Cyclic symmetry (Talk 1) invariant theory (Talk 2), $q-$ and $t-$analogues (Talk 3)

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Talk 1 Outline

I. The cyclic sieving phenomenon (CSP)

II. Example 1: subsets.

III. Keywords

IV. "Bad" / "good" proofs.

V. A good proof via invariant theory
($\rightsquigarrow$ Talk 2).

I. The cyclic sieving phenomenon (CSP) (–, Stanton, and White 2004)

Given

- a finite set X, and
- a polynomial $X(t) \in \mathbb{Z}[t]$, and
- a cyclic group $C = \langle c \rangle \cong \mathbb{Z}/n\mathbb{Z}$ permuting X,

say the triple $(X, X(t), C)$ exhibits the CSP if for any element c^m in C, the number of elements of X which c^m fixes is

$$|X^{c^m}| = [X(t)]_{t=\left(e^{\frac{2\pi i}{n}}\right)^m}$$

In particular, $|X| = X(1)$.

In examples,

– most often $X(t) \in \mathbb{N}[t]$,

– sometimes $X(t)$ is a generating function

for X of the form

$$X(t) = \sum_{x \in X} t^{s(x)},$$

– sometimes a Hilbert series

$$\begin{aligned} X(t) &= \mathrm{Hilb}(U, t) \\ &:= \sum_{d \geq 0} \dim(U_d)\, t^d \end{aligned}$$

for some interesting graded
vector space/ring/representation

$$U = \bigoplus_{d \geq 0} U_d.$$

Special case when $C = \mathbb{Z}/2\mathbb{Z}$:
Stembridge's $t = -1$ phenomenon (1994):

$$[X(t)]_{t=-1} = |X^c|$$

for some involution $c : X \to X$.

This turned out to be useful in organizing some results enumerating plane partitions with symmetry.

I. Example 1– subsets

$X :=$ k-subsets of $\{1, 2, \ldots, n\}$

$X(t) :=$ t-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_t = \frac{[n]!_t}{[k]!_t [n-k]!_t},$$

with $[n]!_t := [n]_t \cdots [2]_t [1]_t$

$$[n]_t := 1 + t + t^2 + \cdots + t^{n-1} = \frac{1 - t^n}{1 - t}$$

$C := \langle (123 \cdots n) \rangle \cong \mathbb{Z}/n\mathbb{Z}$

cyclically permuting $\{1, 2, \ldots, n\}$,

and therefore also permuting k-subsets .

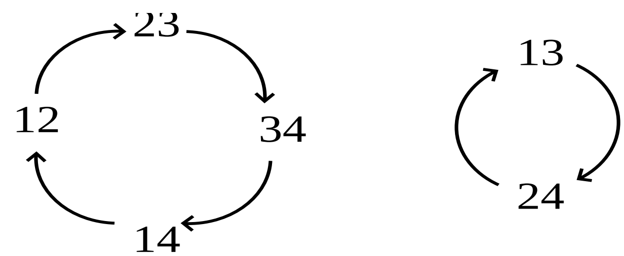
THM (–, Stanton, White 2004)
This triple $(X, X(t), C)$ exhibits the CSP.

Example 1 (continued)

For $n = 4, k = 2$, the set

$$X = \{12, 13, 14, 23, 24, 34\}$$

carries this action of $C = \mathbb{Z}_4$:



$$X(t) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_t = \frac{[4]_t[3]_t}{[2]_t} = 1 + t + 2t^2 + t^3 + t^4$$

evaluates at 4^{th}-roots of unity as

$$X(\omega) = \begin{cases} 6(= |X|) & \text{if } \omega = 1 \\ 2(= |X^{c^2}|) & \text{if } \omega = -1 \\ 0(= |X^c| = |X^{c^3}|) & \text{if } \omega = \pm i. \end{cases}$$

Alternate phrasing of CSP:
in the unique expansion

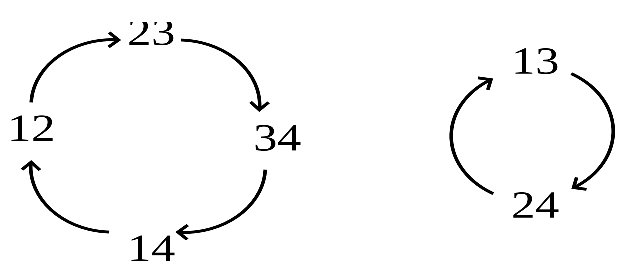
$$X(t) \equiv a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1} \mod t^n - 1$$

a_i counts the C-orbits on X for which the C-stabilizer has order dividing i.

