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HANKEL DETERMINANTS OF SOME SEQUENCES OF POLYNOMIALS

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ABSTRACT. Ehrenborg gave a combinatorial proof of Radoux's theorem which states that the determinant of the $(n+1)\times(n+1)$ dimensional Hankel matrix of exponential polynomials is $x^{n(n+1)/2}\prod_{i=0}^{n} i!$. This proof also shows the result that the $(n+1)\times(n+1)$ Hankel matrix of factorial numbers is $\prod_{k=1}^{n}(k!)^2$. We observe that two polynomial generalizations of factorial numbers also have interesting determinant values for Hankel matrices.

A polynomial generalization of the determinant of the Hankel matrix with entries being fixed-point free involutions on the set $[2n]$ is given next. We also give a bivariate non-crossing analogue of a theorem of Cigler about the determinant of a similar Hankel matrix.

1. INTRODUCTION

For a sequence $A = (a_0, a_1, a_2, \ldots)$, consider the matrix

$$H = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The nth Hankel matrix H_n of A is the top left $(n+1)\times(n+1)$ sized matrix. In this work, the a_i's will be sequences of polynomials.

Let $[n] = \{1, 2, \ldots, n\}$, and let SP_n be the set of set-partitions of $[n]$. For a set-partition $\pi \in \mathsf{SP}_n$, let $|\pi|$ be the number of parts in π. Consider the n-th exponential polynomial $e_n(x) = \sum_\pi x^{|\pi|} = \sum_{k=0}^{n} S(n,k)x^k$, where the $S(n,k)$'s are Stirling numbers of the second kind. The following is known (see Krattenthaler [7, Eqn. (3.54)] or Radoux [9]).

Theorem 1. *The determinant of the nth Hankel matrix of exponential polynomials is*

$$x^{n(n+1)/2}\prod_{k=1}^{n} k!.$$

Ehrenborg [4] gave a combinatorial proof of Theorem 1 which also proves the following classical result.

Theorem 2. *The determinant of the nth Hankel matrix of the factorial numbers (i.e., $\det((i+j)!)_{0\le i,j\le n}$) is $\prod_{k=1}^{n}(k!)^2$.*

We give generalizations of this result where we replace $n!$ by polynomials $p_n(.)$ in variables q, x and t . Consider the Eulerian polynomial $\mathsf{Eu}_n(x) = \sum_{k=0}^{n} A(n,k)x^k$ where $A(n,k)$ is the number of permutations in S_n with k excedances (or k descents; see Section 2 for precise definitions), with $\mathsf{Eu}_0(x) = 1$. In Section 2, we prove the following determinant evaluation.

Theorem 3. *The determinant of the nth Hankel matrix with (i, j)-th entry (for $0 \leq i, j \leq n$) being $\mathsf{Eu}_{i+j}(x)$ is $x^{n(n+1)/2} \prod_{k=1}^{n} k!^2$.*

We also give a bivariate generalisation. Consider the Eulerian-Mahonian polynomial $\mathsf{EI}_n(q, t) = \sum_{\pi \in S_n} t^{\mathsf{exc}(\pi)} q^{\mathsf{inv}(\pi)}$ where again $\mathsf{EI}_0(q, t) = 1$. Then, in Subsection 2.1, we prove the following result.

Theorem 4. *The determinant of the nth Hankel matrix with (i, j)-th entry (for $0 \leq i, j \leq n$) being $\mathsf{EI}_{i+j}(q, t)$ is*

$$t^{n(n+1)/2} q^{n(n+1)(2n+1)/6} \prod_{k=1}^{n} ([k]_q!)^2.$$

Krattenthaler [7] gives a survey of several techniques for evaluating Hankel determinants, one of which uses connections between Hankel determinants and continued fractions. Clarke, Steingrímsson and Zeng (see [2, Corollary 11]) give a continued fraction expansion for the polynomial $EI_n(q, t) = \sum_{\pi \in S_n} t^{\mathsf{exc}(\pi)} q^{\mathsf{inv}(\pi)}$. Thus, Theorem 4, and hence Theorem 3, is already established. In the present paper, we provide an alternative proof. As all other proofs in this paper, it is combinatorial in nature (i.e., it is based on an explicit sign reversing involutions). The virtue of such combinatorial proofs is that they make the result completely transparent, and this is the main purpose of our paper.

