What is the link between ... ?

- fast formulae for computing π, $\frac{1}{\pi}\ldots$
- irrationality of $\zeta(3)$
- Young tableaux of bounded height
- (generalized) hypergeometric functions
- Latin squares
- the triple product identity of Jacobi
- k-regular graphs
- cost of searching in quadtrees, m-ary search trees
- alternating sign matrices
- consecutive records in permutations
- non 3-crossing partitions
- (lot of) random walks in the (quarter) plane
- automatic integration
- "Calabi-Yau" parametrizations
- enumeration and asymptotics in statistical mechanics (polyominoes, etc)
- identities involving symmetric functions
- ...

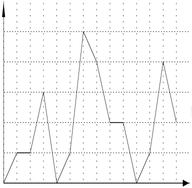
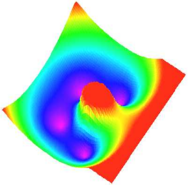
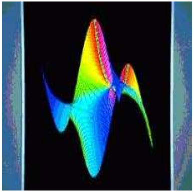
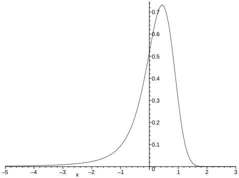
What is the link between ... ?

- fast formulae for computing π, $\frac{1}{\pi}\dots$
- irrationality of $\zeta(3)$ [Apéry, 1978]
- Young tableaux of bounded height
- (generalized) hypergeometric functions
- Latin squares
- the triple product identity of Jacobi
- k-regular graphs
- cost of searching in quadtrees, m-ary search trees [Hwang, Fuchs, Chern, 2006]
- alternating sign matrices
- consecutive records in permutations
- non 3-crossing partitions
- (lot of) random walks in the (quarter) plane
- automatic integration
- "Calabi-Yau" parametrizations [Zudilin, Almkvist & al., 2008]
- enumeration and asymptotics in statistical mechanics (polyominoes, etc) [Guttmann & al., Di Francesco & al.]
- identities involving symmetric functions

ALL OF THEM ARE HOLONOMIC OBJECTS !

New results on asymptotics of holonomic sequences

Cyril Banderier
CNRS, LIPN, Paris XIII (Villetaneuse, France)
http ://lipn.fr/∼banderier
based on work in progress with...
Felix Chern & Hsien-Kuei Hwang, Taipei
SLC'62, Feb. 24, 2009

Holonomic = P-recursive sequences = D-finite functions

Sequence $(a_n)_{n\in\mathbb{N}}$ is P-recursive := it satisfies a linear recurrence with polynomial coefficients in n.

$$(2+n)a_{n+1} - (2+4n)a_n = 0$$

$A(z)$ is D-finite (differentialy finite) := its derivatives span a vector space of finite dimension.
$\Longleftrightarrow$ $A(z)$ satisfies an ODE (= ordinary differential equation) with coefficients polynomials in z.

$$1 + (2z-1)A(z) + (4z^2 - z)A'(z) = 0, \qquad A(z) = \sum_{n\geq 0} a_n z^n$$

These 2 notions are equivalent.
$>$ 25% of the sequences in the Sloane EIS are P-recursive.
$>$ 60% of the special functions in the Abramowitz-Stegun book are D-finite.
The importance of D-finite functions was established in the 80's by Stanley/Gessel/Lipshitz/Zeilberger (which also uses the word "holonomic").

D-finiteness and holonomy

Holonomy is related to the growth rate of the coefficients of the Hilbert function, [Bernstein 1971] :
$A(z) = \sum_{n \in \mathbb{N}} a_n z^n$ is holonomic iff $a_n := \dim_{\mathbb{C}} \{x^i \delta_z^j A(z), i + j = n\} = O(n^d)$.

(kind of minimal "noetherianity"… good, algorithms will terminate ! [Chyzak, 1998])

NB : Holonomy theory is in fact quite general (shift for sequences, differentiation, integration, mahlerian substitutions, for one or several variables), using Ore algebra and Groebner bases allows automatic proof of a lot of identities related to integrals or sums (as in the book "$A = B$").

D-finite functions have a lot of closure properties...

Rational or hypergeometric functions are trivially D-finite (recurrence for the coefficients !).

