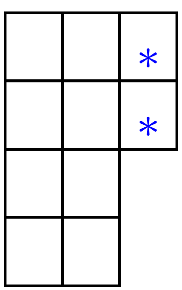


$$\sigma = \begin{pmatrix} 3-2 & 2-2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = (2, 2, 2)$$

$$\zeta' = (3, 3, 2, 2) = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$



$$\tau = \begin{pmatrix} 2-1 & 1-1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} = (2, 2, 2, 2)$$

Proof of second part of Theorem

- Just note that for $p \geq 0$

$$\left| \begin{array}{c} y_{n+1-i}^{j-1+\chi_{j>n} p} \\ \cdots \\ x_{i-n}^{m+n-j+\chi_{j\leq n} p} \end{array} \right| = \left| \begin{array}{ccc} y_{n+1-i}^{j-1} & \vdots & y_{n+1-i}^{j-1+p} \\ \cdots & & \cdots \\ x_{i-n}^{m+n-j+p} & \vdots & x_{i-n}^{m+n-j} \end{array} \right|$$

$$= \left| \begin{array}{ccc} y_{n+1-i}^{j-1} & \vdots & \chi_{j>n-p}\, y_{n+1-i}^{j-1+p} \\ \cdots & & \cdots \\ \chi_{j\leq n+p}\, x_{i-n}^{m+n-j+p} & \vdots & x_{i-n}^{m+n-j} \end{array} \right|$$

- Then use the Lemma with $p = q \geq 0$

Row length restricted Cauchy formula

- For all $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ and $p \geq 0$

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x)\; s_\lambda(y)$$

$$= \frac{1}{\prod_{i=1}^{m}\prod_{a=1}^{n}(1 - x_i y_a)} \cdot \sum_{\zeta}(-1)^{|\zeta|}\; s_{\zeta+p^r}(x)\; s_{\zeta'+p^r}(y)$$

$$= \sum_{\lambda} s_\lambda(x)\; s_\lambda(y) \;\cdot\; \sum_{\zeta}(-1)^{|\zeta|}\; s_{\zeta+p^r}(x)\; s_{\zeta'+p^r}(y)$$

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$$= \sum_\lambda s_\lambda(x)\; s_\lambda(y) \cdot \sum_\zeta (-1)^{|\zeta|}\; s_{\zeta+p^r}(x)\; s_{\zeta'+p^r}(y)$$

- This expresses the row length restricted series as a product of the unrestricted series times a correction factor

Column length restricted Cauchy formula

- Using the involutions $\omega_x : s_\lambda(x) \mapsto s_{\lambda'}(x)$ and $\omega_y : s_\lambda(y) \mapsto s_{\lambda'}(y)$ for all λ, either separately or together, we obtain three more restricted formula.

Column length restricted Cauchy formula

- Using the involutions $\omega_x : s_\lambda(x) \mapsto s_{\lambda'}(x)$ and $\omega_y : s_\lambda(y) \mapsto s_{\lambda'}(y)$ for all λ, either separately or together, we obtain three more restricted formula.
- Using $\omega_x\omega_y$ we find that for all x, y and for all $p \geq 0$

$$\sum_{\lambda:\ell(\lambda)\leq p} s_\lambda(x)\, s_\lambda(y)$$

$$= \sum_\lambda s_\lambda(x)\, s_\lambda(y) \cdot \sum_\zeta (-1)^{|\zeta|}\, s_{(\zeta+p^r)'}(x)\, s_{(\zeta'+p^r)'}(y)$$

$$= \sum_\lambda s_\lambda(x)\, s_\lambda(y) \cdot \sum_\eta (-1)^{|\eta|}\, s_{\eta-p^r}(x)\, s_{\eta'-p^r}(y)$$

where the sum over η is restricted to those η such that both $\eta - p^r$ and $\eta' - p^r$ are partitions

Column length restricted Cauchy formula

- Using our Lemma with both p and q set equal to $-p$, with $p \geq 0$ gives

Theorem Let $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$ with $m, n \geq 1$. Then for all $p \geq 0$ we have

$$\sum_{\lambda:\ell(\lambda)\leq p} s_\lambda(x)\, s_\lambda(y) \;=\; \frac{1}{\left|x_i^{m-j}\right|\;\left|y_a^{n-b}\right|\;\prod_{i=1}^{m}\prod_{a=1}^{n}(1 - x_i y_a)}$$

$$\cdot \begin{vmatrix} y_{n+1-i}^{j-1} & \vdots & \chi_{\,j>n+p}\; y_{n+1-i}^{j-1-p} \\ \cdots & & \cdots \\ \chi_{\,j\leq n-p}\; x_{i-n}^{m+n-j-p} & \vdots & x_{i-n}^{m+n-j} \end{vmatrix}$$

Generalisation to the supersymmetric case

- All our restricted row and column length formula involving symmetric functions may be generalised to the case of supersymmetric functions

Generalisation to the supersymmetric case

- All our restricted row and column length formula involving symmetric functions may be generalised to the case of supersymmetric functions
- Characters of Lie groups and algebras may be expressed in terms of symmetric Schur functions
- Characters of Lie supergroups and superalgebras may be expressed in terms of supersymmetric Schur functions

Supersymmetric functions

- Let $m,\ n$ be fixed positive integers
- Let $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$
- A function $f(x/y)$ is said to be supersymmetric if it is
  - symmetric under permutations of the x_i
  - symmetric under permutations of the y_j
  - independent of t if $x_i = t = -y_j$ for any i and j

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- For each partition λ the supersymmetric Schur function $s_\lambda(x/y)$ may be defined by

$$s_\lambda(x/y) = \sum_\mu s_\mu(x)\, s_{\lambda'/\mu'}(y)$$

Semistandard supertableaux

- Let $\mathcal{T}^\lambda(m/n)$ be the set of $gl(m/n)$-tableaux T obtained by filling the boxes of F^λ with entries from $\{1 < 2 < \ldots < n < 1' < 2' < \ldots < n'\}$ such that

  unprimed entries

  T1 weakly increase across each row from left to right

  T2 stricly increase down each column from top to bottom

  and primed entries

  T'1 strictly increase across each row from left to right

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- Ex: $m = 4,\ n = 6,\ \lambda = (5, 4, 2)$,

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 3 & 3' & 4' \\ \hline 4 & 1' & 2' & 3' \\ \cline{1-4} 1' & 5' \\ \cline{1-2}\end{array}$$

Supersymmetric Schur function

- Since $$s_\lambda(x/y) = \sum_\mu s_\mu(x)\, s_{\lambda'/\mu'}(y)$$

  with $$s_\mu(x) = \sum_{T\in\mathcal{T}^\mu(m)} x^{\mathrm{wgt}\,(T)}$$

  and $$s_{\lambda'/\mu'}(y) = \sum_{T\in\mathcal{T}^{\lambda'/\mu'}(n)} y^{\mathrm{wgt}\,(T)}$$

  we have $$s_\lambda(x/y) = \sum_{T\in\mathcal{T}^\lambda(m/n)} (x\,y)^{\mathrm{wgt}\,(T)}$$

