A Graph Model of the Heisenberg-Weyl algebra

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The Heisenberg-Weyl algebra, underlying virtually all physical representations of Quantum Theory, is considered from a combinatorial point of view. We construct a concrete model of the algebra in terms of graphs endowed with intuitive concepts of composition and decomposition leading to a rich Hopf algebra structure. The latter encompasses the Heisenberg-Weyl algebra which gains a straightforward interpretation as a shadow of natural constructions on graphs. In this way, by drawing attention to the algebraic structure of Quantum Theory we intend to shed light on the combinatorial nature hidden behind its formalism.

Enveloping algebra
\(\mathcal{U}(\mathcal{L}_\mathcal{H})\)

Heisenberg-Weyl algebra
\(\mathcal{H}\)

(Hopf algebra)

( AAU )
The Heisenberg-Weyl algebra, underlying virtually all physical representations of Quantum Theory, is considered from a combinatorial point of view. We construct a concrete model of the algebra in terms of graphs endowed with intuitive concepts of composition and decomposition leading to a rich Hopf algebra structure. The latter encompasses the Heisenberg-Weyl algebra which gains a straightforward interpretation as a shadow of natural constructions on graphs. In this way, by drawing attention to the algebraic structure of Quantum Theory we intend to shed light on the combinatorial nature hidden behind its formalism.
Heisenberg - Weyl algebra revisited

Generators: \( a, a^\dagger \)

Relation: \( a a^\dagger = a^\dagger a + I \)

Basis in \( \mathcal{H} \): \( a^\dagger a^s \)

\[
\mathcal{H} \ni \sum_{r_1, \ldots, r_k, s_1, \ldots, s_k} \alpha_{r_1, \ldots, r_k, s_1, \ldots, s_k} \ a^\dagger r_1 a^{s_1} \cdots a^\dagger r_k a^{s_k}
\]

It is an associative algebra with unit (AAU)

Structure constants:

\[
a^\dagger p a^q a^\dagger k a^l = \sum_i \binom{q}{i} \binom{k}{i} i! a^\dagger p+k-i a^{q+l-i}
\]
**Enveloping algebra**

- Generators: $a, a^\dagger, e$
- Relations: $a a^\dagger = a^\dagger a + e$, $a e = e a$, $a^\dagger e = e a^\dagger$

\[ U(\mathcal{L}_H) = \mathbb{K} \langle a, a^\dagger, e \rangle / \begin{cases} [a, a^\dagger] = e \\ [a, e] = [a^\dagger, e] = 0 \end{cases} \]

- Basis in $U(\mathcal{L}_H)$: $a^\dagger p a^q e^r$
- $U(\mathcal{L}_H)$ is a Hopf algebra.

**Co-product** $\Delta : U(\mathcal{L}_H) \rightarrow U(\mathcal{L}_H) \otimes U(\mathcal{L}_H)$, s.t. on generators $\Delta (x) = x \otimes I + I \otimes x$:

\[ \Delta (a^\dagger p a^q e^r) = \sum_{i,j,k} \binom{p}{i} \binom{q}{j} \binom{r}{k} a^\dagger i a^j e^k \otimes a^\dagger p-i a^q-j e^{r-k} \]

**Co-unit** $\varepsilon : U(\mathcal{L}_H) \rightarrow \mathbb{K}$, given by:

\[ \varepsilon (a^\dagger p a^q e^r) = \begin{cases} 1 & \text{if } p, q, r = 0 \\ 0 & \text{otherwise} \end{cases} \]

**Antipode** $S : U(\mathcal{L}_H) \rightarrow U(\mathcal{L}_H)$, s.t. for generators $S (x) = -x$:

\[ S (a^\dagger p a^q e^r) = (-1)^{p+q+r} e^r a^q a^\dagger p \]
Algebraic "picture"

**Enveloping algebra**
(Hopf algebra)
\[ \mathcal{U}(\mathcal{L}_\mathcal{H}) \]

\[ I \]

\[ e \times a \times a^\dagger \]

**Heisenberg-Weyl algebra**
(AAU)
\[ \mathcal{H} \]

\[ I \]

\[ a \times a^\dagger \]

**Lie algebra**
\[ \mathcal{L}_\mathcal{H} \]

\[ e \times a \times a^\dagger \]

\[ e \rightarrow I \]
Algebraic "picture"

Combinatorial Graph algebra
( Hopf algebra & AAU )

Enveloping algebra
( Hopf algebra )

$\mathcal{U}(\mathcal{L}_H) \xrightarrow{\varphi} I \xrightarrow{e \mapsto I} \mathcal{L}_H \xrightarrow{\bar{\varphi}} \mathcal{H}$

Heisenberg-Weyl algebra
(AAU )

Lie algebra
A directed graph is a collection of edges $E$ and vertices $V$ together with two mappings $h, t : E \rightarrow V$ prescribing how the head and tail of each edge is attached to vertices.