The proofs of all our results are modifications of Ehrenborg's proof (see [4]), with the labelling of certain sets being the difference. One of our objectives is to give more examples where Ehrenborg's proof method is applicable.

1.1. **Fixed point free involutions.** Let I_{2n} be the set of fixed-point free involutions on the set $[2n]$. We denote $a_n = |I_{2n}|$. Clearly, $a_n = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ (with $a_0 = 1$). It is clear that any $M \in I_{2n}$ can also be thought of as a perfect matching in K_{2n}, the complete graph on $2n$ vertices. Thus, we use the terms "fixed-point free involutions" and "perfect matchings" interchangeably.

A well studied statistic on perfect matchings is its crossing number, defined as follows. Let $M = \{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\} \in I_{2n}$ be a perfect matching. Define $\mathsf{cross}(M)$, its crossing number, to be cardinality of the set $\{i < j < k < l : (i, k), (j, l) \in M\}$. The crossing number occurs for example in an expansion for the pfaffian of a skew-symmetric matrix of order $2n$ (see Krattenthaler's survey [7, p. 23]). Let $\mathsf{Cr}_n(q) = \sum_{M \in I_{2n}} q^{\mathsf{cross}(M)}$ be the q-crossing polynomial, where $\mathsf{Cr}_0(q) = 1$. Given these definitions, in Section 3, we prove the following theorem.

Theorem 5. *The determinant of the nth Hankel matrix with (i, j)-th entry (for $0 \leq i, j \leq n$) being $\mathsf{Cr}_{i+j}(q)$ is $\prod_{k=1}^{n} [2k]_q!$.*

1.2. **Non crossing partitions.** Cigler [1] generalised Theorem 1 as follows. Wachs and White [16] defined four statistics on set-partitions, one of which they called $\mathsf{rs}(\cdot)$. This, when added over set-partitions of $[n]$ with k parts, gives a q-analogue of $S(n, k)$, the Stirling numbers of the second kind. Let $S_q(n, k) = \sum_{\pi \in \mathsf{SP}_n, |\pi| = k} q^{\mathsf{rs}(\pi)}$, where q is an indeterminate. Consider the polynomial $E_n(q, x) = \sum_{k=0}^{n} x^k S_q(n, k)$ with $E_0(q, x) = 1$. Given these notations, the generalisation of Cigler [1] reads as follows.

Theorem 6. *The determinant of the nth Hankel matrix with the (i,j)-th entry (for $0 \leq i,j \leq n$) being $E_{i+j}(x,q)$ is*

$$x^{\binom{n+1}{2}} q^{\binom{n+1}{3}} \prod_{k=1}^{n} [k]_q!.$$

Ehrenborg [5] gave a combinatorial proof of Theorem 6 using juggling cards. It is known (see Simion [12]) that the number of non-crossing partitions of $[n]$ is equal to the n-th Catalan number C_n. Recall that non-crossing partitions of $[n]$ are those set-partitions $\pi = B_1|B_2|\cdots|B_k \in \mathsf{SP}_n$ which have the property that for all $1 \leq a < b < c < d \leq n$, whenever $a,c \in B_i$ and $b,d \in B_j$ for $1 \leq i,j \leq k$, then $i = j$; i.e., the blocks do not "cross". Let NC_n be the set of non-crossing partitions of $[n]$. The following is known (see Krattenthaler's second survey [8, Theorem 33]) or Radoux [10]).

