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FLAG-MAJOR INDEX AND FLAG-INVERSION NUMBER ON COLORED WORDS AND WREATH PRODUCT

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Abstract. In [*Proc. Amer. Math. Soc.* **19** (1968), 236–240], Dominique Foata constructed a map Φ, called *second fundamental transformation*, exchanging the integer-valued statistics *inversion number* "inv" and *major index* "maj" on words whose letters are integers. Later, Foata and Han introduced the *flag-inversion number* "finv" and extended Φ on signed words and permutations, showing that the *flag major index* "fmaj" and "finv" were equidistributed. In this paper we give an extension of Φ to *ℓ-colored words*. Using this extension, we show that the bistatistics $(\mathrm{fmaj}, \mathrm{des}^*)$ and $(\mathrm{finv}, \mathrm{pcol})$ are equidistributed, where "pcol" is the sum of color powers and "des^*" is a new statistic derived from "des".

1. Introduction

The *second fundamental transformation*, denoted by Φ and described in [5] by Foata, is defined on finite words whose letters are integers. If $\mathbf{m} = (m_1, \cdots, m_r)$ is a sequence of nonnegative integers, let $R_{\mathbf{m}}$ be the set of all rearrangements $w = x_1\, x_2 \cdots x_m$ of the sequence $1^{m_1} 2^{m_2} \ldots n^{m_r}$ where $m = m_1 + m_2 + \cdots + m_r$. The transformation Φ maps each word w to another word $\Phi(w)$ and has the following properties:

(1) $\mathrm{maj}\ w = \mathrm{inv}\ \Phi(w)$;
(2) $\Phi(w)$ is a rearrangement of w and the restriction of Φ to $R_{\mathbf{m}}$ is a bijection of $R_{\mathbf{m}}$ onto itself.

Further properties were proved later on by Foata and Schützenberger [7], and by Björner and Wachs [3], in particular, when the transformation is restricted to act on the symmetric group S_r.

The purpose of this paper is to extend the transformation Φ to *ℓ-colored words*.

Let C_ℓ be the ℓ-cyclic group generated by $\zeta = e^{2i\pi/\ell}$. By an *ℓ-colored word*, we understand a pair (ε, x), where $\varepsilon \in (C_\ell)^m$ and x is a word of length m whose letters are nonnegative integers. For reasons which will appear later, if $w := (\varepsilon, x)$ is an ℓ-colored word where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$ and $x = x_1\, x_2 \cdots x_m$, we write $w := w_1 w_2 \cdots w_m$ where $w_j = \varepsilon_j x_j$ $(1 \le j \le m)$. For any j with $1 \le j \le m$, ε_j is called the *color* of w_j and, if $\varepsilon_j = \zeta^{k_j}$, k_j is the power of this color. For small values of ℓ, we shall use k_j bars over x_j instead of $\zeta^{k_j} x_j$.

For example, if $w = \zeta^2 3\, \zeta^2 1\, \zeta^0 4\, \zeta 1\, \zeta^2 3$, then we write $w = \bar{\bar{3}}\bar{\bar{1}}4\bar{1}\bar{\bar{3}}$.

Any ℓ-colored word can be considered as a finite word over the alphabet $\Sigma_\ell := \{\xi j;\ \xi \in C_\ell,\ j \ge 1\}$.

Let $w := w_1 w_2 \cdots w_m := \varepsilon_1 x_1\, \varepsilon_2 x_2 \cdots \varepsilon_m x_m$ be an ℓ-colored word. We write

$$\begin{aligned} |w_i| &:= x_i, \qquad\qquad 1 \le i \le m; \\ |w| &:= |w_1||w_2| \cdots |w_m|; \end{aligned} \tag{1.1}$$

and we define the statistic *power-color* "pcol" by

$$\begin{aligned} \operatorname{pcol}_i w &:= \sum_{0 \le j \le \ell-1} j \chi(\varepsilon_i = \zeta^j), \qquad\qquad 1 \le i \le m; \\ \operatorname{pcol} w &:= \sum_{1 \le i \le m} \operatorname{pcol}_i w. \end{aligned} \tag{1.2}$$