Example:

- We shall consider classes of graphs up to isomorphism, i.e. simply pictures.
- Graphs embedded in a plane are called planar graphs.
- Following a cycle in a graph one ends at the starting point.
Combinatorial Concepts

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Heisenberg-Weyl graphs

Definition

Combinatorial class of Heisenberg-Weyl graphs consists of planar directed graphs \( \Gamma \) which \textbf{do not have cycles} and may be \textbf{partially-defined}.
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Edges in a graph may have one of the ends free (but not both)
Heisenberg-Weyl graphs

**Definition**

Combinatorial class of Heisenberg-Weyl graphs consists of planar directed graphs $\Gamma$ which do not have cycles and may be partially-defined.

- Edges in a graph may have one of the ends free (but not both)
- It has three sorts of edges: inner, ingoing and outgoing ones

\[ \Gamma^+ \text{ outgoing edges} \]
\[ \Gamma^0 \text{ inner edges} \]
\[ \Gamma^- \text{ ingoing edges} \]
Heisenberg-Weyl graphs

Definition

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- It has three sorts of edges: inner, ingoing and outgoing ones

Size of a graph:

$$d(\Gamma) = 2|\Gamma^0| + |\Gamma^+| + |\Gamma^-|$$
### Graph composition

**Definition**

For two graphs $\Gamma_2$ and $\Gamma_1$ and a matching $m \in \Gamma_2^- \leftrightarrow \Gamma_1^+$. the composite graph, denoted as $\Gamma_2 \overset{m}{\leftrightarrow} \Gamma_1$, is constructed by joining the edges coupled by the matching $m$.

- A matching $A \leftrightarrow B$ of two sets $A$ and $B$ is a choice of pairs $(a, b) \in A \times B$ such that no component appear twice.
- The number of matchings consisting of $i$ pairs (of edges) is given by

$$\# \Gamma_2^- \leftrightarrow \Gamma_1^+ = \binom{|\Gamma_2^-|}{i} \binom{|\Gamma_1^+|}{i} i!$$
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Graph composition - Properties

Let \( \Gamma_2 \bowtie \Gamma_1 \) denote the set of all possible compositions of the graph \( \Gamma_2 \) with \( \Gamma_1 \), i.e.

\[
\Gamma_2 \bowtie \Gamma_1 = \biguplus_{m \in \Gamma_2^- \otimes \Gamma_1^+} \Gamma_2^m \bowtie \Gamma_1
\]
Let $\Gamma_2 \blacklozenge \Gamma_1$ denote the set of all possible compositions of the graph $\Gamma_2$ with $\Gamma_1$, i.e.

$$\Gamma_2 \blacklozenge \Gamma_1 = \biguplus \Gamma_2 \blacklozenge \Gamma_1 \quad m \in \Gamma_2^- \blacklozenge \Gamma_1^+$$

**Finiteness**

$$\# \Gamma_2 \blacklozenge \Gamma_1 < \infty$$
Graph composition - Properties

Let $\Gamma_2 \blacktriangleleft \Gamma_1$ denote the set of all possible compositions of the graph $\Gamma_2$ with $\Gamma_1$, i.e.

$$\Gamma_2 \blacktriangleleft \Gamma_1 = \bigoplus_{m \in \Gamma_2^- \blacktriangleright \Gamma_1^+} \Gamma_2^m \blacktriangleleft \Gamma_1$$

- Finiteness
  $$\# \Gamma_2 \blacktriangleleft \Gamma_1 < \infty$$

- Triple composition
  $$(\Gamma_3 \blacktriangleleft \Gamma_2) \blacktriangleleft \Gamma_1 = \Gamma_3 \blacktriangleleft (\Gamma_2 \blacktriangleleft \Gamma_1)$$
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  $$(\Gamma_3 \triangleright \Gamma_2) \triangleright \Gamma_1 = \Gamma_3 \triangleright (\Gamma_2 \triangleright \Gamma_1)$$

- Neutral (void) graph
  $$\Gamma \triangleright \emptyset = \emptyset \triangleright \Gamma = \Gamma$$
Graph composition - Properties

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- **Neutral (void) graph**
  \[ \Gamma \bowtie \emptyset = \emptyset \bowtie \Gamma = \Gamma \]