In particular,
a_0 is the number of C-orbits in total,
a_1 is the number of C-orbits which are free.

E.g. above

$$\begin{aligned} X(t) &= 1 + t + 2t^2 + t^3 + t^4 \\ &\equiv 2 + t + 2t^2 + t^3 \mod t^4 - 1 \end{aligned}$$



A few remarks on Example 1...

REMARK:

One also has the CSP for $(X, X(t), C)$ with
same set X equal to all k-subsets of $\{1, 2, \ldots, n\}$
same set $X(t) = \begin{bmatrix} n \\ k \end{bmatrix}_t$,
different cyclic group

$$C = \langle (123 \cdots n-1)(n) \rangle \cong \mathbb{Z}/(n-1)\mathbb{Z}.$$

But then it fails for any other
cyclic subgroup C of permutations which is
not a subgroup of $\langle (123 \cdots n) \rangle$
or $\langle (123 \cdots n-1)(n) \rangle$!

REMARK:

$X(t) = \begin{bmatrix} n \\ k \end{bmatrix}_t$ has many interpretations;
we emphasize one from invariant theory...

Let $S := \mathbb{C}[x_1, \ldots, x_n]$, with symmetric group $\mathfrak{S}_n$ permuting variables. Then one has

$$
\begin{aligned}
X(t) &= \begin{bmatrix} n \\ k \end{bmatrix}_t \\
&= \frac{1}{(1-t)\cdots(1-t^k)\cdot(1-t)\cdots(1-t^{n-k})} \\
&\qquad \Big/ \frac{1}{(1-t)\cdots(1-t^n)} \\
&= \mathrm{Hilb}(S^{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}, t)/\mathrm{Hilb}(S^{\mathfrak{S}_n}, t) \\
&= \mathrm{Hilb}(S^{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}/(S_+^{\mathfrak{S}_n}), t).
\end{aligned}
$$

Note that one can think of our set X as

$$k-\text{subsets of } \{1, 2, \ldots, n\} \quad \longleftrightarrow \quad \mathfrak{S}_n/(\mathfrak{S}_k \times \mathfrak{S}_{n-k}).$$

III. Keywords

Some examples of CSP's we have encountered, conjecturally in at least one case:

– $X = k$-dimensional subspaces of $(\mathbb{F}_q)^n$
(that is, q-Example 1, which led to Talks 2, 3)
– $X =$multisets
– $X =$Polya colorings
– $X =$polygon triangulations/dissections
($\rightsquigarrow W$-clusters)
– $X =$noncrossing partitions
($\rightsquigarrow W$-noncrossing partitions)
– $X =$nonnesting partitions
($\rightsquigarrow W$-nonnesting partitions)
–$X =$rectangular-shaped tableaux
–$X =$alternating sign matrices

IV. “Bad” versus “Good” proofs

Given $(X, X(t), C)$, a “bad” way to prove

$$|X^{c^m}| = [X(t)]_{t=\left(e^{\frac{2\pi i}{n}}\right)^m}$$

(i) evaluates the right side
(often via a product formula for $X(t)$,

(ii) counts the left side,
(often via good ol’ combinatorics),

(iii) equates the answers!