If $\mathbf{m} = (m_1, \cdots, m_r)$ is a sequence of nonnegative integers such that $m_1 + \cdots + m_r = m$, let $G_{\ell,\mathbf{m}}$ be the set of all ℓ-colored words $w = w_1\, w_2 \cdots w_m$ such that $|w| \in R_{\mathbf{m}}$. The class $G_{\ell,\mathbf{m}}$ contains $\ell^m \binom{m}{m_1, m_2, \ldots, m_r}$ ℓ-colored words. When $m_1 = m_2 = \cdots = m_r = 1$, the class $G_{\ell,\mathbf{m}}$ is the wreath product $C_\ell \wr S_r$ denoted by $G_{\ell,r}$. We define an order relation on Σ_ℓ as follows:

$$\zeta^j i > \zeta^{j'} i' \iff [j < j'] \quad \text{or} \quad [(j = j') \quad \text{and} \quad (i > i')]. \tag{1.3}$$

The restriction of this order to the class of ordinary words (with nonnegative letters) is the usual order.

As in [6], the statistics "inv" and "maj" must be adapted to *ℓ-colored words* and correspond to classical statistics when applied to ordinary words. Let

$$(\omega; q)_n := \begin{cases} 1 & \text{if } n = 0; \\ (1-\omega)(1-\omega q) \cdots (1-\omega q^{n-1}) & \text{if } n \ge 1; \end{cases}$$

denote the usual *q-shifted factorial*, and let

$$\begin{bmatrix} m_1 + m_2 + \cdots + m_r \\ m_1, m_2, \ldots, m_r \end{bmatrix}_q := \frac{(q;q)_{m_1+m_2+\cdots+m_r}}{(q;q)_{m_1} (q;q)_{m_2} \cdots (q;q)_{m_r}}$$

be the *q-multinomial coefficient*.

With the order relation defined in (1.3), the natural extensions of the *flag-major index* "fmaj" and the *flag-inversion number* "finv" introduced by Foata and Han [6] to *ℓ-colored* *words* are defined as follows: for all ℓ-colored word $w := w_1w_2\cdots w_m$,

$$\operatorname{fmaj} w := \ell\sum_{i=1}^{m-1} i\chi(w_i > w_{i+1}) + \operatorname{pcol} w;$$
$$\operatorname{finv} w := \sum_{\substack{1\le i<j\le m\\ \xi\in C_\ell}} \chi(\xi w_i > w_j) + \operatorname{pcol} w.$$

Foata and Han defined $(-q;q)_m \left[{m \atop m_1,m_2,\dots,m_r}\right]_q$ as a q-analog of $2^m \binom{m}{m_1,m_2,\dots,m_r}$. By analogy, $\frac{(q^\ell;q^\ell)_m}{(q;q)_m}\left[{m \atop m_1,m_2,\dots,m_r}\right]_q$ is a natural q-analog of $\ell^m\binom{m}{m_1,m_2,\dots,m_r}$.

We claim that

$$\frac{(q^\ell;q^\ell)_m}{(q;q)_m}\begin{bmatrix} m \\ m_1,m_2,\dots,m_r\end{bmatrix}_q = \sum_{w\in G_{\ell,\mathbf{m}}} q^{\operatorname{finv} w}. \tag{1.4}$$

This can be established by induction on m. Indeed, let us consider the bijective transformation

$$\begin{aligned}\varphi:\quad & G_{\ell,\mathbf{m}} \longrightarrow \{0,1,\dots,\ell-1\}\times\bigcup_{k=1}^{r} G_{\ell,\mathbf{m\text{-}1}_k},\\ & w := w_1w_2\cdots w_m \longmapsto (s,w') := (\operatorname{pcol}_m w, w_1w_2\cdots w_{m-1})\end{aligned}$$

where $\mathbf{m\text{-}1}_k = (m_1,m_2,\dots,m_{k-1},m_k-1,m_{k+1},\dots,m_r)$. We have $k=|w_m|$ and

$$\begin{aligned}\operatorname{finv}\, w &= \operatorname{finv}\, w' + s + \sum_{\substack{1\le i\le m-1\\ 0\le j\le \ell-1}} \chi(\zeta^j|w_i| > w_m)\\ &= \operatorname{finv}\, w' + ms + (m_{k+1}+\cdots+m_r)\chi(k<r).\end{aligned}$$