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- Ex: $m=4$, $n=6$, $\lambda=(5,4,2)$

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 3 & 3' & 4' \\ \hline 4 & 1' & 2' & 3' \\ \cline{1-4} 1' & 5' \\ \cline{1-2}\end{array}$$

$$(x\,y)^{\mathrm{wgt}\,(T)} = x_1\, x_3^2\, x_4\, {y_1'}^2\, y_2'\, {y_3'}^2\, y_4'\, y_5'$$

Littlewood-Richardson coefficients

- Let $x = (x_1, \ldots, x_m)$ with $m \in \mathbb{N}$
- In Λ_m the ring of symmetric polynomial functions

$$s_\lambda(x)\, s_\mu(x) = \sum_\nu c^\nu_{\lambda\mu}\, s_\nu(x)$$

where the coefficients $c^\nu_{\lambda\mu}$ are non-negative integers – the Littlewood-Richardson coefficients

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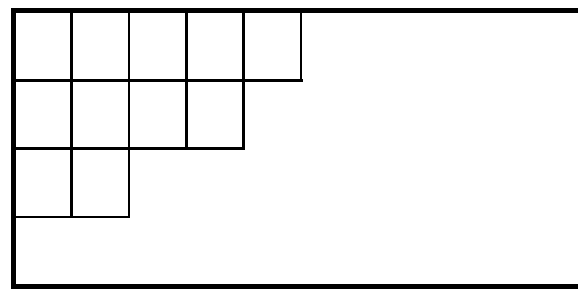
where the coefficients $c^\nu_{\lambda\mu}$ are non-negative integers – the Littlewood-Richardson coefficients
- Let $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$ with $m, n \in \mathbb{N}$
- In $\Lambda_{(m/n)}$ the ring of supersymmetric polynomial functions
$$s_\lambda(x/y)\, s_\mu(x/y) = \sum_\nu c^\nu_{\lambda\mu}\, s_\nu(x/y)$$
where the same Littlewood-Richardson coefficients occur.

Constraints on supersymmetric Schur functions

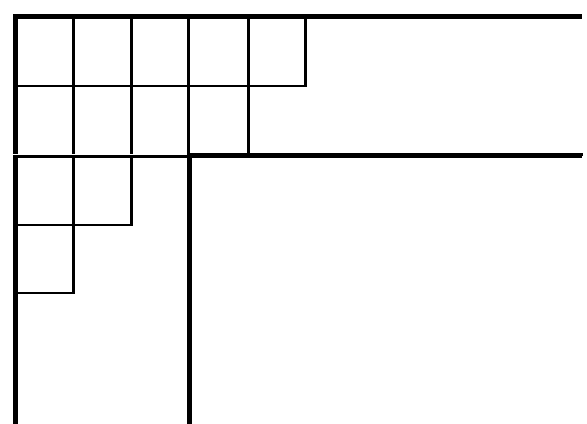
- Notice that $s_\nu(x) = s_\nu(x_1,\dots,x_m) = 0$ if $\lambda'_1 > m$ while $s_\nu(x/y) = s_\nu(x_1,\dots,x_m/y_1,\dots,y_n) = 0$ if $\lambda'_{n+1} > m$

Constraints on supersymmetric Schur functions

- Notice that $s_\nu(x) = s_\nu(x_1, \ldots, x_m) = 0$ if $\lambda'_1 > m$ while $s_\nu(x/y) = s_\nu(x_1, \ldots, x_m/y_1, \ldots, y_n) = 0$ if $\lambda'_{n+1} > m$
- that is $s_\nu(x) \neq 0$ iff F^ν lies within a horizontal strip of depth m

  Ex: $m = 4$ $F^\nu =$ 

- and $s_\nu(x/y) \neq 0$ iff F^ν lies within a hook with arm width m and leg width n

  Ex: $m = 2$, $n = 3$ $F^\nu =$ 

upersymmetric row and column restricted identiti

- With respect to the bases $s_\lambda(x)$ and $s_\lambda(x/y)$ the rings Λ_n and $\Lambda_{(m/n)}$ coincide modulo the horizontal strip and hook shape restrictions on λ
- It follows that any identity expressed in terms of Schur functions $s_\lambda(x)$ takes exactly the same form in terms of supersymmetric Schur functions $s_\lambda(x/y)$
- However, the generating functions for Schur function series require amendment for the corresponding supersymmetric Schur function series

Supersymmetric Schur function series

$$\sum_{\lambda} s_\lambda(x/y)$$

$$= \frac{\prod_i \prod_a (1 + x_i y_a)}{\prod_i (1 - x_i) \prod_{j<k} (1 - x_j\, x_k) \prod_a (1 - y_a) \prod_{b<c} (1 - y_b\, y_c)}$$

$$\sum_{\lambda:\ell(\lambda')\leq p} s_\lambda(x/y)$$

$$= \frac{\prod_i \prod_a (1 + x_i y_a) \sum_{\mu\in\mathcal{P}_p} (-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x/y)}{\prod_i (1 - x_i) \prod_{j<k} (1 - x_j\, x_k) \prod_a (1 - y_a) \prod_{b<c} (1 - y_b\, y_c)}$$

$$\sum_{\lambda:\ell(\lambda)\leq p} s_\lambda(x/y)$$

$$= \frac{\prod_i \prod_a (1 + x_i y_a) \sum_{\mu\in\mathcal{P}_{-p}} (-1)^{[|\mu|-r(\mu)(p-1)]/2}\, s_\mu(x/y)}{\prod_i (1 - x_i) \prod_{j<k} (1 - x_j\, x_k) \prod_a (1 - y_a) \prod_{b<c} (1 - y_b\, y_c)}$$

Supersymmetric Schur function series

$$\sum_{\lambda \text{ even}} s_\lambda(x/y) = \frac{\prod_i \prod_a (1 + x_i y_a)}{\prod_{j \le k}(1 - x_j\, x_k)\ \prod_{b<c}(1 - y_b\, y_c)}$$

$$\sum_{\lambda \text{ even}\,:\ell(\lambda') \le 2p} s_\lambda(x/y)$$

$$= \frac{\prod_i \prod_a (1 + x_i y_a)\ \sum_{\mu \in \mathcal{P}_{2p+1}} (-1)^{[|\mu| - r(\mu)(2p)]/2}\, s_\mu(x/y)}{\prod_{j \le k}(1 - x_j\, x_k)\ \prod_{b<c}(1 - y_b\, y_c)}$$

$$\sum_{\lambda' \text{ even}\,:\ell(\lambda) \le 2p} s_\lambda(x/y)$$

$$= \frac{\prod_i \prod_a (1 + x_i y_a)\ \sum_{\mu \in \mathcal{P}_{-2p-1}} (-1)^{[|\mu| - r(\mu)(2p)]/2}\, s_\mu(x/y)}{\prod_{j < k}(1 - x_j\, x_k)\ \prod_{b \le c}(1 - y_b\, y_c)}$$