Here's a "good" way to prove

$$|X^{c^m}| = [X(t)]_{t=\left(e^{\frac{2\pi i}{n}}\right)^m}.$$

(i) Find a natural graded vector space

$$U = \oplus_{d \geq 0} U_d$$

with

$$X(t) = \mathrm{Hilb}(U, t).$$

Then the C-action on U defined by having c act as the scalar $(e^{\frac{2\pi i}{n}})^d$ on U_d has the trace of c^m on U equal to

$$\sum_{d \geq 0} \dim(U_d)\,(e^{\frac{2\pi i}{n}})^{dm} = [X(t)]_{t=\left(e^{\frac{2\pi i}{n}}\right)^m}$$

(ii) Define a permutation representation $\mathbb{C}[X]$ of C having $\mathbb{C}$-basis elements

$$\{e_x\}_{x \in X}$$

and C-action by permuting the basis:

$$c(e_x) = e_{c(x)}.$$

Then the trace of c^m on $\mathbb{C}[X]$ equals $|X^{c^m}|$.

(iii) Prove that as C-representations,

$$\mathbb{C}[X] \cong U.$$

Then c^m should have the same trace in both:

$$|X^{c^m}| = [X(t)]_{t=\left(e^{\frac{2\pi i}{n}}\right)^m}.$$

Harder than it looks, of course!

Sadly, many of our CSP proofs are "bad", but some have been replaced by "good" ones.

MORAL: t is a grading variable in many CSP's.

V. Example 1, the "good" way via invariant theory

Let $V = \mathbb{C}^n$, and
W a finite subgroup of $GL(V) = GL_n(\mathbb{C})$.

Then W acts on $S = \mathbb{C}[x_1, \ldots, x_n]$
via linear substitutions variables.

THM (Shephard-Todd, Chevalley 1955)
When the group W is generated by reflections (= elements r with V^r a hyperplane), there is an isomorphism of W-representations between the coinvariant algebra and the left-regular represenation:

$$S/(S_+^W) \cong \mathbb{C}[W].$$

We need more....

Say that an element c in a finite reflection group W is regular if it has an eigenvector v that avoids all of the reflection hyperplanes. Hence $c(v) = \omega \cdot v$ for a root-of-unity ω in $\mathbb{C}$.

THM (T.A. Springer 1972)
Let $C = \langle c \rangle$ be generated by a regular element c in a finite reflection group W.

Then the Shephard-Todd/Chevalley isomorphism

$$S/(S_+^W) \cong \mathbb{C}[W].$$

extends to one of $W \times C$-representations, with W acting as before, but C acting...
– on left, via scalar substitutions

$$c(x_i) = \omega x_i,$$

– on right, via right-translation: $c(e_w) = e_{wc}$.

Now given any subgroup W' of W
(think $W = \mathfrak{S}_n$ and $W' = \mathfrak{S}_k \times \mathfrak{S}_{n-k}$)
take the W'-fixed spaces
in Springer's $W \times C$-isomorphism,
leaving a C-isomorphism:

$$\left(S/(S_+^W)\right)^{W'} \cong \mathbb{C}[W]^{W'}$$

Then say some magic words turning this into...

$$S^{W'}/(S^W{}_+) \cong \mathbb{C}[W' \backslash W]$$

The left side is our U modelling

$$X(t) = \mathsf{Hilb}(S^{W'}/(S_+^W), t) = \frac{\mathsf{Hilb}(S^{W'}, t)}{\mathsf{Hilb}(S^W, t)}$$

The right side is $\mathbb{C}[X]$ where $X = W' \backslash W$,
and C acts by right-translating cosets:

$$c(W'w) = W'wc.$$

Equating traces of c^m on both sides gives...

COR(−,Stanton,White 2004)
For a regular element c in a
complex reflection group W,
and any subgroup W',
the triple $(X, X(t), C)$ in which

$$X = W/W'$$

$C = \langle c \rangle$ left-translating cosets

$$X(t) = \mathrm{Hilb}(S^{W'}/(S^W_+), t) = \frac{\mathrm{Hilb}(S^{W'}, t)}{\mathrm{Hilb}(S^W, t)}$$

always exhibits the CSP.