Proposition [Comtet, 70's] : Algebraic functions are D-finite.
Proof : Differentiating $P(z, F(z)) = 0$ and using Bezout identity between P and P' implies that F' belongs to $\mathbb{C}(z) \oplus \mathbb{C}(z)F \oplus \cdots \oplus \mathbb{C}(z)F^{d-1}$, then proceed by recurrence.

Proposition [folklore/Gessel/Stanley/Lipshitz/Zeilberger..., 80's] Closure by

- addition,
- product (and therefore nested sums $\sum_{j=1}^{m} \sum_{k=1}^{n} f_{n,i} \ldots$),
- Hadamard product $(a_n b_n)$,
- diagonal $(f_{n,n,n,n,n})$, (cf generalisation of Delannoy numbers)
- algebraic substitution,
- Laplace/Borel (inverse) transform $(n! a_n, a_n/n!)$,
- shuffle (cf Pólya drunkard),
- manipulation of symmetric functions.

$\Rightarrow$ A very good class for computer algebra !

Holonomy ⇒ automatic proof of combinatorial identities

Ex. 1 : irrationality of $\zeta(3) = \sum_{n\geq 1} \frac{1}{n^3}$ [Apéry, 1978] :

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$

Ex. 2 : Mehler's identity for Hermite polynomials :

as $\displaystyle\sum_{n\geq 0} H_n(x) \frac{z^n}{n!} = \exp(z(2x-z))$ then $\displaystyle\sum_{n\geq 0} H_n(x) H_n(y) \frac{z^n}{n!} = \frac{\exp\left(\frac{4z(xy-z(x^2+y^2))}{1-4z^2}\right)}{\sqrt{1-4z^2}}$

Advertising for useful programs proving/guessing combinatorial identities :
Combstruct and Gfun/Mgfun/Rate packages in Maple/Mathematica
[Flajolet/Salvy/Zimmermann/Chyzak/Krattenthaler].

Computational complexity of the coefficients

Rational functions : $O(d^3 \ln(n))$ [using binary exponentiation on the associated matrix]
Algebraic functions : $O(dn)$ [because they're D-finite !]
Special functions from physics : $O(n)$ time for computing n coefficients of their Taylor expansions give the key for a fast plot of their graph !

Why do a computer scientist care for asymptotics of a_n ?

It is crucial for average case analysis of algorithms !

This is the message of Knuth in The art of computer programming :

$$\text{algorithms} = \text{data structures} = \text{combinatorial structures}$$
$$\text{recursivity} = \text{recurrence}$$
$$\text{cost} = \text{asymptotics}$$
$$\Rightarrow$$
$$\text{good programs} = \text{good mathematical analysis of the hidden combinatorial structures.}$$

Not only you can then decide which algorithm will almost always be the faster (on my laptop, I prefer $a_n = .5n \ln n$ than $a_n = 30n \ln n$) but you can then tune some algorithms in an optimal way !

Recent applications : uniform random generation of combinarial objects !
before until size 100, now, thanks to analytic combinatorics : until size 10^6,
Boltzmann method [Flajolet & al.]

The unreasonable efficiency of complex analysis

Hecke : "Es ist eine Tatsache, daβ die genauere Kenntnis des Verhaltens einer analytischen Funktion in der Nähe ihrer singulären Stellen eine Quelle von arithmetischen Sätzen ist."

Hadamard : "The shortest path between two assertions in the real world goes through the complex world."

Moral :
insight on the singularities (landscape) of $A(z)$ = insight on the coefficients a_n's

Asymptotics are related to the singularities

A singularity can be : a pole $1/z$, a branching point $\ln(z)$, $\sqrt{z}$, essential singularity $\exp(1/z)$, a natural boundary point $\Pi_{k\geq 1}\dfrac{1}{1-z^k}$, ...

R dominant singularity (=radius of convergence) of $F(z) = \sum f_n z^n$
$\Longrightarrow$ F_n grows like $1/R^n$.

Power of complex analysis gives much more !