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$$= \frac{\prod_i \prod_a (1 + x_i y_a)\ \sum_{\mu\in\mathcal{P}_{p-1}:r(\mu)\text{even}} (-1)^{[|\mu|-r(\mu)p]/2}\, s_\mu(x/y)}{\prod_{j<k}(1 - x_j\, x_k)\ \prod_{b\le c}(1 - y_b\, y_c)}$$

$$\sum_{\lambda \text{ even}\,:\ell(\lambda)\le p} s_\lambda(x/y)$$

$$= \frac{\prod_i \prod_a (1 + x_i y_a)\ \sum_{\mu\in\mathcal{P}_{-p+1}:r(\mu)\text{even}} (-1)^{[|\mu|-r(\mu)p]/2}\, s_\mu(x/y)}{\prod_{j\le k}(1 - x_j\, x_k)\ \prod_{b<c}(1 - y_b\, y_c)}$$

Supersymmetric form of the Cauchy identities

- Let $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_d)$, $\mathbf{w} = (w_1, \ldots, w_e)$ with $m, n, d, e \in \mathbb{N}$, then

$$\sum_{\lambda} s_\lambda(x/y)\, s_\lambda(z/\mathbf{w}) = \frac{\prod_{j,l}(1 + x_i\, w_l)\ \prod_{j,k}(1 + y_j\, z_k)}{\prod_{i,k}(1 - x_i\, z_k)\ \prod_{j,l}(1 - y_j\, w_l)}$$

$$\sum_{\lambda:\ell(\lambda')\le p} s_\lambda(x/y)\, s_\lambda(z/\mathbf{w}) = \sum_{\lambda} s_\lambda(x/y)\, s_\lambda(z/\mathbf{w}) \cdot \sum_{\zeta} (-1)^{|\zeta|}\, s_{\zeta+p^r}(x/y)\, s_{\zeta'+p^r}(z/\mathbf{w})$$

$$\sum_{\lambda:\ell(\lambda)\le p} s_\lambda(x/y)\, s_\lambda(z/\mathbf{w}) = \sum_{\lambda} s_\lambda(x/y)\, s_\lambda(z/\mathbf{w}) \cdot \sum_{\eta} (-1)^{|\eta|}\, s_{\eta-p^r}(x/y)\, s_{\eta'-p^r}(z/\mathbf{w})$$

Dual pairs of Lie supergroups

- Howe's original work on dual pairs encompassed Lie supergroups, such as $GL(m/n)$ and $OSp(m/n)$
- Thus all our supersymmetric identities should be placed within this context
- They may be derived from the following dual pairs [Cheng and Zhang 04, Kwon 08]

Dual pairs of Lie supergroups

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Ex: Dual pairs supercentralising one another in the given module

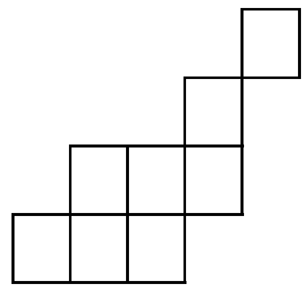
$$\begin{aligned}
\mathcal{S}(\mathbb{C}^{m/n} \otimes \mathbb{C}^{d/e}) &\quad : \quad GL(m/n) \times GL(d/e) \\
\Lambda(\mathbb{C}^{m/n} \otimes \mathbb{C}^{d/e}) &\quad : \quad GL(m/n) \times GL(d/e) \\
\mathcal{S}(\mathbb{C}^{m/n} \otimes \mathbb{C}^{d}) &\quad : \quad OSp(m/n) \times O(d) \\
\mathcal{S}(\mathbb{C}^{m/n} \otimes \mathbb{C}^{d}) &\quad : \quad OSp(m/n) \times Sp(d)
\end{aligned}$$

Jacobi-Trudi identities

- So far we have discussed infinite series of Schur functions, including their expression in determinantal form
- These have been used to provide generalisations of formulae of both Littlewood and Cauchy
- All the formulae have arisen from the expression of a Schur function as ratio of two alternants
- It is natural to ask if similar results can be obtained from the expression of a Schur function in Jacobi-Trudi form

Jacobi-Trudi identities

- For partitions λ and μ, we write $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i.
- If $\mu \subseteq \lambda$ then the skew Young diagram $F^{\lambda/\mu}$ is defined to be $F^{\lambda} \backslash F^{\mu}$.

Ex: $F^{5443/431} =$ 

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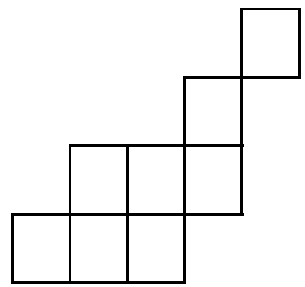
Ex: $F^{5443/431} =$ 

Schur functions:

$$s_\lambda(x) = \left| \, h_{\lambda_i - i + j}(x) \, \right| = \left| \, s_{\lambda_i - i + j}(x) \, \right| \, ,$$

Jacobi-Trudi identities

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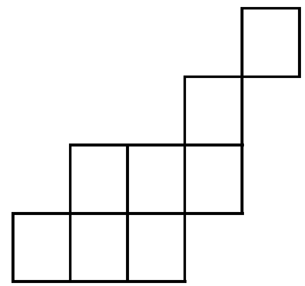
- If $\mu \subseteq \lambda$ then the skew Young diagram $F^{\lambda/\mu}$ is defined to be $F^{\lambda} \backslash F^{\mu}$.

Ex: $F^{5443/431} =$



Schur functions:

$$s_\lambda(x) = \left| h_{\lambda_i - i + j}(x) \right| = \left| s_{\lambda_i - i + j}(x) \right| ,$$

Skew Schur functions:

$$s_{\lambda/\mu}(x) = \left| h_{\lambda_i - \mu_j - i + j}(x) \right| = \left| s_{\lambda_i - \mu_j - i + j}(x) \right| ,$$

Bressoud-Wei identities

- Bressoud and Wei [1992]   For all integers $t \geq -1$:

$$2^{(t-|t|)/2} \left| h_{\lambda_i - i + j}(x) + (-1)^{(t+|t|)/2} h_{\lambda_i - i - j + 1 - t}(x) \right|$$

$$= \sum_{\sigma \in \mathcal{P}_t} (-1)^{[|\sigma| + r(\sigma)(|t|-1)]/2} s_{\lambda/\sigma}(x)$$

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$$= \sum_{\sigma \in \mathcal{P}_t} (-1)^{[|\sigma| + r(\sigma)(|t|-1)]/2} \; s_{\lambda/\sigma}(x)$$

- Hamel and K [2008] For all integers t and all q:

$$\left| \, h_{\lambda_i - i + j}(x) + q \; \chi_{j > -t} \; h_{\lambda_i - i - j + 1 - t}(x) \, \right|$$

$$= \sum_{\sigma \in \mathcal{P}_t} (-1)^{[|\sigma| - r(\sigma)(t+1)]/2} \; q^{r(\sigma)} \; s_{\lambda/\sigma}(x)$$