Example 1 comes from

$W = \mathfrak{S}_n$,
$W' = \mathfrak{S}_k \times \mathfrak{S}_{n-k}$,
$c = (123 \cdots n)$ or $c = (123 \cdots n-1)(n)$:

Note that setting $\zeta_n := e^{\frac{2\pi i}{n}}$,
then $c = (123 \cdots n)$ is regular because
it has ζ_n-eigenvector

$$(1, \zeta_n^{1}, \zeta_n^{2}, \ldots, \zeta_n^{n-1})$$

while $c = (123 \cdots n-1)(n)$ is regular because
it has ζ_{n-1}-eigenvector

$$(1, \zeta_{n-1}^{1}, \zeta_{n-1}^{2}, \ldots, \zeta_{n-1}^{n-2}, 0).$$

Talk 2: Invariant theory
Outline

I. Example 1: subsets.

II. q-Example 1: subspaces.

III. A general Springer-type theorem
(with Bram Broer,
Larry Smith,
and Peter Webb)

I. Recall the CSP and Example 1

Recall $(X, X(t), C)$ exhibits the CSP if for any element c^m in C, the number of elements of X which c^m fixes is

$$|X^{c^m}| = [X(t)]_{t=\left(e^{\frac{2\pi i}{n}}\right)^m}$$

Example 1 was

$$X = k\text{-subsets of}\{1,2,\ldots,n\} = \mathfrak{S}_n/(\mathfrak{S}_k \times \mathfrak{S}_{n-k})$$

$$C = \langle(123\cdots n)\rangle$$

$$X(t) = \begin{bmatrix} n \\ k \end{bmatrix}_t = \frac{\mathrm{Hilb}(S^{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}, t)}{\mathrm{Hilb}(S^{\mathfrak{S}_n}, t)}$$
$$= \frac{1}{(1-t)\cdots(1-t^k)\cdot(1-t)\cdots(1-t^{n-k})}$$
$$\Big/ \frac{1}{(1-t)\cdots(1-t^n)}$$

where $S = \mathbb{C}[x_1, \ldots, x_n]$
and $S^{\mathfrak{S}_n} = \mathbb{C}[e_1(\mathbf{x}), e_2(\mathbf{x}), \ldots, e_n(\mathbf{x})]$ with

$$e_i(\mathbf{x}) = \sum_{|I|=i} \left(\prod_{i \in I} x_i \right).$$

I. q-Example 1

For the q-analogue, we take

$$X = k\text{-dimensional subspaces of } \mathbb{F}_q^n = G/P$$

which carries a transitive action of

$$G := GL_n(\mathbb{F}_q) = GL_{\mathbb{F}_q}(\mathbb{F}_q^n)$$

and P is the parabolic subgroup
fixing some particular k-subspace.

Where do we get a cyclic action on X?

Any element c inside $G = GL_n(\mathbb{F}_q)$ could be taken to generate the cyclic group C.

But the correct q-analogue of $c = (123 \cdots n)$ turns out to be a Singer cycle c, that is, a generator for the (cyclic!) group

$$\mathbb{F}_{q^n}^{\times} \cong \mathbb{Z}/(q^n - 1)/\mathbb{Z}$$

embedded into

$$G := GL_n(\mathbb{F}_q) \cong GL_{\mathbb{F}_q}(\mathbb{F}_q^n) \cong GL_{\mathbb{F}_q}(\mathbb{F}_{q^n})$$

by picking any $\mathbb{F}_q$-vector space isomorphism $\mathbb{F}_q^n \cong \mathbb{F}_{q^n}$.

What $X(t)$ will we take with $X = G/P$?

Let $S := \mathbb{F}_q[x_1, \ldots, x_n]$.

Then the group $G = GL_n(\mathbb{F}_q)$ acts on S
by linear substitutions of variables,
and so does the subgroup P.

Not surprisingly perhaps, we choose

$$X(t) = \frac{\mathrm{Hilb}(S^P, t)}{\mathrm{Hilb}(S^G, t)}$$

But what is this $X(t)$ explicitly?