So,

$$\begin{aligned}\sum_{w\in G_{\ell,\mathbf{m}}} q^{\operatorname{finv} w} &= \sum_{0\le s\le \ell-1} q^{ms}\left(\sum_{w'\in G_{\ell,\mathbf{m\text{-}1}_r}} q^{\operatorname{finv} w'} + \sum_{1\le k\le r-1} q^{m_{k+1}+\cdots+m_r}\sum_{w'\in G_{\ell,\mathbf{m\text{-}1}_k}} q^{\operatorname{finv} w'}\right)\\ &= \frac{1-q^{\ell m}}{1-q^m}\left(\sum_{w'\in G_{\ell,\mathbf{m\text{-}1}_r}} q^{\operatorname{finv} w'} + \sum_{1\le k\le r-1} q^{m_{k+1}+\cdots+m_r}\sum_{w'\in G_{\ell,\mathbf{m\text{-}1}_k}} q^{\operatorname{finv} w'}\right).\end{aligned}$$

By induction, for each $1\le k\le r$, we have

$$\begin{aligned}\sum_{w'\in G_{\ell,\mathbf{m\text{-}1}_k}} q^{\operatorname{finv} w'} &= \frac{(q^\ell;q^\ell)_{m-1}}{(q;q)_{m-1}}\frac{(q;q)_{m-1}}{(q;q)_{m_1}\cdots(q;q)_{m_{k-1}}(q;q)_{m_k-1}(q;q)_{m_{k+1}}\cdots(q;q)_{m_r}}\\ &= \frac{(q^\ell;q^\ell)_{m-1}(1-q^{m_k})}{(q;q)_m}\frac{(q;q)_m}{(q;q)_{m_1}\cdots(q;q)_{m_k}\cdots(q;q)_{m_r}}.\end{aligned}$$

Thus,

$$\sum_{1\le k\le r-1} q^{m_{k+1}+\cdots+m_r} \sum_{w'\in G_{\ell,\mathbf{m}-\mathbf{1}_k}} q^{\operatorname{finv} w'}$$
$$= \frac{(q^\ell;q^\ell)_{m-1}}{(q;q)_m} \frac{(q;q)_m}{(q;q)_{m_1}\cdots(q;q)_{m_k}\cdots(q;q)_{m_r}} \sum_{1\le k\le r-1} q^{m_{k+1}+\cdots+m_r}(1-q^{m_k})$$
$$= \frac{(q^\ell;q^\ell)_{m-1}(q^{m_r}-q^m)}{(q;q)_m} \frac{(q;q)_m}{(q;q)_{m_1}\cdots(q;q)_{m_k}\cdots(q;q)_{m_r}}$$

and

$$\sum_{w\in G_{\ell,\mathbf{m}}} q^{\operatorname{finv} w} = \frac{1-q^{\ell m}}{1-q^m}\Bigg(\frac{(q^\ell;q^\ell)_{m-1}(1-q^{m_r})}{(q;q)_m} \frac{(q;q)_m}{(q;q)_{m_1}\cdots(q;q)_{m_k}\cdots(q;q)_{m_r}}$$
$$+ \frac{(q^\ell;q^\ell)_{m-1}(q^{m_r}-q^m)}{(q;q)_m} \frac{(q;q)_m}{(q;q)_{m_1}\cdots(q;q)_{m_k}\cdots(q;q)_{m_r}}\Bigg)$$
$$= \frac{(q^\ell;q^\ell)_{m}}{(q;q)_m} \frac{(q;q)_m}{(q;q)_{m_1}\cdots(q;q)_{m_k}\cdots(q;q)_{m_r}}$$
$$= \frac{(q^\ell;q^\ell)_{m}}{(q;q)_m} \begin{bmatrix} m \\ m_1, m_2, \ldots, m_r\end{bmatrix}_q .$$

This concludes the proof of the claim in (1.4).

We construct the extension $\widehat{\Phi}$ of the second fundamental transformation Φ to ℓ-colored words in the next section. Define

$$\operatorname{des}^* w = \ell \operatorname{des} w - \operatorname{des} |w| + \operatorname{pcol}_1 w, \tag{1.5}$$

where $\operatorname{des} w := \sum_{i=1}^{m-1} \chi(w_i > w_{i+1})$.