Algebraic proof

$$
\begin{aligned}
\left|\, h_{\lambda_i-i+j}(x) + q\ \chi_{j>-t}\ h_{\lambda_i-i-j+1-t}(x) \,\right| \\
&= \sum_{r=0}^{n} \sum_{\kappa} q^r \ \left|\, h_{\lambda_i-i+j-\kappa_j}(x) \,\right| \\
&= \sum_{\sigma\in\mathcal{P}_t} (-1)^{(j_r-1)+\cdots+(j_2-1)+(j_1-1)}\ q^r\ \left|\, h_{\lambda_i-i+j-\sigma_j}(x) \,\right|
\end{aligned}
$$

Algebraic proof

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&\qquad = \sum_{\sigma\in\mathcal{P}_t} (-1)^{(j_r-1)+\cdots+(j_2-1)+(j_1-1)}\ q^r\ \left|\, h_{\lambda_i-i+j-\sigma_j}(x)\,\right|
\end{aligned}
$$

- $\kappa_j = 2j-1+t$ for $j \in \{j_1, j_2, \ldots, j_r\}$ and $\kappa_j = 0$ otherwise
- with $n \geq j_1 > j_2 > \cdots > j_r \geq 1 - \chi_{t<0} t$
- $\sigma = \begin{pmatrix} j_1-1+t & j_2-1+t & \cdots & j_r-1+t \\ j_1-1 & j_2-1 & \cdots & j_r-1 \end{pmatrix} \in \mathcal{P}_t$
- $r = r(\sigma)$

Combinatorial proof

- Lattice path interpretation of determinant

$$\left| h_{\lambda_i - i + j}(x) + q\, \chi_{j > -t}\, h_{\lambda_i - i - j + 1 - t}(x) \right|$$
$$= \sum_{\pi \in S_n} (-1)^{\pi} \prod_{i=1}^{n} \left(h_{\lambda_i - i + \pi(i)}(x) + q\, \chi_{\pi(i) > -t}\, h_{\lambda_i - i - \pi(i) + 1 - t}(x) \right)$$

Combinatorial proof

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$$\left|\, h_{\lambda_i-i+j}(x) + q\,\chi_{j>-t}\,h_{\lambda_i-i-j+1-t}(x)\,\right|$$
$$= \sum_{\pi\in S_n}(-1)^{\pi}\prod_{i=1}^{n}\left(\,h_{\lambda_i-i+\pi(i)}(x) + q\,\chi_{\pi(i)>-t}\,h_{\lambda_i-i-\pi(i)+1-t}(x)\,\right)$$

- Each π defines a set of n-tuples of north-east paths
- For $i=1,2,\ldots,n$ the ith path goes
  - from $P_{\pi(i)}=(n+1-\pi(i),1)$ or $P'_{\pi(i)}=(n+t+\pi(i),1)$
  - to $Q_i=(n+1+\lambda_i-i,n)$
  - each step east at height k carries weight x_k
  - each path from $P'_{\pi(i)}$ (rather than $P_{\pi(i)}$) carries weight q

An n-tuple of lattice paths

- Ex.1 $\quad n = 4, \quad t = 2 \quad \lambda = (6, 4, 4, 2) \quad \pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3' & 1' & 2 & 4 \end{pmatrix}$
- Contribution $(-1)^{2+0}\, q^2\, (x_2)\, (1)\, (x_1 x_3^2)\, (x_3 x_4)$

An n-tuple of lattice paths

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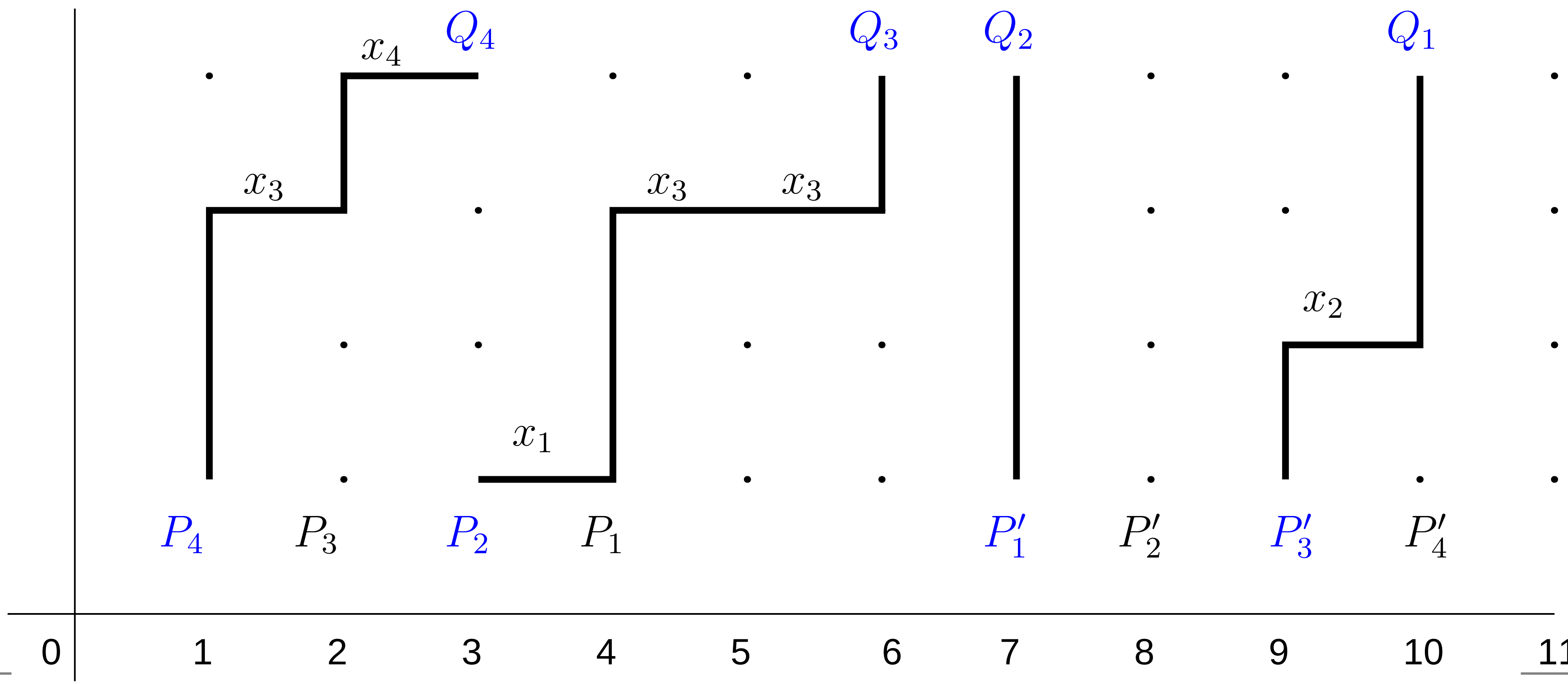
  $\lambda = (6, 4, 4, 2)$