THM (L.E. Dickson 1911) The invariant ring

$$S^G = \mathbb{F}_q[D_{n,0}, D_{n,1}, \ldots, D_{n,n-1}]$$

for $G = GL_n(\mathbb{F}_q)$ is a polynomial algebra, whose generators $D_{n,i}$ have degrees $q^n - q^i$, and can be written

$$D_{n,i} = \sum_{\substack{i\text{-dim'l subspaces} \\ U \subset (\mathbb{F}_q^n)^*}} \left(\prod_{\ell(\mathbf{x}) \notin U} \ell(\mathbf{x}) \right).$$

Hence one has $\mathrm{Hilb}(S^G, t) = \frac{1}{n!_{q,t}}$ where

$$n!_{q,t} = (1 - t^{q^n - 1})(1 - t^{q^n - q}) \cdots (1 - t^{q^n - q^{n-1}})$$

This was generalized by Mui (1975) to a result for all of the parabolic subgroups P, showing that

$$\text{Hilb}(S^P, t) = \frac{1}{k!_{q,t} \cdot (n-k)!_{q,t^{q^k}}}$$

Hence their quotient gives an explicit product formula for

$$\begin{aligned} X(t) &= \frac{\text{Hilb}(S^P, t)}{\text{Hilb}(S^G, t)} \\ &= \frac{1}{k!_{q,t} \cdot (n-k)!_{q,t^{q^k}}} \\ &=: \begin{bmatrix} n \\ k \end{bmatrix}_{q,t} \\ &= \text{ the } (q,t)\text{-binomial coefficient.} \end{aligned}$$

THM
(−, Stanton, White 2004, via "bad" proof!)

The triple

$$X = G/P = k\text{-subspaces of } \mathbb{F}_q^n$$
$$X(t) = \begin{bmatrix} n \\ k \end{bmatrix}_{q,t}$$
$$C = \mathbb{F}_{q^n}^{\times} = \langle c \rangle \cong \mathbb{Z}/(q^n - 1)\mathbb{Z}$$

exhibits the CSP.

We wanted a better proof,
that explained more examples over $\mathbb{F}_q$,
involving other subgroups of $G = GL_n(\mathbb{F}_q)$.

III. A more general Springer theorem

Recall that Springer's theorem
was about (complex) reflection groups.

INTERESTING FACT:
$G = GL_n(\mathbb{F}_q)$ is a reflection group!

THM (Serre 1967)
For *any* field $\mathbb{F}$,
if a finite subgroup G of $GL_n(\mathbb{F})$
acting on $S := \mathbb{F}[x_1, \ldots, x_n]$ has
the invariant ring S^G a polynomial algebra,
then G must be generated by reflections.

The converse is false generally,
but true in characteristic zero (Chevalley 1955)

Here "reflections" are still elements r
for which the fixed space $(\mathbb{F}^n)^r$ is a hyperplane.

But in positive characteristic, it allows
for r to be a transvection, that is,
non-semisimple, of determinant 1, e.g.

$$r = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note one can generate $G = GL_n(\mathbb{F}_q)$
using transvections and semisimple reflections.

When S^G is polynomial,
so that G is generated by reflections,
define a regular element c in G
(as before) to be one with an eigenvector v
that avoids all the reflecting hyperplanes.

PROP
An element c in $GL_n(\mathbb{F}_q)$ is regular
$\Leftrightarrow$ c is a power of a Singer cycle, that is,
c is in the image of some embedding

$$\mathbb{F}_{q^n}^{\times} \hookrightarrow GL_n(\mathbb{F}_q)$$

THM(Broer, −, Smith, Webb, 2007)
Let $\mathbb{F}$ be any field, and $S = \mathbb{F}[x_1, \ldots, x_n]$.
Let G be a finite subgroup of $GL_n(\mathbb{F})$
with S^G polynomial.
Let C be the cyclic subgroup generated by
a regular element c in G.
Let H be any subgroup of G.

Then the triple

$$X = G/H$$

$$X(t) = \frac{\mathrm{Hilb}(S^H, t)}{\mathrm{Hilb}(S^G, t)}$$

$$C = \langle c \rangle \text{ left-translating cosets } gH$$

always exhibits the CSP.

MORAL:
This $X(t)$ is the right way to
introduce a grading variable into a set
$X = G/H$ that has a transitive G-action.

Some ideas of the proof...