Theorem 7. *The determinant of the nth Hankel matrix with the (i,j)-th entry (for $0 \leq i,j \leq n$) being C_{i+j} is 1.*

We prove the following analogue of Theorem 6 for non-crossing partitions, thereby generalising Theorem 7. Simion [12] considered Wachs and White's $\mathsf{rs}(.)$ statistic when restricted to non-crossing partitions and gave an alternate definition for the statistic $\mathsf{rs}(\pi)$ when $\pi \in \mathsf{NC}_n$. Let $\pi = B_1|B_2|\ldots|B_k \in \mathsf{NC}_n$ be the standard representation of π, i.e., a representation where π has k blocks, $B_1,\ldots,B_k$ and, if the least elements of B_i are f_i for $1 \leq i \leq k$, then $f_1 < f_2 < \cdots < f_k$. Similarly, let ℓ_i be the largest element in B_i for $1 \leq i \leq k$. Simion ([12, Lemma 2.1]) showed that

$$\mathsf{rs}(\pi) = \left(\sum_{i=1}^{k} (\ell_i - f_i)\right) - n + k. \tag{1}$$

Let $\mathsf{NC}(n,k)$ be the set of non-crossing set-partitions of $[n]$ into k parts. It is known that the cardinality of $\mathsf{NC}(n,k)$ is given by the Narayana numbers $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ (see Edelman [3]). Consider the q-analogue of the Narayana numbers defined as $\mathsf{NC}_q(n,k) = \sum_{\pi \in \mathsf{NC}(n,k)} q^{\mathsf{rs}(\pi)}$ and the bivariate polynomial $N_n(x,q) = \sum_{k=0}^{n} x^k \mathsf{NC}_q(n,k)$. In Section 4, we prove the following theorem.

Theorem 8. *The determinant of the nth Hankel matrix with the (i,j)-th entry (for $0 \leq i,j \leq n$) being $N_{i+j}(x,q)$ is $x^{\binom{n+1}{2}} q^{2\binom{n+1}{3}}$.*

2. THE EULERIAN ANALOGUE

We recall that, given a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$, an *excedance* is an index i such that $\pi_i > i$. The number of excedances of π is denoted as $\mathsf{exc}(\pi)$. It is well known (see [15]) that the excedance number of a permutation is an Eulerian statistic. Consider the n-th Hankel matrix $E_n = (e_{i,j})$ where $e_{i,j} = \mathsf{Eu}_{i+j}(x)$.

Proof of Theorem 3. Let $R_0, R_1, \ldots, R_n$ and $C_0, C_1, \ldots, C_n$ be pairwise disjoint sets with $|R_i| = |C_i| = i$. Let P be the set of tuples $(\sigma, \pi_0, \pi_1, \ldots, \pi_n)$, where σ is a permutation on the set $[n_0] = \{0, 1, \ldots, n\}$ and π_i is a permutation of $R_i \cup C_{\sigma(i)}$. It is clear that only such tuples index terms in the determinant evaluation of PP_n and that such a tuple contributes $(-1)^{\mathsf{inv}(\sigma)} x^{\sum_{i=0}^{n} \mathsf{exc}(\pi_i)}$ to the determinant value. We assume that the elements of R_i and C_i for $1 \leq i \leq n$ are totally ordered. We also assume that all elements of R_i (respectively C_i) are less than all elements of R_j (respectively C_j) for $j > i$ and that any element of C_i is greater than any element of R_j for all pairs i, j.

We represent all permutations π_i in two-line notation. Since we have permutations which are themselves indexed as π_i, we denote the r-th letter of π_i as $(\pi_i)_r$. Let T be the set of terms indexed by tuples $(\mathsf{id}, \pi_0, \pi_1, \ldots, \pi_n)$ where id is the identity permutation on $[n_0]$ and π_i is such that $\pi_i(R_i) = C_i$ and $\pi_i(C_i) = R_i$ for each $1 \le i \le n$. Clearly there are $\prod_{k=1}^{n}(k!)^2$ such tuples in T. Since $\mathsf{sgn}(\mathsf{id}) = 1$, and since each element of C_k is larger than each element of R_k for all k, all the π_k's in T will have $\mathsf{exc}(\pi_k) = k$. Since this happens for all k, the contribution of T to the determinant of E_n is $\prod_{k=0}^{n} k!^2 x^{\sum_{k=0}^{n} k} = x^{n(n+1)/2}\prod_{k=0}^{n}(k!)^2$.

