Pattern-avoiding fillings of Young diagrams

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Introduction

Let λ be a Young diagram.

Definition

A ⅃-diagram of shape λ is a 0-1 filling of λ such that: for any 0 in the diagram, all cells to its left contain 0, or all cells above it contain 0.

Example

0	0	1	0	1
1	0	1	0	1
1	1	1	0	
0	0	1		
1				

(A. Postnikov, positive Grassmann cells)
(G. Cauchon, primes in quantum algebras)

Equivalently, they are the 0-1 fillings of λ such that the patterns $\begin{smallmatrix}11\\10\end{smallmatrix}$ and $\begin{smallmatrix}01\\10\end{smallmatrix}$ are forbidden.

Definition

A permutation tableau is a Ⅎ-diagram with at least a 1 per column.

Proposition

There is a bijection between between permutation tableaux and permutations.

Another kind of pattern-avoiding fillings are the 0-1 tableaux (De Medicis, Stanton and White)
The only condition is that there is exactly a 1 per column.

Example

0	0	0	0	1
1	0	0	1	
0	1	1		
0	0			

They are in bijection with set partitions.

Another kind of pattern-avoiding fillings are the rook placements. The condition is that there at most a 1 per row, at most a 1 per column.

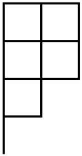
Example

0	0	0	1	0
0	0	1	0	0
1	0	0	0	
0	1	0		
0	0			

They are in bijection with partial involutions.

We define a graph G_λ with:

- a vertex for each row or column of λ,
- an edge for each cell of λ



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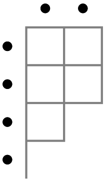

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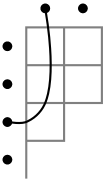

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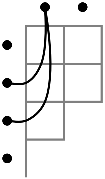

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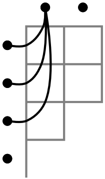

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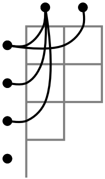

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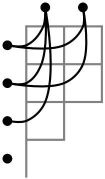
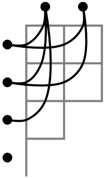




We define a graph G_λ with:

- a vertex for each row or column of λ,
- an edge for each cell of λ

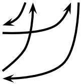
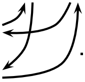
Proposition

The orientations of G_λ are in bijection with the 0-1 fillings of λ.

For example, 0 correspond to  and 1 to 

Proposition

The acyclic orientations of G_λ are in bijection with the 0-1 fillings of λ avoiding the patterns $\begin{smallmatrix}10\\01\end{smallmatrix}$ and $\begin{smallmatrix}01\\10\end{smallmatrix}$.

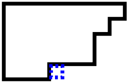
These two patterns correspond to the 4-cycles  and .

An orientation of G_λ is acyclic iff there is no 4-cycle.

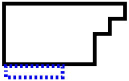
Part 1:

Recursive enumeration of pattern-avoiding 0-1 fillings

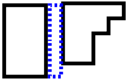
From a Young diagram λ we define strictly smaller Young diagrams:



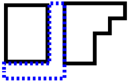
$$\lambda^{(1)} = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_k - 1)$$



$$\lambda^{(2)} = (\lambda_1, \ldots, \lambda_{k-1})$$



$$\lambda^{(3)} = (\lambda_1 - 1, \ldots, \lambda_k - 1)$$



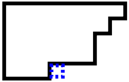
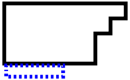
$$\lambda^{(4)} = (\lambda_1 - 1, \ldots, \lambda_{k-1} - 1)$$

Proposition

The number F_λ of $\rfloor$-diagrams of shape λ satisfies:

$$F_\lambda = F_{\lambda^{(1)}} + F_{\lambda^{(2)}} + F_{\lambda^{(3)}} - F_{\lambda^{(4)}}$$

Proof.

$\lambda^{(1)} =$ 
$\lambda^{(2)} =$ 

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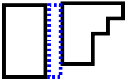
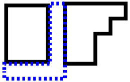
$\lambda^{(3)} =$ 
$\lambda^{(4)} =$ 

The number of ⅃-diagrams of shape λ with a 1 in the corner is

$$F_{\lambda^{(1)}}.$$

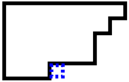
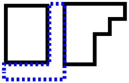
The number of ⅃-diagrams of shape λ with a 0 in the corner is

$$F_{\lambda^{(2)}} + F_{\lambda^{(3)}} - F_{\lambda^{(4)}}.$$

$\square$

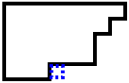
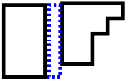
There are similar recurrence relation for rook placements:

$$R_\lambda = R_{\lambda^{(1)}} + R_{\lambda^{(4)}}$$

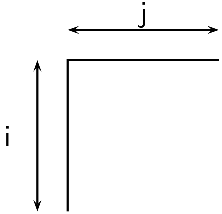
$\lambda^{(1)} =$  $\lambda^{(4)} =$ 

and for 0-1 tableaux:

$$P_\lambda = P_{\lambda^{(1)}} + P_{\lambda^{(3)}}$$

$\lambda^{(1)} =$  $\lambda^{(3)} =$ 

Let λ be the Young diagram with i empty rows, j empty columns, and $|\lambda| = 0$. The initial conditions are:



$$P_\lambda = T_\lambda = \delta_{j0}$$

for permutation tableaux, 0-1 tableaux (*i.e.* when we require at least a 1 per column), and:

$$A_\lambda = F_\lambda = R_\lambda = 1$$

for acyclic orientations, ⅃-diagrams, rook placements.

Proposition (Postnikov)

The number A_λ of acyclic orientations of the graph G_λ satisfies:

$$A_\lambda = A_{\lambda^{(1)}} + A_{\lambda^{(2)}} + A_{\lambda^{(3)}} - A_{\lambda^{(4)}}$$

Proof.

We have $A_\lambda = \chi_\lambda(-1)$ where χ_λ is the chromatic polynomial of G_λ (Stanley).
We prove the result for $\chi_\lambda(x)$ when $x \geq 0$ (enumeration of proper colorings).
Then we specialize at $x = -1$. □

Corollary

For any λ, $A_\lambda = F_\lambda$.

This result means that the number of pattern-avoiding fillings of λ are the same, if the patterns are:

- $\begin{smallmatrix}10\\01\end{smallmatrix}$ and $\begin{smallmatrix}01\\10\end{smallmatrix}$ (acyclic orientations)
- or $\begin{smallmatrix}11\\10\end{smallmatrix}$ and $\begin{smallmatrix}01\\10\end{smallmatrix}$ (Ⅎ-diagrams).

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- $\begin{smallmatrix}10\\01\end{smallmatrix}$ and $\begin{smallmatrix}01\\10\end{smallmatrix}$ (acyclic orientations)
- or $\begin{smallmatrix}11\\10\end{smallmatrix}$ and $\begin{smallmatrix}01\\10\end{smallmatrix}$ (⅃-diagrams).

Proposition (Postnikov, Spiridonov)

The same holds for the patterns $\begin{smallmatrix}11\\11\end{smallmatrix}$ and $\begin{smallmatrix}10\\11\end{smallmatrix}$, $\begin{smallmatrix}11\\11\end{smallmatrix}$ and $\begin{smallmatrix}01\\11\end{smallmatrix}$, $\begin{smallmatrix}10\\11\end{smallmatrix}$ and $\begin{smallmatrix}01\\11\end{smallmatrix}$, $\begin{smallmatrix}10\\11\end{smallmatrix}$ and $\begin{smallmatrix}11\\10\end{smallmatrix}$ (and patterns obtained by transposition, complement).

Part 2:

Bijections

Definition

A row in a 0-1 filling of λ is unrestricted if it contains no 0 with a 1 above it.

Proposition

The number of unrestricted rows in permutation tableaux is equidistribued with the number of cycles in permutations.

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Theorem

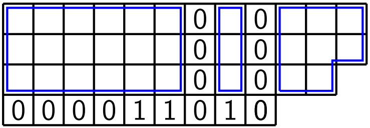
There is a bijection Φ between acyclic orientations of G_λ and $\reflectbox{L}$-diagrams of shape λ, preserving the set of unrestricted rows and the set of zero-columns (there is also a bijection preserving the set of zero-rows and zero-columns).

Definition

A mixed diagram of shape λ is a 0-1 filling such that:

- the $k-1$ first rows avoid the patterns $\begin{smallmatrix}01\\10\end{smallmatrix}$ and $\begin{smallmatrix}10\\01\end{smallmatrix}$,
- for any 0 in the bottom row, either all entries to its left contain 0 or all entries above contain 0.

Example



where the blue region avoids $\begin{smallmatrix}10\\01\end{smallmatrix}$ and $\begin{smallmatrix}01\\10\end{smallmatrix}$

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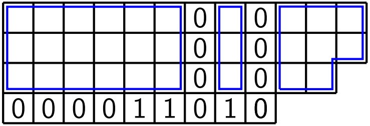


where the blue region avoids $\begin{smallmatrix}10\\01\end{smallmatrix}$ and $\begin{smallmatrix}01\\10\end{smallmatrix}$

Proposition

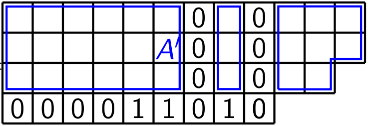
There is a bijection φ between acyclic orientations of G_λ and mixed diagrams of shape λ, preserving the set of unrestricted rows and the set of zero-columns (there is also a bijection preserving the set of zero-rows and zero-columns).