IDEA 1 Because $char(\mathbb{F})$ might not be zero, and S^H is not always Cohen-Macaulay,

$$X(t) = \frac{\text{Hilb}(S^H, t)}{\text{Hilb}(S^G, t)}$$
$$\neq \text{Hilb}(\underbrace{\underbrace{S^H/(S^G_+)}_{S^H \otimes_{S^G} \mathbb{F}}}_{\text{Tor}_0^{S^G}(S^H,\mathbb{F})}, t)$$

However the following corrects this:

$$\begin{aligned} X(t) &= \text{Hilb}(\text{Tor}_0^{S^G}(S^H, \mathbb{F}), t) \\ &\quad - \text{Hilb}(\text{Tor}_1^{S^G}(S^H, \mathbb{F}), t) \\ &\quad + \text{Hilb}(\text{Tor}_2^{S^G}(S^H, \mathbb{F}), t) - \cdots \\ &= \sum_{i=0}^{n} (-1)^i \text{Hilb}(\text{Tor}_i^{S^G}(S^H, \mathbb{F}), t) \end{aligned}$$

So work with all of $\text{Tor}_*^{S^G}(S^H, \mathbb{F})$
not just $\text{Tor}_0^{S^G}(S^H, \mathbb{F}) = S^H/(S^G_+)$ as in Springer.

IDEA 2

Let $G \subset GL_n(\mathbb{F})$ act on $V := \mathbb{F}^n$,
and on $S = \mathbb{F}[x_1, \ldots, x_n]$.

Then the surjection $V \xrightarrow{\pi} V/G$
corresponds to the inclusion $S^G \hookrightarrow S$.

(Same for $V \to V/H \to V/G$
and $S^G \hookrightarrow S^H \to S$.)

Then $S/(S_+^G)$ is the coordinate ring
of the fiber $\pi^{-1}(\pi(0))$.

Compare it with the fiber $\pi^{-1}(\pi(v))$, where
v is the eigenvector of the regular element c.

The latter fiber $\pi^{-1}(\pi(v))$ has a free G-action,
and even a fairly simple $G \times C$-action.

Talk 3: q- and t-analogues
Outline

We'll see examples of ...

$$\begin{array}{ccccc}
 & & |X| \in \mathbb{N} & & \\
q=1 \nearrow & & & & \nwarrow t=1 \\
|X_q| \in \mathbb{N}[q] & & \overset{t \leftrightarrow q}{\longleftrightarrow} & & X(t) \in \mathbb{N}[t] \\
t=1 \nwarrow & & & & \nearrow t \mapsto t^{\frac{1}{q-1}}, q=1 \\
 & & X_q(t) & &
\end{array}$$

with CSP for $(X, X(t), C)$ in which
$C = \langle c \rangle$ for c an n-cycle in $\mathfrak{S}_n$,

and CSP for $(X_q, X_q(t), C_q)$ in which
$C_q = \langle c_q \rangle$ for c_q a Singer cycle in $GL_n(\mathbb{F}_q)$.

We've seen one such example already with

$$\begin{array}{lll} X & = k\text{-subsets of } \{1,2,\ldots,n\} & = \mathfrak{S}_n/(\mathfrak{S}_k \times \mathfrak{S}_{n-k}) \\ X_q & = k\text{-subspaces of } \mathbb{F}_q^n & = G/P \end{array}$$

$$\begin{array}{ccc} & |X| = \binom{n}{k} & \\ q=1 \nearrow & & \nwarrow t=1 \\ |X_q| = \begin{bmatrix} n \\ k \end{bmatrix}_q & \overset{t\leftrightarrow q}{\longleftrightarrow} & X(t) = \begin{bmatrix} n \\ k \end{bmatrix}_t \\ & & = \frac{\mathrm{Hilb}(S^{\mathfrak{S}_k \times \mathfrak{S}_{n-k}},t)}{\mathrm{Hilb}(S^{\mathfrak{S}_n},t)} \\ t=1 \nwarrow & & \nearrow t \mapsto t^{\frac{1}{q-1}}, q=1 \\ & X_q(t) = \begin{bmatrix} n \\ k \end{bmatrix}_{q,t} & \\ & = \frac{\mathrm{Hilb}(S^P,t)}{\mathrm{Hilb}(S^G,t)} & \end{array}$$