- Contribution $(-1)^{2+0}\, q^2\, (x_2)\, (1)\, (x_1 x_3^2)\, (x_3 x_4)$



Combinatorial proof contd.

$$\sum_{\pi \in S_n} (-1)^\pi \prod_{i=1}^{n} \left(h_{\lambda_i - i + \pi(i)}(x) + q\, \chi_{\pi(i) > -t}\, h_{\lambda_i - i - \pi(i) + 1 - t}(x) \right)$$

Combinatorial proof contd.

$$\sum_{\pi \in S_n} (-1)^{\pi} \prod_{i=1}^{n} \left(h_{\lambda_i - i + \pi(i)}(x) + q\, \chi_{\pi(i) > -t}\, h_{\lambda_i - i - \pi(i) + 1 - t}(x) \right)$$

- Path from $P_{\pi(i)}$ to Q_i contributes to $h_{\lambda_i - i + \pi(i)}(x)$
- Path from $P'_{\pi(i)}$ to Q_i contributes to $h_{\lambda_i - i - \pi(i) + 1 - t}(x)$
- Sign changing involution removes contributions from intersecting paths
- All paths in n-tuple non-intersecting implies $\pi =$

$$\begin{pmatrix} 1 & 2 & \cdots & r & r+1 & r+2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(r) & \pi(r+1) & \pi(r+2) & \cdots & \pi(n) \end{pmatrix}$$

with $\pi(1) > \pi(2) > \cdots > \pi(r)$ for $P'_{\pi(i)} Q_i$ paths

and $\pi(r+1) < \pi(r+2) < \cdots < \pi(n)$ for $P_{\pi(i)} Q_i$ paths

Combinatorial proof contd.

- Eastward distance P_i to $Q_i = \lambda_i$ for $i = 1, \ldots, n$
- Let distance P_i to $P'_{\pi(i)} = \sigma_i$ for $i = 1, \ldots, r$
- Let distance P_i to $P_{\pi(i)} = \sigma_i$ for $i = r+1, \ldots, n$
- Then, in Frobenius notation

$$\sigma = \begin{pmatrix} \pi(1) - 1 + t & \pi(2) - 2 + t & \cdots & \pi(r) - r + t \\ \pi(1) - 1 & \pi(2) - 2 & \cdots & \pi(r) - r \end{pmatrix}$$

Combinatorial proof contd.

- Eastward distance P_i to $Q_i = \lambda_i$ for $i = 1, \ldots, n$
- Let distance P_i to $P'_{\pi(i)} = \sigma_i$ for $i = 1, \ldots, r$
- Let distance P_i to $P_{\pi(i)} = \sigma_i$ for $i = r+1, \ldots, n$
- Then, in Frobenius notation

$$\sigma = \begin{pmatrix} \pi(1)-1+t & \pi(2)-2+t & \cdots & \pi(r)-r+t \\ \pi(1)-1 & \pi(2)-2 & \cdots & \pi(r)-r \end{pmatrix}$$

- Now re-interpret ith path monomial as contribution to ith row of an $s_{\lambda/\sigma}(x)$ semistandard tableau, so that our determinant reduces to

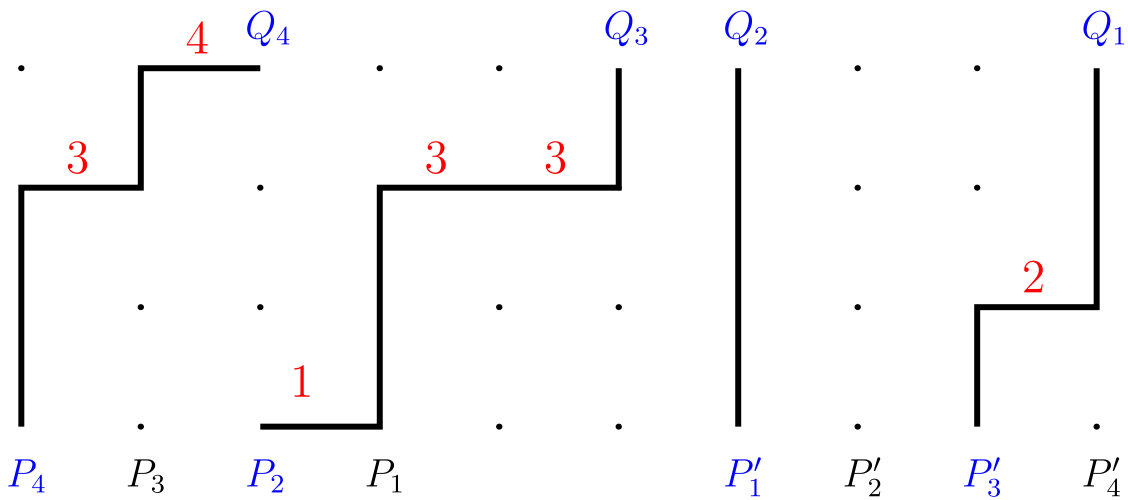
$$\sum_{\sigma \in \mathcal{P}_t} (-1)^{(\pi(r)-1)+\cdots+(\pi(2)-1)+(\pi(1)-1)} \, q^r \, \left| h_{\lambda_i - i + j - \sigma_j}(x) \right|$$

as required

Semistandard skew tableaux

- Each n-tuple of non-intersecting paths defines a semistandard skew tableaux

- Ex.1 $\quad n=4,\ \ t=2,\quad \lambda=(6,4,4,2)\qquad \pi=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3' & 1' & 2 & 4 \end{pmatrix}$



0	0	0	0	0	2
0	0	0	0		
0	1	3	3		
3	4				

Semistandard skew tableaux

- Each n-tuple of non-intersecting paths defines a semistandard skew tableaux

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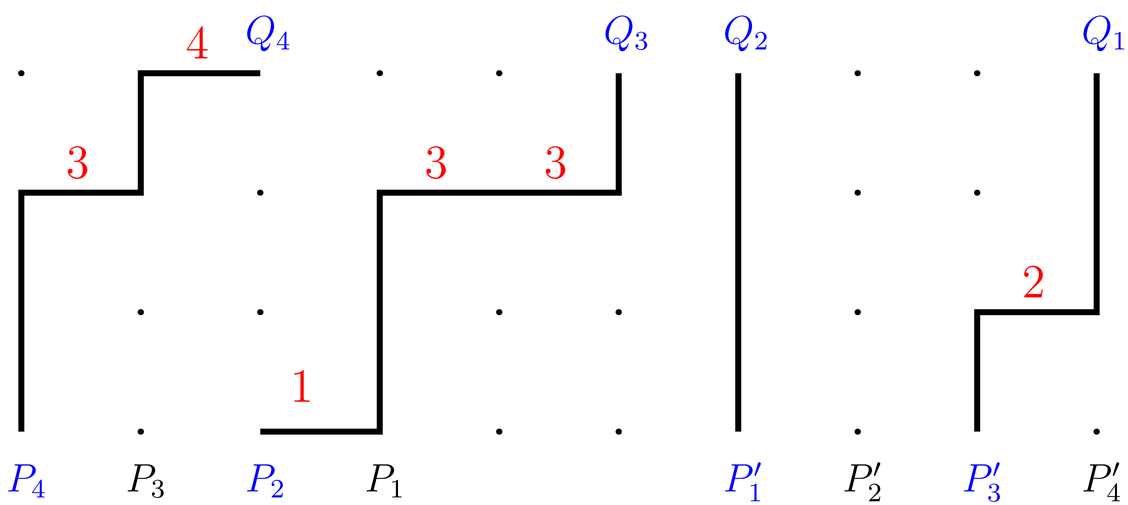