The main purpose of this paper is to prove the following theorem.

Theorem 1.1. *The transformation $\widehat{\Phi}$ constructed in Section 2 has the following properties*

1. *For every ℓ-colored word w, $(\operatorname{fmaj}, \operatorname{des}^*)\, w = (\operatorname{finv}, \operatorname{pcol})\, \widehat{\Phi}(w)$;*
2. *The restriction of $\widehat{\Phi}$ to each class $G_{\ell,\boldsymbol{m}}$ is a bijection of $G_{\ell,\boldsymbol{m}}$ onto itself.*

Corollary 1.2. *For each $m = (m_1, m_2, \ldots, m_r)$, the bistatistics* $(\operatorname{fmaj}, \operatorname{des}^*)$ *and* (finv, pcol) *are equidistributed on $G_{\ell,\boldsymbol{m}}$.*

Example 1.3. Let us consider the hyperoctahedral group of order 2.

w	$1\,2$	$\bar1\,2$	$1\,\bar2$	$\bar1\,\bar2$	$2\,1$	$\bar2\,1$	$2\,\bar1$	$\bar2\,\bar1$
fmaj w	0	1	3	2	2	1	3	4
des* w	0	1	2	1	1	0	1	2
finv w	0	1	2	3	1	2	3	4
pcol w	0	1	1	2	0	1	1	2

Now consider the statistic Rfinv defined on the hyperoctahedral group of order n as follows:

$$\mathrm{Rfinv}\ w = \mathrm{inv}\, w + \sum_{i=1}^{n} |w_i| \chi(w_i < 0).$$

If one uses the natural order relation on $[-n, n]$ given by

$$-n < -(n-1) < \cdots < -1 < 1 < \cdots < (n-1) < n, \tag{1.6}$$

Brenti [4] shows that finv coincides with the traditional *length function*, and Adin and Roichman [1] proved that Rfinv and fmaj are equidistributed on the hyperoctahedral group.

Back to the order relation (1.3) on $[-n, n]$, i.e.,

$$-1 < \cdots < -(n-1) < -n < 1 < \cdots < (n-1) < n$$

one has

$$\textit{length function} \neq \mathrm{Rfinv} \quad \text{and} \quad \mathrm{finv} \neq \mathrm{Rfinv},$$

but we observe that Rfinv remains equidistributed with fmaj, and we prove that its extension to the wreath product is also Mahonian. We have the following theorem.

Theorem 1.4. *The statistic* Rfinv *defined on the wreath product* $C_\ell \wr S_n$ *by*

$$\mathrm{Rfinv}\ w = \mathrm{inv}\ w + \sum_{i=1}^{n} |w_i| \,\mathrm{pcol}_i\, w \tag{1.7}$$

is Mahonian.

By a result of Haglund, Loehr and Remmel [8], we obtain the following corollary.

Corollary 1.5. *We have*

$$\sum_{\sigma \in G_{\ell,n}} q^{\mathrm{Rfinv}\,\sigma} = \sum_{\sigma \in G_{\ell,n}} q^{\mathrm{finv}\,\sigma} = \frac{(q^\ell; q^\ell)_n}{(1-q)^n}. \tag{1.8}$$

2. The construction of the transformation $\widehat{\Phi}$

Let us recall the second fundamental transformation Φ from [5]. First, for each integer x, we recall the transformation γ_x. Let $w = x_1 x_2 \cdots x_m$ be a word with positive letters. The first (respectively last) letter x_1 (respectively x_m) is denoted by $F(w)$ (respectively $L(w)$). If $L(w) \le x$ (respectively $L(w) > x$), w admits the unique factorization

$$(u_1 y_1, u_2 y_2, \cdots, u_p y_p)$$

called its *x-right-to-left factorization* having the following properties:

(1) each y_i $(1 \le i \le p)$ is a letter verifying $y_i \le x$ (respectively $y_i > x$);
(2) each u_i $(1 \le i \le p)$ is a factor which is either empty or has all its letters greater than (respectively smaller than or equal to) x.