Given a tuple $(\sigma, \pi_0, \pi_1, \ldots, \pi_n)$, for each $1 \le i \le n$ let a_i be the sum of the number of elements of R_i which under π_i go to $C_{\sigma(i)}$ and the number of elements of $C_{\sigma(i)}$ which under π_i go to R_i. We note that all the a_i's are even and if $a_i = 2p$, then p elements of R_i are written under $C_{\sigma(i)}$ and p elements of $C_{\sigma(i)}$ are written under R_i in the two line notation of π_i. Furthermore, it is clear that the a_i's are distinct for a term indexed by $(\sigma, \pi_0, \pi_1, \ldots, \pi_n)$ if and only if $(\sigma, \pi_0, \pi_1, \ldots, \pi_n) \in T$. For each $(\sigma, \pi_0, \pi_1, \ldots, \pi_n) \in P - T$, a partner tuple $(\sigma', \pi'_0, \pi'_1, \ldots, \pi'_n)$ with an opposite sign will be constructed so that this map is a sign reversing involution on the set of tuples in $P - T$. This will show that the tuples in $P - T$ contribute nothing to the determinant value.

If $(\sigma, \pi_0, \pi_1, \ldots, \pi_n) \in P - T$, then at least two of the a_i's must be identical. Choose the lexicographically minimum pair s, t such that $a_s = a_t$. Let σ' be a permutation on the set $[n_0]$ such that $\sigma'(s) = \sigma(t)$, $\sigma'(t) = \sigma(s)$ and $\sigma'(i) = \sigma(i)$ for $i \ne s, t$.

Let $2a = a_s = a_t$. Let π'_i be identical to π_i on $R_i \cup C_{\sigma(i)}$ for $i \ne s, t$. For the two line representation of π'_s (a permutation on $R_s \cup C_{\sigma(t)}$), since $C_{\sigma(t)}$ and $C_{\sigma(s)}$ are ordered, list the a elements of $C_{\sigma(t)}$ below R_s in the order of the a elements of $C_{\sigma(s)}$ in π_s and list the a elements of R_s below $C_{\sigma(t)}$ in the same order as in π_s. (See the example below.)

$$\pi_s = \left(\begin{array}{ccc|cccc} & R_s & & & & C_{\sigma(s)} & \\ 1 & 2 & 3 & 51 & 52 & 53 & 54 \\ 53 & 3 & 52 & 1 & 54 & 51 & 2 \end{array}\right) \qquad \pi_t = \left(\begin{array}{cccc|ccc} & R_t & & & & C_{\sigma(t)} & \\ 10 & 11 & 12 & 13 & 56 & 57 & 58 \\ 56 & 57 & 13 & 10 & 12 & 11 & 58 \end{array}\right)$$

$$\mathsf{inv}(\pi_s) = \mathbf{13}, \mathsf{exc}(\pi_s) = \mathbf{4} \qquad \mathsf{inv}(\pi_t) = \mathbf{12}, \mathsf{exc}(\pi_t) = \mathbf{3}$$

$$\pi'_s = \left(\begin{array}{ccc|ccc} & R_s & & & C_{\sigma(t)} & \\ 1 & 2 & 3 & 56 & 57 & 58 \\ 57 & 3 & 56 & 1 & 2 & 58 \end{array}\right) \qquad \pi'_t = \left(\begin{array}{cccc|cccc} & R_t & & & & C_{\sigma(s)} & & \\ 10 & 11 & 12 & 13 & 51 & 52 & 53 & 54 \\ 52 & 53 & 13 & 10 & 12 & 54 & 51 & 11 \end{array}\right)$$

$$\mathsf{inv}(\pi'_s) = \mathbf{8}, \mathsf{exc}(\pi'_s) = \mathbf{3} \qquad \mathsf{inv}(\pi'_t) = \mathbf{17}, \mathsf{exc}(\pi'_t) = \mathbf{4}$$