Singularity analysis [Flajolet-Odlyzko] If $F(z) \sim A(z)$, then with $A(z)$

- algebraic : $(1-z)^\alpha = \sum_{k\geq 1}\binom{\alpha}{k}(-z)^k$ (kind of continuous version of Newton binomial formula) $f_n \sim C/\Gamma(-\alpha)R^{-n}n^{-1-\alpha}$
- Alg-log functions : $(1-z)^\alpha \ln^\beta \frac{1}{1-z}$: $\qquad f_n \sim n^{-\alpha-1}\ln(n)^\beta$

Dominant singularities : one has to add the contribution of each of them.

$$F(z) = 1/(1-z^2) \Longrightarrow f_n = 1^n + (-1)^n$$

Frobenius method

Classifications of singularities for differential equations Fuchs 1866, Fabry 1885.
Poincaré expansions of P-recursive sequences Birkhoff and his student Trjitzinsky 1932

Trjitzinsky-Birkhoff method "ressurected" by Wimp & Zeilberger 1985
$f_n \sim n!^r \exp(n^q) n^\alpha \ln n^h$ with $r, q \in \mathbb{Q}, \alpha \in \mathbb{C}, h \in \mathbb{N}$
non rigourous matched asymptotics : plug and identify...

Frobenius method Frobenius 1873, Wasow : If F is D-finite, then
$F(z) \sim$ linear combination of $\exp(z^r) z^\alpha \ln(z)^i A(z^s)$ with
$r \in \mathbb{Q}, i \in \mathbb{N}, \alpha, s \in \mathbb{C}, A \in \mathbb{Q}[[z]]$.

The GF approach has some advantages : $f_n = n^{\sqrt{17}}$ is not holonomic but is the asymptotic of some holonomic sequence (Quadtrees).
$\ln(n), p_n, \pi(n)$ are not holonomic [via GF]. [Flajolet/Gerhold/Salvy, 2005]. Bernoulli numbers. Bell numbers $\exp(\exp(x) - 1)$. Cayley tree function $C(z) = z \exp(C(z))$. irreducible polynomials on a finite field (=Lyndon words). ("en passant" : not context free).

regular and irregular singularities of DE

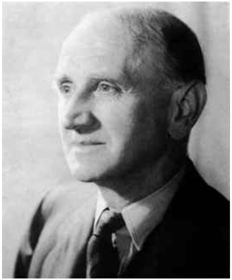
regular singularity (=Fuchsian) : degree of the indicial equation equals the order of the ODE
irregular singularity : degree of the indicial equation smaller than the order of the ODE

$\partial_z^{10} + \cdots + z^{10} F(z)$: regular
$\partial_z^{10} + \cdots + z^{11} F(z)$: irregular

roots differ by integer : "resonance" implies ln

Famous open problem : Frobenius method gives a linear combination, but with which coefficients, i.e. it is unknown if we can get the value of K in $a_n \sim K A^n n^{\alpha}$! ! !
One solution : numerical approximation ! Another solution : Banderier-Chern-Hwang

Sequence accelaration schemes



A first trick (if no "resonance") gives the follow pattern for most of D-finite sequences : $c_n := \frac{f_n^2}{n^\alpha f_{2n}} = K + K_1/n + K_2/n^2 + \dots$.

Aitken Δ^2 method doubles the precision ! : $b_n := c_n - \frac{(c_{n+1}-c_n)^2}{c_{n+1}-2c_n+c_{n-1}} = K + \frac{K_1}{2n} + \dots$

iterated Aitken : $lg(n)$ iterations leads to $O(1/n^2)$.

This often allows to go from 3-4 correct digits to $\sim$ 8 digits.

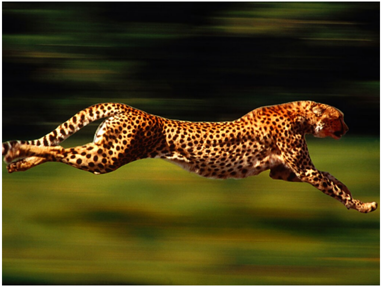
For some specific sequences, it is possible to get more :

Richardson (clever linear combinations), generalized Richardson, (when applied to integrals = Romberg, ...links with Simpson).

This often allows to get $\sim$ 20 digits. It is possible to get more ?

yes : the Acinonyx Jubatus algorithm.

The cheetah algorithm



acinonyx jubatus, aka cheetah.

$$K = \sum_{i=1}^{n} (-1)^{d-i} \frac{i^d}{i!(d-i)!} a(i) + O(1/n^n)$$

This accelaration scheme needs to be adapted if "resonance". This acceleration scheme is impressive when other singularities are "far away".