0	0	0	0	0	2
0	0	0	0		
0	1	3	3		
3	4				

- $\mu = (5,4,1) = \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix} \in \mathcal{P}_2$

An n-tuple of lattice paths

- Ex.2 $n=4, \quad t=-2$ $\quad \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5' & 3' & 1 & 2 & 4 & 6 \end{pmatrix}$

  $\lambda = (5,4,4,3,3,2)$
- Contribution $(-1)^{4+2}\, q^2\, (x_1x_6)\, (x_1x_2)\, (x_3^2)\, (x_4)\, (x_3x_6)\, (x_1x_5)$

An n-tuple of lattice paths

- Ex.2 $\quad n = 4, \quad t = -2 \qquad \lambda = (5,4,4,3,3,2) \qquad \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5' & 3' & 1 & 2 & 4 & 6 \end{pmatrix}$
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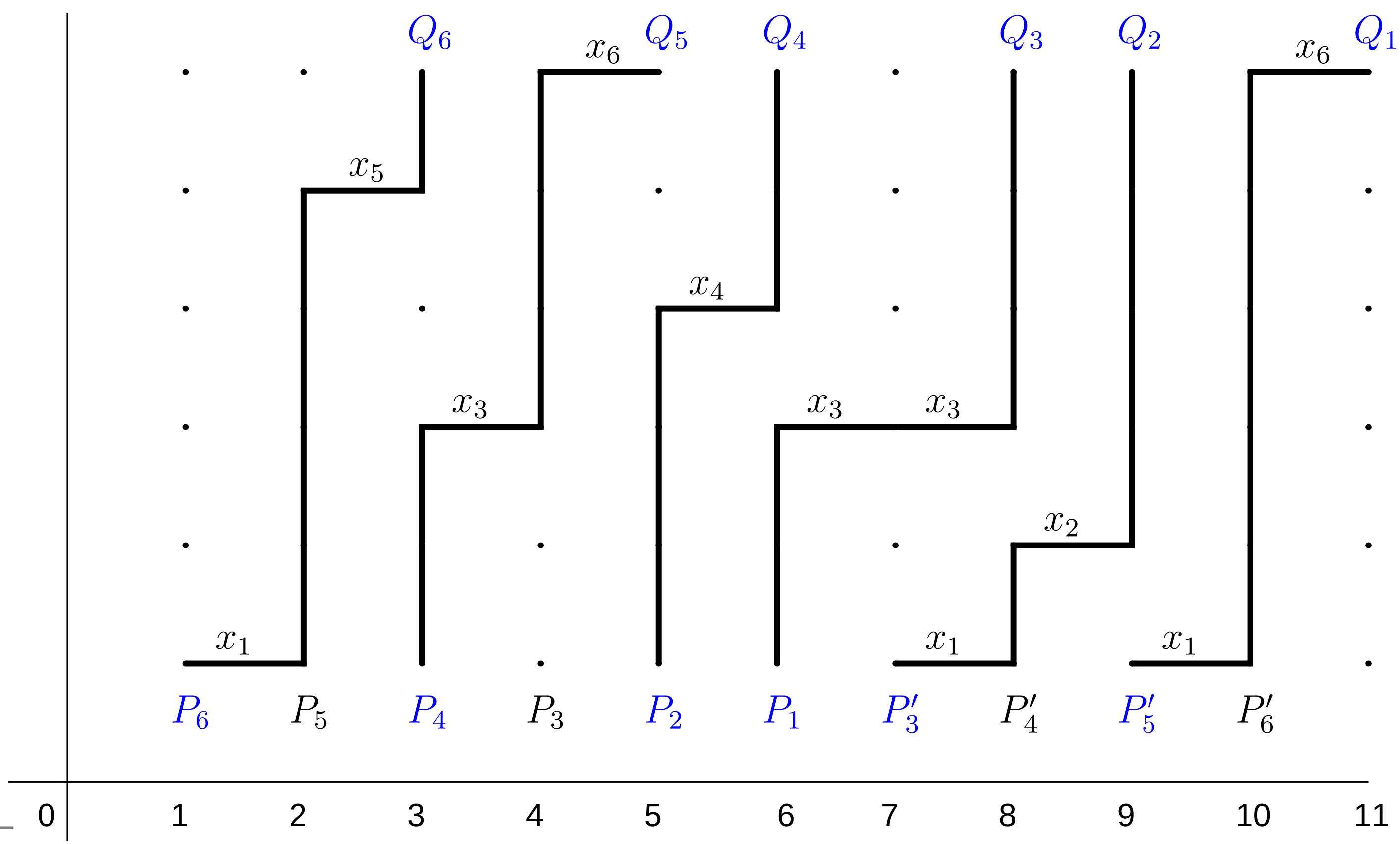