Similarly, to get the two line representation of π'_t (a permutation on the set $R_t \cup C_{\sigma(s)}$), list the a elements of $C_{\sigma(s)}$ below R_t in the order of the a elements of R_t in π_t and list the a elements of R_t below $C_{\sigma(s)}$ as they appeared in π_t. From the above, it is easy to see that the term $(\sigma', \pi'_0, \pi'_1, \ldots, \pi'_n)$ has the same a_i's as the term $(\sigma, \pi_0, \pi_1, \ldots, \pi_n)$ and thus these terms map to each other, showing that we have an involution. We show now that

$$\mathsf{exc}(\pi_s) + \mathsf{exc}(\pi_t) = \mathsf{exc}(\pi'_s) + \mathsf{exc}(\pi'_t) \tag{2}$$

Since a elements of $C_{\sigma(t)}$ and $C_{\sigma(s)}$ (which are larger than all elements of R_s and R_t) get swapped in π_s and π_t to give π'_s and π'_t, the two quantities on each side of Equation (2) are equal. Furthermore, as $\mathsf{sgn}(\sigma) = -\mathsf{sgn}(\sigma')$, all terms in $P - T$ cancel out in the determinant expansion. Thus the only terms that contribute to the determinant are those in T, completing the proof. $\square$

2.1. **Bivariate analogue of Theorem 2.**

Proof of Theorem 4. Consider the same bijection and the same set T fixed under the involution. We show that

(1) terms in $P - T$ have the same $(\mathsf{exc}, \mathsf{inv})$ joint distribution under the sign reversing involution
(2) and $t^{n(n+1)/2} q^{n(n+1)(2n+1)/6} \prod_{k=1}^{n} ([k]_q!)^2$ is the sum of the joint distribution of $(\mathsf{exc}, \mathsf{inv})$ over terms of T.

PROOF OF ITEM 1. Straightforward and thus omitted, just that this time terms π and π' that are mapped to each other under the bijection satisfy $t^{\mathsf{exc}(\pi)} q^{\mathsf{inv}(\pi)} = t^{\mathsf{exc}(\pi')} q^{\mathsf{inv}(\pi')}$ and have opposite signs.

PROOF OF ITEM 2. We show that the sum of the joint distribution of $(\mathsf{exc}, \mathsf{inv})$ over the π_i's is $t^i q^{i^2} [i]_q!$. To see this, we note that each π_i maps R_i to C_i and vice-versa (so there are $i!^2$ such π_i's) and, due to our labelling, all π_i's have i excedances. Hence we can pull out a t^i factor and sum $q^{\mathsf{inv}(\pi_i)}$ over these $(i!)^2$ permutations. It is clear from the two-line representation of these permutations and $\sum_{\pi \in S_i} q^{\mathsf{inv}(\pi)} = [i]_q!$ (see [6, Section 2, p. 15]) that permutations π_i contribute $t^i q^{i^2} ([i]_q!)^2$ to the determinant's value. Taking the product over $1 \le i \le n$ completes the proof. □

Remark 1. From Theorem 4, it is clear that if $p_n(q) = [n]_q!$ with $p_0(q) = 1$, then the n-th Hankel matrix whose (i,j)-th term for $0 \le i, j \le n$ is $p_{i+j}(q)$ has determinant

$$q^{n(n+1)(2n+1)/6} \prod_{k=1}^{n} ([k]_q!)^2.$$

This can also be seen from [7, Proposition 1]. By replacing q by q^2 or more generally by q^d for a positive integer d, we get analogues for the hyperoctahedral group and the wreath-product group $S_n \wr \mathbb{Z}_d$.

Remark 2. The same proposition can be used to show that, if $s_n(x) = x(x+1)\cdots(x+n-1)$, then the determinant of the n-th Hankel matrix with (i,j)-th entry being $s_{i+j}(x)$ is $\prod_{k=1}^{n} (k! s_k(x))$. More generally, for positive integers a, r, consider the arithmetic progression $A = (a, a+r, a+2r, \ldots,)$ (i.e., $a_1 = a$ and $a_i = a + (i-1)r$) and define a sequence of polynomials $p_n(x,y) = \prod_{k=1}^{n} (a_1 x + (k-1) r y)$ with $p_0(x,y) = 1$. The determinant of the n-th Hankel matrix with (i,j)-th entry being $p_{i+j}(x,y)$ is