E.g. $n = 2$ and $k = 1$ looks like this...

$$|X| = \binom{2}{1}$$

$$q = 1 \nearrow \qquad\qquad\qquad\qquad \nwarrow t = 1$$

$$|X_q| = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = q + 1 \qquad \overset{t \leftrightarrow q}{\longleftrightarrow} \qquad X(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_t = t + 1$$

$$t = 1 \nwarrow \qquad\qquad\qquad\qquad \nearrow t \mapsto t^{\frac{1}{q-1}}, q = 1$$

$$\begin{aligned} X_q(t) &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{q,t} \\ &= \frac{2!_{q,t}}{1!_{q,t} \cdot 1!_{q,t^q}} \\ &= \frac{(1-t^{q^2-1})(1-t^{q^2-q})}{(1-t^{q-1})(1-t^{q^2-q})} \\ &= \frac{1-t^{(q-1)(q+1)}}{1-t^{q-1}} \\ &= [q+1]_{t^{q-1}} \end{aligned}$$

An interesting extra feature in this example... Think of X as partitions λ whose Ferrers diagram fits inside a $k \times (n-k)$ rectangle. Then

$$X(t) = \begin{bmatrix} n \\ k \end{bmatrix}_t = \sum_{\lambda \in X} t^{|\lambda|}$$

$$|X_q| = \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \in X} q^{|\lambda|}$$

THM (–, Stanton 2008) One has

$$X_q(t) = \begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \sum_{\lambda \in X} \mathrm{wt}(\lambda; q, t)$$

where

$$\mathrm{wt}(\lambda; q, t) = \prod_{\text{cells } x \text{ of } \lambda} t^{a(x)} [q]_{t q^{b(x)} - q^{c(x)}}.$$

In particular, $\mathrm{wt}(\lambda; q, t) \to q^{|\lambda|}, t^{|\lambda|}$ under the two kinds of limits that send $X_q(t)$ to $|X_q|, X(t)$.

This all persists in more general examples.

For any composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of n, consider the Young subgroup

$$\mathfrak{S}_\alpha := \mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_\ell}$$

inside $\mathfrak{S}_n$,

and the corresponding parabolic subgroup P_α inside $G = GL_n(\mathbb{F}_q)$ that stabilizes some particular flag of subspaces having dimensions

$$D(\alpha) := (\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots)$$

One then finds the same story with

$$X = \mathfrak{S}_n/\mathfrak{S}_\alpha$$

$$X_q = G/P_\alpha$$

together with the usual $q-$ or $t-$multinomial coefficients

$$X(t) = \begin{bmatrix} n \\ \alpha \end{bmatrix}_t$$

$$|X_q| = \begin{bmatrix} n \\ \alpha \end{bmatrix}_q$$

and the (q,t)-multinomial

$$X_q(t) = \begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} := \frac{\mathsf{Hilb}(S^{P_\alpha}, t)}{\mathsf{Hilb}(S^G, t)}.$$

Here one can think of X as

$$X = \{w \in \mathfrak{S}_n : \mathrm{Des}(w) \subseteq D(\alpha)\}$$

where $\mathrm{Des}(w)$ is the usual descent set of a permutation w. Then

$$X(t) = \begin{bmatrix} n \\ \alpha \end{bmatrix}_t = \sum_{w \in X} t^{\ell(w)}$$

$$|X_q| = \begin{bmatrix} n \\ \alpha \end{bmatrix}_q = \sum_{w \in X} q^{\ell(w)}$$

with $\ell(w)$ the length/inversion number of w.

THM (–, Stanton 2008) One has

$$X_q(t) = \begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} = \sum_{w \in X} \mathrm{wt}(w; q, t)$$

where $\mathrm{wt}(w; q, t)$ has a summation-of-products expression as before.

This suggests consideration of the more refined descent classes

$$X = \{w \in \mathfrak{S}_n : \mathrm{Des}(w) = D(\alpha)\}$$

and their length generating functions

$$X(t) := \sum_{w \in X} t^{\ell(w)}$$
$$|X_q| := \sum_{w \in X} q^{\ell(w)}$$

as well as

$$X_q(t) := \sum_{w \in X} \mathrm{wt}(w; q, t)$$

where $\mathrm{wt}(w; q, t)$ is the same weight that appeared before.