Then, the bijective transformation γ_x maps $w = u_1 y_1 u_2 y_2 \dots u_p y_p$ to the word

$$\gamma_x(w) = y_1 u_1 y_2 u_2 \cdots y_p u_p.$$

Foata defined $\Phi(w)$ by induction on the length of w. If w has length one, then $\Phi(w) = w$. If it has more than one letter, write the word as vx where x is the last letter and define $\Phi(vx)$ to be the juxtaposition product

$$\Phi(vx) := \gamma_x(\Phi(v))x. \tag{2.1}$$

We now define $\widehat{\Phi}$ as follows. For each word $u = x_1\, x_2\, \cdots\, x_m$ with nonnegative letters and each element $\epsilon := (\epsilon_1, \cdots, \epsilon_m)$ of $(\mathcal{C}_\ell)^m$, we denote by $\Psi_u(\epsilon)$ the element $\epsilon' = (\epsilon'_1, \cdots, \epsilon'_m)$ of $(\mathcal{C}_\ell)^m$ defined as follows:

$$\begin{cases} \epsilon'_i = \frac{\epsilon_i}{\epsilon_{i+1}} \zeta^{-\chi(x_i > x_{i+1})} & \text{if } i < m, \\ \epsilon'_m = \epsilon_m & \text{if } i = m. \end{cases} \tag{2.2}$$

Let $w := (\epsilon, u)$ be an ℓ-colored word ($u = |w|$). Define

$$\widehat{\Phi}(w) = (\Psi_u(\epsilon), \Phi(u)). \tag{2.3}$$

Example 2.1. Let us take $\ell = 4$ and $w = \bar{\bar{3}}\bar{1}4\bar{1}\bar{\bar{3}}$. We have

$$w = ((\zeta^2, \zeta, 1, \zeta, \zeta^2), 31413).$$

By construction of Φ (relation (2.1)), we have

$$\begin{aligned} \Phi(3) &= 3, \\ \Phi(31) &= \gamma_1(\Phi(3))1 = \gamma_1(3)1 = 31, \\ \Phi(314) &= \gamma_4(\Phi(31))4 = \gamma_4(31)4 = 314, \\ \Phi(3141) &= \gamma_1(\Phi(314))1 = \gamma_1(314)1 = 3411, \\ \Phi(31413) &= \gamma_3(\Phi(3141))3 = \gamma_3(3411)3 = 31413. \end{aligned}$$

In the other hand,

$$\epsilon'_1 = \frac{\zeta^2}{\zeta}\zeta^{-1} = 1, \quad \epsilon'_2 = \frac{\zeta}{1} = \zeta, \quad \epsilon'_3 = \frac{1}{\zeta^1}\zeta^{-1} = \zeta^2, \quad \epsilon'_4 = \frac{\zeta^1}{\zeta^2} = \zeta^3, \quad \epsilon'_5 = \zeta^2.$$

Therefore,

$$w' = \widehat{\Phi}(\bar{\bar{3}}\bar{1}4\bar{1}\bar{\bar{3}}) = 3\bar{1}\bar{\bar{4}}\bar{\bar{\bar{1}}}\bar{\bar{3}}.$$

We have $(\mathrm{fmaj}, \mathrm{des}^*)w = (\mathrm{finv}, \mathrm{pcol})w' = (34, 8)$.

3. Proof of Theorem 1.1

Lemma 3.1. *Let $w := w_1 w_2 \cdots w_m$ be an ℓ-colored word. With notation in relation* (1.2)*, we have*

$$\mathrm{finv}\ w = \mathrm{inv}\ |w| + \sum_{i=1}^{m} i\, \mathrm{pcol}_i\, w. \tag{3.1}$$

Proof. For all integers i, j, k such that $1 \le i < j \le m$ and $0 \le k \le \ell - 1$, one has:

$$\chi(\zeta^k |w_i| > w_j) = \chi(k < \operatorname{pcol}_j w) + \chi(k = \operatorname{pcol}_j w)\chi(|w_i| > |w_j|).$$

Thus,

$$\operatorname{finv}\ w = \sum_{j=1}^{m} (j-1) \operatorname{pcol}_j w + \operatorname{inv}\ |w| + \sum_{j=1}^{m} \operatorname{pcol}_j\ w = \operatorname{inv}\ |w| + \sum_{j=1}^{m} j \operatorname{pcol}_j w.$$