$$\prod_{k=1}^{n} \big(ax + (k-1)ry\big) \prod_{k=1}^{n} (ry)^k k!.$$

3. FIXED POINT FREE INVOLUTIONS

Let M be a perfect matching (or fixed-point free involution) on the set $[2n]$. We call each pair $(i,j) \in M$ an edge of M. We assume without loss of generality that the edges of M are $(i_1, j_i), (i_2, j_2), \ldots, (i_n, j_n)$ where $i_r < j_r$ for $1 \le r \le n$ and $i_r < i_{r+1}$ for $1 \le r < n$.

Proof of Theorem 5. For each positive integer $1 \le i \le n$, let R_i and C_i be disjoint sets of size $2i$. As before we label them such that any element of any C_j is larger than any element of any R_i. It is easy to see that terms in the determinant are indexed by the set P, with elements being tuples of the form $p = (\pi, m_0, m_1, \ldots, m_n)$, where $\pi \in S_{n_0}$ and m_i is a perfect matching on the set $R_i \cup C_{\pi(i)}$. Such a tuple p contributes $(-1)^{\mathsf{sgn}(\pi)} q^{\sum_{i=0}^n \mathsf{cross}(m_i)}$ to the determinant value.

As done before, let $T \subseteq P$ be those terms indexed by $(\mathsf{id}, m_0, m_1, \ldots, m_n)$ where in m_i, each element of R_i is paired with an element of C_i. We omit the routine sign reversing involution on the terms in $P - T$ and show that $\sum_{(\mathsf{id}, m_0, \ldots, m_n) \in T} q^{\sum_{i=0}^n \mathsf{cross}(m_i)} = \prod_{k=1}^n [2k]_q!$. Let PM_{2i} denote the set of perfect matchings on the set $R_i \cup C_i$ (where $|R_i| = |C_i| = 2i$) with the property that every edge of PM_{2i} is of the type (r, c) where $r \in R_i$ and $c \in C_j$. Clearly, $|\mathsf{PM}_{2i}| = (2i)!$. It suffices to show that $\sum_{M \in \mathsf{PM}_{2i}} q^{\mathsf{cross}(M)} = [2i]_q!$. To see this, write M as a $2i$ sized permutation π_M in two line notation with the elements of R_i above and those of C_i below (π_M contains complete information about M). Then it is simple to see that $\mathsf{inv}(\pi_M) = \binom{2i}{2} - \mathsf{cross}(M)$. Since each of the $(2i)!$ permutation π_M occur this way, $\sum_{M \in \mathsf{PM}_{2i}} q^{\mathsf{cross}(M)} = \sum_{\pi \in S_{2i}} q^{\binom{2i}{2} - \mathsf{inv}(\pi)}$. This is clearly $[2i]_q!$ (see [15, page 21]). The proof is complete after taking the product over the n different PM_{2i}'s. □

4. NON CROSSING PARTITIONS

We begin with a slightly modified definition of the statistic $\mathsf{rs}(\cdot)$. If a set of size n is not labelled by $[n]$, but by some set $T_n \subseteq \mathbb{Z}$ of n positive integers, then from Equation (1), we see that the statistic $\mathsf{rs}(\pi)$ for a non-crossing partition π depends on the set T_n, which is not desirable. Thus, whenever we have a non-crossing partition π on a set T_n, we map its elements to the set $\{1, 2, \ldots, n\}$ in an order-preserving (with respect to the natural order on $\mathbb{Z}$) way, and then compute its $\mathsf{rs}(\cdot)$ value.

Proof of Theorem 8. We have disjoint sets R_i for $0 \leq i \leq n$ with $|R_i| = i$ and another completely disjoint family of sets C_i for $0 \leq i \leq n$, again with $|C_i| = i$. We label the elements of R_i in increasing order of i and then, starting from a larger number L, number the elements of C_i in *decreasing* order of i. (This is done so that, for any $0 \leq i, j \leq n$, if the elements of R_i are written in increasing order from left-right and those of C_j are written below in increasing order from right-left, the elements of $R_i \cup C_j$ appear in increasing order when viewed clockwise.)