Can we say anything meaningful about these?

Yes- two things. Firstly,
MacMahon's determinantal formula for descent class sizes

$$|X| = n! \det \left(\frac{1}{(\alpha_i + \alpha_{i+1} + \cdots + \alpha_j)!} \right)_{i,j=1,\ldots,\ell}$$

which was generalized by Stanley to

$$X(t) = [n]!_t \det \left(\frac{1}{[\alpha_i + \alpha_{i+1} + \cdots + \alpha_j]!_t} \right)_{i,j=1,\ldots,\ell}$$

$$|X_q| = [n]!_q \det \left(\frac{1}{[\alpha_i + \alpha_{i+1} + \cdots + \alpha_j]!_q} \right)_{i,j=1,\ldots,\ell}$$

generalizes further to

THM(–, Stanton 2008)

$$X_q(t) = [n]!_{q,t} \det \left(\frac{1}{[\alpha_i + \alpha_{i+1} + \cdots + \alpha_j]!_{q,t^{q^{\sum_{m=1}^{i-1} \alpha_i}}}} \right)_{i,}$$

where

$$[n]!_{q,t} := (1 - t^{q^n - 1})(1 - t^{q^n - q}) \cdots (1 - t^{q^n - q^{n-1}}).$$

Secondly, one has homological and invariant theory interpretations.

The size of the descent class $|X|$ gives the dimension of the top (and only) homology group for the α-rank-selected subcomplex of the Coxeter complex for $\mathfrak{S}_n$, or the order complex of the Boolean algebra. Call this homology $\mathfrak{S}_n$-representation χ^α.

The polynomial $|X_q| = \sum_{w \in X} q^{\ell(w)}$ was shown by Björner (1984) to give the dimension of the top (and only) homology group for the α-rank-selected subcomplex of the of the Tits building for $GL_n(\mathbb{F}_q)$, or the order complex of the subspace lattice. Call this homology $GL_n(\mathbb{F}_q)$-representation χ_q^α.

On the other hand, one can show the following

THM(–, Stanton 2008)

$$X(t) := \sum_{w \in X} t^{\ell(w)} = \frac{\mathrm{Hilb}(M, t)}{\mathrm{Hilb}(S^{\mathfrak{S}_n}, t)}$$

where $M := \mathrm{Hom}_{\mathfrak{S}_n}(\chi^{\alpha}, S)$, and

$$X_q(t) := \sum_{w \in X} \mathrm{wt}(w; q, t) = \frac{\mathrm{Hilb}(M^q, t)}{\mathrm{Hilb}(S^G), t}$$

where $M^q := \mathrm{Hom}_G(\chi_q^{\alpha}, S)$.

In the special case $\alpha = 1^n$,
this last result is related to work of
the topologists N. Kuhn and S. Mitchell (1984).

They were interested in knowing exactly
how many copies of the Steinberg module
of $GL_n(\mathbb{F}_q)$ occur in each graded component
of $S = \mathbb{F}_q[x_1, \ldots, x_n]$.

An incomplete picture for column-strict tableaux

Let X be all column-strict tableaux of a skew-shape λ/μ with entries in $\{0, 1, \ldots, n\}$.

An appropriate t-analogue is the principally specialized Schur function

$$X(t) := s_{\lambda/\mu}(1, t, t^2, \ldots, t^n).$$

This can then be generalized to a suitable (q, t)-analogue $X_q(t)$ that has many of the good properties we have seen, including a product formulae, and $X(t)$ as an appropriate limit.

These polynomials $X_q(t)$ in fact are lifts from $\mathbb{F}_q[t]$ to $\mathbb{Z}[t]$ of principal specializations of Macdonald's "7th variation" on Schur functions from SLC 1992.

QUESTION
What is the algebraic meaning (e.g. invariant-theoretic, Hilbert series) for these (q,t)-analogues $X_q(t)$?