□

Now, slightly abusing notation, let $w = (\epsilon, |w|)$ be an ℓ-colored word of length m and $w' = (\epsilon', |w'|) := \widehat{\Phi}(w)$. For each i such that $1 \le i \le m-1$, we have

- if $|w_i| \le |w_{i+1}|$, then $\epsilon'_i = \dfrac{\epsilon_i}{\epsilon_{i+1}} = \zeta^{\operatorname{pcol}_i w - \operatorname{pcol}_{i+1} w}$;
- if $|w_i| > |w_{i+1}|$, then $\epsilon'_i = \dfrac{\epsilon_i}{\epsilon_{i+1}}\zeta^{-1} = \zeta^{\operatorname{pcol}_i w - \operatorname{pcol}_{i+1} w - 1}$.

So,

$$\begin{aligned}
\operatorname{pcol}_i w' &= [\operatorname{pcol}_i w - \operatorname{pcol}_{i+1} w + \ell\chi(\operatorname{pcol}_i w < \operatorname{pcol}_{i+1} w)]\chi(|w_i| \le |w_{i+1}|) \\
&\qquad + [\operatorname{pcol}_i w - \operatorname{pcol}_{i+1} w - 1 + \ell\chi(\operatorname{pcol}_i w \le \operatorname{pcol}_{i+1} w)]\chi(|w_i| > |w_{i+1}|) \\
&= \operatorname{pcol}_i w - \operatorname{pcol}_{i+1} w + \ell\chi(\operatorname{pcol}_i w < \operatorname{pcol}_{i+1} w) \\
&\qquad + \ell\chi(\operatorname{pcol}_i w = \operatorname{pcol}_{i+1} w)\chi(|w_i| > |w_{i+1}|) - \chi(|w_i| > |w_{i+1}|).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\operatorname{pcol}\ w' &= \sum_{i=1}^{m} \operatorname{pcol}_i w' \\
&= \sum_{i=1}^{m} \operatorname{pcol}_i w - \sum_{i=2}^{m} \operatorname{pcol}_i w + \ell \sum_{i=1}^{m-1} [\chi(\operatorname{pcol}_i w < \operatorname{pcol}_{i+1} w) \\
&\qquad + \chi(\operatorname{pcol}_i w = \operatorname{pcol}_{i+1} w)\chi(|w_i| > |w_{i+1}|)] - \sum_{i=1}^{m-1} \chi(|w_i| > |w_{i+1}|) \\
&= \operatorname{pcol}_1 w + \ell \operatorname{des}\ w - \operatorname{des}\ |w| \\
&= \operatorname{des}^*\ w,
\end{aligned}$$

and, by Φ,

$$
\begin{aligned}
\operatorname{finv} w' &= \operatorname{inv} |w'| + \sum_{i=1}^{m} i \operatorname{pcol}_i w' \\
&= \operatorname{maj} |w| + \sum_{i=1}^{m} i \operatorname{pcol}_i w - \sum_{i=1}^{m} (i-1) \operatorname{pcol}_i w \\
&\qquad + \ell \sum_{i=1}^{m-1} i[\chi(\operatorname{pcol}_i w < \operatorname{pcol}_{i+1} w) + \chi(\operatorname{pcol}_i w = \operatorname{pcol}_{i+1} w)\chi(|w_i| > |w_{i+1}|)] \\
&\qquad - \sum_{i=1}^{m-1} i\chi(|w_i| > |w_{i+1}|) \\
&= \operatorname{maj} |w| + \sum_{i=1}^{m} \operatorname{pcol}_i w + \ell \sum_{i=1}^{m-1} i\chi(w_i > w_{i+1}) - \operatorname{maj} |w| \\
&= \ell \sum_{i=1}^{m-1} i\chi(w_i > w_{i+1}) + \operatorname{pcol} w \\
&= \operatorname{fmaj} w.
\end{aligned}
$$

Finally, we show that $\widehat{\Phi}$ is a bijection of $G_{\ell,\mathbf{m}}$ onto itself. Indeed, let $w' := (\epsilon', u')$ be an element of $G_{\ell,\mathbf{m}}$. By the relation (2.2), if $w := (\epsilon, u)$ is an element of $G_{\ell,\mathbf{m}}$ such that $\widehat{\Phi}(w) = w'$, then $u = \Phi^{-1}(u')$, $\epsilon_m = \epsilon'_m$ and, for $i < m$,

$$
\epsilon_i = \epsilon'_m \prod_{i \le j \le m-1} \epsilon'_j \zeta^{\chi(x_j > x_{j+1})},
$$

where $u := x_1 x_2 \cdots x_m$.