NB : This shemes is also working if $K_1, K_2, \ldots$ are large ! $b_n = K + K_1/n + K_2/n^2 + \ldots$

An explicit formula for the constant

$$K = \frac{1}{P_0'(\alpha)} \sum_{j\geq 0} B_j r^*(\alpha + j)$$

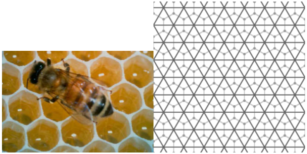
$$B_j = \sum_{k=1}^{d} \frac{P_k(\alpha+k)}{P_0(\alpha+k)} B_{j-k} \qquad\qquad B_0 = 1$$

key ideas : non homogeneous differential equations, change of variables + Cauchy-Euler type, "inverting" the equation by integration, Mellin integral :

$$r^*(s) = \int_0^1 (1-x)^{s-1} r(x) dx$$

Using A=B techniques often allows to evaluate this sum ! Since 1930, it was an open problem (thought to be undecidable) to get the constant K, and we can now prove formulae like $K = 2/\pi, K = \Gamma(1/3), \ldots$ for infinitely many cases ! And we have fast numerical schemes for the remaining cases !

Walks on the honeycomb lattice



Theorem (Banderier 2008)

hexagonal lattice : nice links with Calabi-Yau & number theory.
xy-Manhattan lattice : on $\mathbb{Z}^2$: EllipticK on $\mathbb{N}^2$; excursions $= C_n^2 =$ "bishop moves on $\mathbb{N}^2$" see also [Mishna & Bousquet-Mélou 2008].
x-Manhattan lattice : on $\mathbb{Z}^2$: Heun general function

$$e^{\mathbb{N}}(4n) = \sum_{k=1}^{n} \sum_{i=k}^{n} \binom{n-1}{i-1} \binom{n}{i-1} \binom{i-1}{k-1} \binom{i}{k-1} / (ik)$$

triangular lattice :

$$e^{\mathbb{Z}}(2n) = \sum_{k=0}^{n} \binom{2k}{k} \binom{n}{k}^2 \qquad e^{\mathbb{N}}(2n) = \sum_{k=0}^{n} \binom{2k}{k} \binom{n}{k} \binom{n+1}{k} / (k+1)^2$$

Calabi-Yau equations and number theory

Calabi (1954) conjecture existence of a given Einstein-Kähler metric on compact complex manifolds proven by Yau in 1978. Key step for superstring theory/mirror symmetries (perhaps confirmed by the LHC in the CERN). Huge activities for understanding those equations (kind of generalisation of elliptic curves).... [image].

Intriguing links with number theory : a_n= number of solutions in $\mathbb{Z}/n\mathbb{Z}$.
An associated L-function leads to function whose inverse has some nice properties (rationality/D-finiteness...).

In number theory, those functions first appeared in the work of Beukers (kind of generalisation of the work of Apéry for irrationality of $\zeta(k)$).

Zagier, Zudilin and Almkvist (2008) give a large list of "Calabi-Yau" equations ($\approx$ D-finite equations).

Few digits of Flajolet's constant...

asymptotics for walks on hexagonal lattice... K=a constant which is not in the Plouffe invertor, and for which the Maple "identify(x,all=true)" command finds nothing (LLL/Bailey and Ferguson's PSLQ (Partial Sum of Least Squares) algorithm).
K=1.329553190629908759684153747517674395292135776614883518014551786058118390
1986234122606951694394093641106317406158447247891644240983877209843386697
4988806504131049807028957234718262510710436781197417042063830601898586510
5503354396586243644607903280088302664637101353666792743998428953080760
4852797474903881924061923669438486384328722821830720314450097232604159 4
4117911307016350904025227449807186157980691036817380097177653579150873521
0623417448496044833873654672844810095475969297458071208166612629430473 4
9952510023682607831217758747019694437475007564240536198294821701819061307
3780315664996581087927814743474775518468456198389146677922210294651683 1
1383702825850374744533223642303419594492222653354261950140954742355291 4
3589273086181201224737948134108664635280568428140444158991300559075910 21
1444463742357565086959282891030457462790621842573672215118135450832453 0
6762734846945449189463910996978143341354553319082458805116890485545614 3
6958382232810160002907366818623076194013104839856789344252172963870...(and 10000 more digits!)