It is clear that the terms involved in the determinant expansion of the nth Hankel matrices with (i,j)-th entry being $N_{i+j}(x, q)$ are indexed by tuples $(\sigma, \pi_0, \pi_1, \ldots, \pi_n)$, where σ is a permutation of $[n_0]$ and π_i is a non-crossing partition on the set $R_i \cup C_{\sigma(i)}$, and such a term contributes $(-1)^{\mathsf{sgn}(\sigma)} x^{\sum_{i=0}^n |\pi_i|} q^{\sum_{i=0}^n \mathsf{rs}(\pi_i)}$. Let P be the set of such tuples. For a tuple $(\sigma, \pi_0, \pi_1, \ldots, \pi_n) \in P$, let a_i be the number of blocks of the non-crossing partition π_i which intersect both R_i and $C_{\sigma(i)}$. Let T consist of the tuples indexed by $(\mathsf{id}, \pi_0, \pi_1, \ldots, \pi_n)$ where π_i is a non-crossing partition on $R_i \cup C_i$ with i blocks intersecting both R_i and C_i. Thus, there is exactly one tuple in T.

It is simple to note that the a_i's are all distinct for a tuple if and only if it is in T. We thus need to exhibit a fixed point free sign-reversing involution on the tuples in $P - T$. As before, for a tuple $(\sigma, \pi_0, \pi_1, \ldots, \pi_n) \in P - T$, let (s, t) be the lexicographically smallest pair with $a_s = a_t$. Define $(\sigma', \pi'_0, \pi'_1, \ldots, \pi'_n)$ such that $\sigma(i) = \sigma'(i)$ for $i \in [n_0], i \neq s, t$. Let $\sigma'(s) = \sigma(t)$ and $\sigma'(t) = \sigma(s)$.

Let $\pi'_i = \pi_i$ for $i \neq s, t$. Since $a = a_s = a_t$, there are a blocks of π_s, π_t which intersect both $R_s \cup C_{\sigma(s)}$ and $R_t \cup C_{\sigma(t)}$ (see Figure 1 below). Each of the a blocks of π_s (respectively π_t) contains some element of R_s (respectively R_t) and since the elements of R_s and R_t are totally ordered, each of the a blocks contains a minimum element (of R_s or R_t). Order the

a non-crossing blocks of π_s as $P_1, P_2, \ldots, P_a$ in increasing order of the minimum element of R_s contained in P_i and likewise, order the blocks of π_t as $Q_1, Q_2, \ldots, Q_a$ in increasing order of the minimum element of R_t contained in Q_i. Let $P'_i = (P_i \cap R_s) \cup (Q_i \cap C_{\sigma(t)})$ and $Q'_i = (Q_i \cap R_t) \cup (P_i \cap C_{\sigma(s)})$ for $1 \le i \le a$. Let the blocks of π'_s (a non-crossing partition on the set $R_s \cup C_{\sigma(t)}$), be obtained by replacing P_i by P'_i and similarly let the blocks of π'_t be obtained by replacing Q_i by Q'_i for all $1 \le i \le a$. It is simple to see that both π'_s and π'_t are non-crossing partitions and that $\mathsf{sgn}(\sigma) = -\mathsf{sgn}(\sigma')$. Furthermore, it is easy to see that $|\pi_s| = |\pi'_s|$ and $|\pi_t| = |\pi'_t|$.

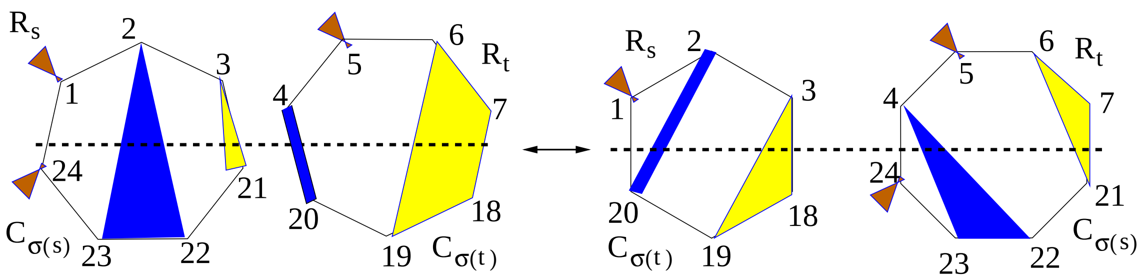

FIGURE 1. An illustration of the bijection (shaded regions are the blocks of the partition).