This concludes the proof of Theorem 1.1.

4. Proof of Theorem 1.4

Consider the following transformations:

▷ ***Transformation*** ρ

$$
\begin{aligned}
\rho : G_{\ell,n} &\longrightarrow G_{\ell,n} \\
w = (\epsilon, |w|) &\longmapsto \rho(w) = w' = (\epsilon', |w'|),
\end{aligned}
$$

where

$$
|w'| = |w|^{-1} \quad \text{and} \quad \epsilon'_i = \epsilon_{|w|^{-1}(i)};
$$

▷ ***Transformation*** τ

For each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $[0, \ell-1]^n$, put $\Sigma_\alpha = \{\zeta^{\alpha_1}, \zeta^{\alpha_2}2, \ldots, \zeta^{\alpha_n}n\}$, and let G_α be the class of *ℓ-colored permutations* whose letters are in Σ_α, i.e.,

$$
G_\alpha = \{w = w_1 w_2 \cdots w_n \in G_{\ell,n} : \operatorname{pcol}_i w = \alpha_i \text{ for all } i \in [n]\}.
$$

Note that $\sharp G_\alpha = n!$. We denote by I_α the increasing bijection from $[n]$ to Σ_α, and we define τ for each class G_α by $\tau(w) = w'_1 w'_2 \dots w'_n$, where

$$w'_i = I_\alpha(|w_i|).$$

Lemma 4.1. *For all $w \in G_{\ell,n}$, we have*

$$\operatorname{finv} w = \operatorname{Rfinv} \tau \circ \rho(w).$$

Proof of Lemma 4.1. Let $w = w_1 w_2 \cdots w_n \in G_{\ell,n}$. Consider the auxiliary statistics

$$\begin{aligned} \wp(w) &:= \sum_{i=1}^{n} i \operatorname{pcol}_i w; \\ \Im(w) &:= \sum_{i=1}^{n} |w_i| \operatorname{pcol}_i w; \\ |\mathrm{inv}|\, w &:= \operatorname{inv} |w|. \end{aligned}$$

It is easy to see that ρ is an involution preserving $|\mathrm{inv}|$ and transforming $\wp$ into $\Im$ and vice versa:

$$(|\mathrm{inv}|, \wp)w = (|\mathrm{inv}|, \Im)\rho(w),$$

and τ preserves $\Im$ and transforms $|\mathrm{inv}|$ into inv, i.e.,

$$(|\mathrm{inv}|, \Im)w = (\mathrm{inv}, \Im)\rho(w).$$

By Lemma 3.1, we have

$$\begin{aligned} \operatorname{finv} w &= |\mathrm{inv}|\, w + \wp(w) \\ &= |\mathrm{inv}|\rho(w) + \Im\, \rho(w) = \operatorname{inv} \tau \circ \rho(w) + \Im\, \tau \circ \rho(w) \\ &= \operatorname{Rfinv} \tau \circ \rho(w). \end{aligned}$$

$\square$

Example 4.2. $w = 5\bar{\bar{3}}\bar{1}2\bar{4}$, $\operatorname{finv} w = 18$; $\rho(w) = \bar{3}4\bar{\bar{2}}\bar{5}1$. Let $\alpha = (0, 2, 1, 0, 1)$. I_α is defined as follows:

i	1	2	3	4	5
$I_\alpha(i)$	$\bar{2}$	$\bar{3}$	$\bar{5}$	1	4

So

$$\tau \circ \rho(w) = w' = \bar{5}1\bar{3}4\bar{\bar{2}} \qquad \text{and} \qquad \operatorname{Rfinv} w' = 18.$$

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