Semistandard skew tableaux

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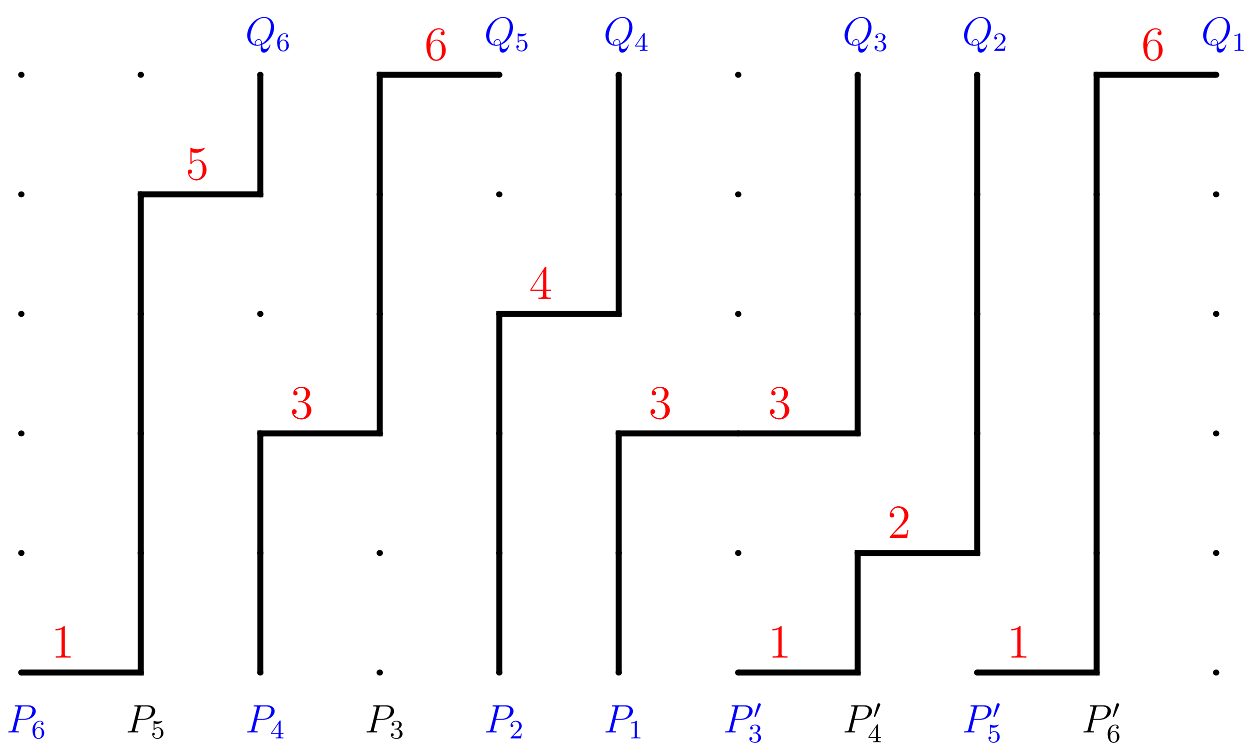
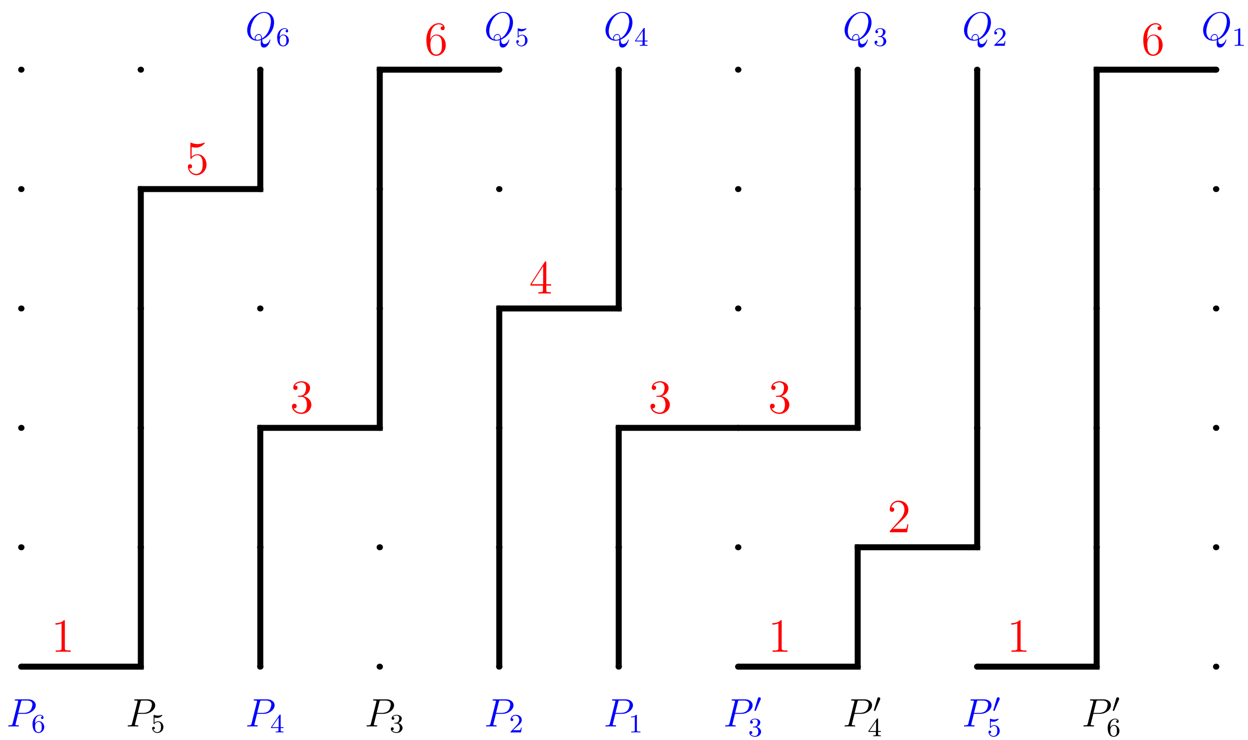


0	0	0	1	6
0	0	1	2	
0	0	3	3	
0	0	4		
0	3	6		
1	5			

Semistandard skew tableaux

- Ex.2 $\quad n = 4, \quad t = -2, \quad \lambda = (5, 4, 4, 3, 3, 2) \qquad \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5' & 3' & 1 & 2 & 4 & 6 \end{pmatrix}$



0	0	0	1	6
0	0	1	2	
0	0	3	3	
0	0	4		
0	3	6		
1	5			

- $\mu = (3, 2, 2, 2, 1) = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix} \in \mathcal{P}_{-2}$

Cauchy-type Jacobi-Trudi expansion

Theorem [Hamel and K, 2008]

- Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$
- Let λ and μ have lengths $\ell(\lambda) \leq m$ and $\ell(\mu) \leq n$
- Then for each pair of integers p and q we have

$$\left| \begin{array}{c|c} h_{\mu_{n+1-i}+i-j}(y) & \chi_{j>n-q}\, h_{\mu_{n+1-i}+i-j-q}(y) \\ \cdots & \cdots \\ \chi_{j\leq n+p}\, h_{\lambda_{i-n}-i+j-p}(x) & h_{\lambda_{i-n}-i+j}(x) \end{array} \right|$$

$$= \sum_{\zeta \subseteq n^m} (-1)^{|\zeta|}\, s_{\lambda/(\zeta+p^{r(\zeta)})}(x)\, s_{\mu/(\zeta'+q^{r(\zeta)})}(y)$$

Cauchy-type extension

- where the determinant is $(m+n) \times (m+n)$, and is partitioned after the nth row and nth column
- and if $\zeta = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \in (n^m)$
- with $a_1 < n$, $b_1 < m$ and $r = r(\zeta)$, then
  - $\zeta + p^r = \begin{pmatrix} a_1{+}p & a_2{+}p & \cdots & a_r{+}p \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$
  - $\zeta' + q^r = \begin{pmatrix} b_1{+}q & b_2{+}q & \cdots & b_r{+}q \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$
  - with $a_r \geq \max\{0, -p\}$ and $b_r \geq \max\{0, -q\}$