In π_s and π_t, since any element of any C_j is greater than any element of any R_i, it is straightforward to note that if $\alpha = |R_s|$ and $\beta = |R_t|$, then we can write, for some c_i, $\ell(P_i) = \alpha + c_i$ for all $1 \le i \le a$ (i.e., the largest element of P_i is the c_i-th largest element above α). Similarly, for some d_i, we can write $\ell(Q_i) = \beta + d_i$.

In π'_s, we note that $\ell(P'_i) = \alpha + d_i$ for $1 \le i \le a$, while in π'_t, $\ell(Q'_i) = \beta + c_i$ for all i. It is straightforward to note that $f(P_i) = f(P'_i)$ for all i and $f(Q_i) = f(Q'_i)$ for all i. We have also observed that $|\pi_s| = |\pi'_s|$ and $|\pi_t| = |\pi'_t|$. Hence, $\mathsf{rs}(\pi_s) + \mathsf{rs}(\pi_t) = \mathsf{rs}(\pi'_s) + \mathsf{rs}(\pi'_t)$ and thus $q^{\sum_{i=0}^n \mathsf{rs}(\pi_i)} = q^{\sum_{i=0}^n \mathsf{rs}(\pi'_i)}$. i.e., the involution preserves both $x^{\sum_{i=0}^n |\pi_i|}$ and $q^{\sum_{i=1}^n \mathsf{rs}(\pi_i)}$ simultaneously.

Since the involution is fixed-point free on $P - T$, the set $P - T$ contributes nothing to the determinant. As $(\mathsf{id}, \pi_0, \pi_1, \ldots, \pi_n)$ is the only tuple in T, it is simple to see that the exponent of x contributed is $n(n+1)/2$ (as each π_i contributes i parts for $0 \le i \le n$). For the exponent of q, we note that π_i has i-blocks and for the k-th block (where $1 \le k \le i$) $f_k = k$, $\ell_k = 2i - k + 1$. Thus, $\mathsf{rs}(\pi_i) = \sum_{k=1}^i (2k-1) - 2i + i = i^2 - i$ and $\sum_{i=0}^n i^2 - i = 2\binom{n+1}{3}$. Thus, the determinant has value $x^{\binom{n+1}{2}} q^{2\binom{n+1}{3}}$, completing the proof. □

We end with a few remarks about Theorem 8. Simion [12] has shown that NC_n is a graded lattice and that the Narayana numbers are its Whitney numbers. Theorem 8 is thus about the Hankel determinant of a q-analogue of the Whitney polynomial of NC_n.

Reiner [11] has defined non-crossing partitions for all classical reflection groups and hence type-B analogues of NC_n, denoted NC_n^B exist. NC_n^B is also a graded lattice with k-th Whitney number being $\binom{n}{k}^2$ (see [11]). Simion [13] has given analogues $\mathsf{rs}^B(\sigma)$ for type-B non-crossing partitions σ of the statistic $\mathsf{rs}(.)$. Consider the type-B Whitney polynomial $p_n^B(x) = \sum_{k=0}^n \binom{n}{k}^2 x^k$. The determinant of the n-th Hankel matrix with (i,j)-th entry being the polynomial $p_{i+j}^B(x)$ is $2^n x^{n(n+1)/2}$ (see Paul Barry's contribution in [14, id: A000984]). However, a q-analogue similar to Theorem 8 does not hold with respect to the statistic $\mathsf{rs}^B(.)$.

It would be very interesting to have a statistic $\mathsf{st}^B(\sigma)$ on a type-B non-crossing partition σ such that a type-B analogue of Theorem 8 holds.

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