For any acyclic orientation A with k rows, the Ⅎ-diagram $\Phi(A)$ is recursively obtained as follows: take the mixed diagram $\varphi(A)$, and replace the $k-1$ first rows with their image by Φ.

Example

A and A' are acyclic orientation of G_λ.

$\varphi(A) =$



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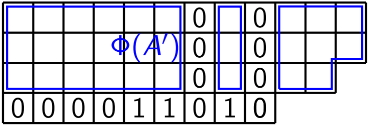
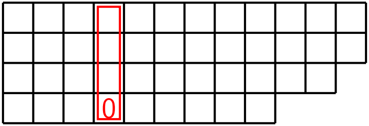


For an acyclic orientation A, the mixed diagram $\varphi(A)$ is defined as follows.

Definition

The pivot column of A is a column

- containing a 0 in bottom position
- containing a maximum number of 1's
- in leftmost position



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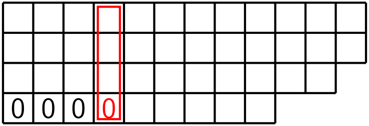

We put 0's on the left of the pivot column.

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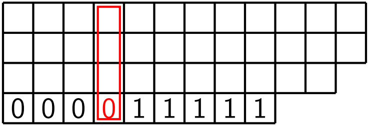

We put 0's on the left of the pivot column.
We put 1's on the right of the pivot column (exception: a zero-column stays a zero-column, a copy of the pivot column becomes a column with a single 1 in bottom position)

Example

1	1	1	1	1	1	1	0	1	1	1	1
1	0	0	0	1	0	0	0	0	0	0	1
1	0	1	1	1	1	0	0	1	0	1	1
1	1	1	1	1	1	1	0	1	1	1	
1	0	1	0	1	1	0	0	0			

1	1	1	1	1	1	1	0	0	1	1	1
1	0	0	0	1	0	0	0	0	0	0	1
1	0	1	1	1	1	0	0	0	0	1	1
1	1	1	1	1	1	1	0	0	1	1	
0	0	0	0	1	1	1	0	1			

The inverse bijection φ^{-1} is also easy to describe.
The pivot column is the rightmost non-zero column with a 0 in bottom position.

Example

	1		1		1						
	1		0		0						
	1		1		1						
	1		1		1						
					0						

It is possible to recover the 1's transformed in 0's : they are in columns containing more 1's than the pivot column.
There is a similar criterion to recover the 0's transformed in 1's.

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There is a similar criterion to recover the 0's transformed in 1's.

The bijection preserving the zero-rows and zero-columns is defined similarly, but we exchange 0 and 1 in the definition of the pivot column:

Definition

The pivot column of A is a column

- containing a 1 in bottom position
- containing a maximum number of 0's
- in leftmost position

Generalizations

A similar method gives bijections for other pattern-avoiding fillings, for example:

- Ⅎ-diagrams and $\left(\begin{smallmatrix}01\\11\end{smallmatrix}, \begin{smallmatrix}10\\11\end{smallmatrix}\right)$-avoiding fillings,
- Ⅎ-diagrams and $\left(\begin{smallmatrix}10\\01\end{smallmatrix}, \begin{smallmatrix}11\\01\end{smallmatrix}\right)$-avoiding fillings,

and all other patterns obtained by symetry, complement.

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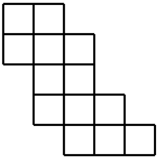
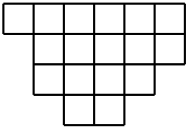
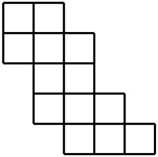
and all other patterns obtained by symetry, complement.
The bijection between ⅃-diagrams and acyclic orientations is extended to other kinds of shapes (for example, skew shapes, stack polyominoes...) This gives bijective proofs for results of Spiridonov.

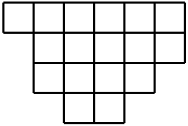




Generalizations to other shapes

The bijection between Ⅎ-diagrams and acyclic orientations is extended to other kinds of shapes (for example, skew shapes, stack polyominoes...) This gives bijective proofs for results of Spiridonov.

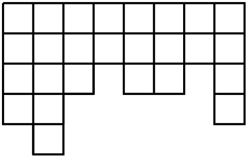




Method: consider the maximal rectangles included in these shapes, and intersecting the bottom row.
Take as a pivot column, the rightmost pivot column of these rectangles.
Make a column-by-column transformation as in the previous case.

The bijection also works for "comb" polyominoes.

Example



Corollary

In this case the number of ⅃-diagrams only depends on the column lengths.

Proof.

The number of $\left(\begin{smallmatrix}10\\01\end{smallmatrix},\begin{smallmatrix}01\\10\end{smallmatrix}\right)$-avoiding fillings only depends on the columns lengths (we can permute the columns). So it is a consequence of the previous bijection. $\square$