Example

- $m = 3$, $n = 4$, $p = -2$, $q = -1$ $\lambda = (5, 3, 2)$, $\mu = (4, 3, 2, 2)$
- Let $\{k\} = h_k(x)$ and $\{k\} = h_k(y)$ for all integers k

Example

- $m = 3$, $n = 4$, $p = -2$, $q = -1$ $\lambda = (5, 3, 2)$, $\mu = (4, 3, 2, 2)$
- Let $\{k\} = h_k(x)$ and $\{k\} = h_k(y)$ for all integers k

$$
\left|\begin{array}{cccc:cccc}
\{2\} & \{1\} & \{0\} & - & - & - & - \\
\{3\} & \{2\} & \{1\} & \{0\} & - & - & - \\
\{5\} & \{4\} & \{3\} & \{2\} & - & \{1\} & \{0\} \\
\{7\} & \{6\} & \{5\} & \{4\} & - & \{3\} & \{2\} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\{3\} & \{4\} & - & - & \{5\} & \{6\} & \{7\} \\
\{0\} & \{1\} & - & - & \{2\} & \{3\} & \{4\} \\
- & - & - & - & \{0\} & \{1\} & \{2\}
\end{array}\right|
$$

Typical term in Laplace expansion

- $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 6 & 2 & 5 & 7 \end{pmatrix} \quad (-1)^{\pi} = (-1)^{0+1+1+2} = +1$

Typical term in Laplace expansion

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- $$\begin{vmatrix} \{2\} & \{0\} & - & - \\ \{3\} & \{1\} & \{0\} & - \\ \{5\} & \{3\} & \{2\} & \{1\} \\ \{7\} & \{5\} & \{4\} & \{3\} \end{vmatrix} \times \begin{vmatrix} \{4\} & \{5\} & \{7\} \\ \{1\} & \{2\} & \{4\} \\ - & \{0\} & \{2\} \end{vmatrix}$$

Typical term in Laplace expansion

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- $\begin{vmatrix} \{2\} & \{0\} & - & - \\ \{3\} & \{1\} & \{0\} & - \\ \{5\} & \{3\} & \{2\} & \{1\} \\ \{7\} & \{5\} & \{4\} & \{3\} \end{vmatrix} \times \begin{vmatrix} \{4\} & \{5\} & \{7\} \\ \{1\} & \{2\} & \{4\} \\ - & \{0\} & \{2\} \end{vmatrix}$

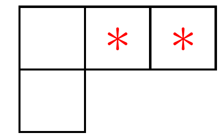
- $(\zeta'_n, \ldots, \zeta'_1 \,|\, \zeta_1, \ldots, \zeta_m) =$
  $(1-1, 3-2, 4-3, 6-4 \,|\, 5-2, 6-5, 7-7) = (0, 1, 1, 2 \,|\, 3, 1, 0)$

- $\zeta = (3, 1, 0) = \begin{pmatrix} 5-2-1 \\ 6-4-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \quad (-1)^{|\zeta|} = +1$

- $r(\zeta) = 1$

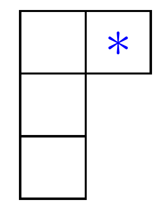
Determination of $\zeta + p^r$ and $\zeta' + q^r$

- $\zeta = (310) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad r = r(\zeta) = 1 \quad p = -2$



$$\Longrightarrow \zeta + p^r = (310) - (200) = (110) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

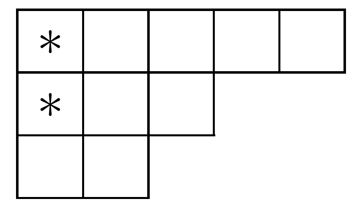
- $\zeta' = (2110) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad r = r(\zeta) = 1 \quad q = -1$



$$\Longrightarrow \zeta' + q^r = (2110) - (1000) = (1110) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

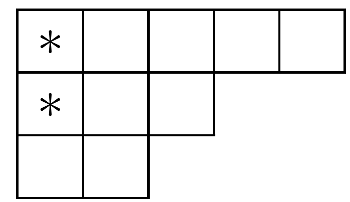
Identification of constituent determinants

- Recall that $\lambda = (532)$ and $\mu = (4322)$
- while $\zeta + p^r = (110)$ and $\zeta' + q^r) = (1110)$
- $$\begin{vmatrix} \{4\} & \{5\} & \{7\} \\ \{1\} & \{2\} & \{4\} \\ - & \{0\} & \{2\} \end{vmatrix} = s_{532/110}(x)$$

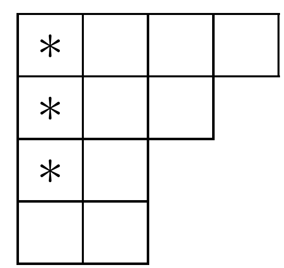


Identification of constituent determinants

- Recall that $\lambda = (\textcolor{red}{532})$ and $\mu = (\textcolor{blue}{4322})$
- while $\zeta + \textcolor{red}{p}^r = (\textcolor{red}{110})$ and $\zeta' + \textcolor{blue}{q}^r) = (\textcolor{blue}{1110})$
- $$\begin{vmatrix} \{\textcolor{blue}{4}\} & \{\textcolor{red}{5}\} & \{\textcolor{red}{7}\} \\ \{\textcolor{blue}{1}\} & \{\textcolor{red}{2}\} & \{\textcolor{red}{4}\} \\ - & \{\textcolor{red}{0}\} & \{\textcolor{red}{2}\} \end{vmatrix} = s_{532/110}(\textcolor{red}{x})$$



- $$\begin{vmatrix} \{\textcolor{blue}{2}\} & \{\textcolor{blue}{0}\} & - & - \\ \{\textcolor{blue}{3}\} & \{\textcolor{blue}{1}\} & \{\textcolor{blue}{0}\} & - \\ \{\textcolor{blue}{5}\} & \{\textcolor{blue}{3}\} & \{\textcolor{blue}{2}\} & \{\textcolor{red}{1}\} \\ \{\textcolor{blue}{7}\} & \{\textcolor{blue}{5}\} & \{\textcolor{blue}{4}\} & \{\textcolor{red}{3}\} \end{vmatrix} = s_{4322/1110}(\textcolor{blue}{y})$$



Conclusions

- Both the classical Schur function series of Littlewood and the Cauchy identity may be restricted with respect to row lengths or column lengths through determinantal formulae
- In each case the correction factors to the original multiplicative formulae may be expressed as a signed sum of Schur functions or pairs of Schur functions specified by partitions having a particularly simple form in Frobenius notation
- Each row or column restricted Schur function series is nothing other than the character of some (rather simple) finite or infinite-dimensional irrep of a classical group

Conclusions

- To evaluate these characters (and thereby derive the restricted Schur function series) use may be made of Howe dual pairs with respect to spin and metaplectic representations of (the covering groups) of the orthogonal and symplectic groups
- All the Schur function identities may be extended to the case of supersymmetric Schur functions
- The dual pair approach enables many other characters to be evaluated, although in doing so it is usually necessary to invoke classical group modification rules

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