- **No symmetry**
  \[ \Gamma_2 \bowtie \Gamma_1 \neq \Gamma_1 \bowtie \Gamma_2 \]
Graph composition - Properties

Let $\Gamma_2 \triangleleft \Gamma_1$ denote the set of all possible compositions of the graph $\Gamma_2$ with $\Gamma_1$, i.e.

$$\Gamma_2 \triangleleft \Gamma_1 = \biguplus_{m \in \Gamma_2^\rightarrow \triangleleft \Gamma_1^\leftarrow} \Gamma_2^m \triangleleft \Gamma_1$$

- Finiteness
  $$\# \Gamma_2 \triangleleft \Gamma_1 < \infty$$

- Triple composition
  $$\left( \Gamma_3 \triangleleft \Gamma_2 \right) \triangleleft \Gamma_1 = \Gamma_3 \triangleleft \left( \Gamma_2 \triangleleft \Gamma_1 \right)$$

- Neutral (void) graph
  $$\Gamma \triangleleft \emptyset = \emptyset \triangleleft \Gamma = \Gamma$$

- No symmetry
  $$\Gamma_2 \triangleleft \Gamma_1 \neq \Gamma_1 \triangleleft \Gamma_2$$

- Compatible with size
  $$d(\Gamma_2^m \triangleleft \Gamma_1) = d(\Gamma_2) + d(\Gamma_1)$$
Graph decomposition

Definition

Decomposition of a graph $\Gamma$ is a splitting $\Gamma \rightsquigarrow (\Gamma|_L, \Gamma|_R)$ induced by an ordered partition of its edges $L + R = E_\Gamma$.

- A sub-graph $\Gamma|_L$ is a restriction of the head and tail mappings to the subset $L \subset E_\Gamma$.
- Enumeration of all decompositions according to the number of lines in the left component:

$$\# \left\{ (\Gamma|_L, \Gamma|_R) \in \langle \Gamma \rangle : \begin{array}{c} |\Gamma^+_L|=i \\ |\Gamma^-_L|=j \\ |\Gamma^0_L|=k \end{array} \right\} = \binom{|\Gamma^+|}{i} \binom{|\Gamma^-|}{j} \binom{|\Gamma^0|}{k}$$
### Graph decomposition

**Definition**

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|\Gamma|_L^+ = i \\
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Graph decomposition - Properties

Let \( \langle \Gamma \rangle \) denote the multi-set of all possible decompositions \( \Gamma \sim (\Gamma|_L, \Gamma|_R) \) of the graph, i.e.

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\langle \Gamma \rangle = \biguplus_{L+R=E_\Gamma} \{ (\Gamma|_L, \Gamma|_R) \}
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Graph decomposition - Properties

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- Finiteness $\# \langle \Gamma \rangle < \infty$
Graph decomposition - Properties

Let \( \langle \Gamma \rangle \) denote the multi-set of all possible decompositions \( \Gamma \leadsto (\Gamma|_L, \Gamma|_R) \) of the graph, i.e.

\[
\langle \Gamma \rangle = \bigcup_{L+R=E_\Gamma} \{(\Gamma|_L, \Gamma|_R)\}
\]

- Finiteness \( \# \langle \Gamma \rangle < \infty \)

- Triple decomposition

\[
\begin{align*}
\Gamma & \xrightarrow{\text{first}} (\Gamma'', \Gamma_1) \\
\Gamma & \xrightarrow{\text{second}} (\Gamma_3, \Gamma_2, \Gamma_1)
\end{align*}
\]
Let \( \langle \Gamma \rangle \) denote the multi-set of all possible decompositions \( \Gamma \sim (\Gamma_L, \Gamma_R) \) of the graph, i.e.

\[
\langle \Gamma \rangle = \biguplus_{L+R=E_{\Gamma}} \{ (\Gamma_L, \Gamma_R) \}
\]

- **Finiteness** \( \# \langle \Gamma \rangle < \infty \)

- **Triple decomposition**

- **Void graph**

\[
\Gamma \rightarrow (\emptyset, \Gamma) \quad \& \quad \Gamma \rightarrow (\Gamma, \emptyset)
\]

unique decomposition
Let $\langle \Gamma \rangle$ denote the multi-set of all possible decompositions $\Gamma \rightsquigarrow (\Gamma|_L, \Gamma|_R)$ of the graph, i.e.

$$\langle \Gamma \rangle = \biguplus_{L+R=E_\Gamma} \{ (\Gamma|_L, \Gamma|_R) \}$$

- **Finiteness**  \# $\langle \Gamma \rangle < \infty$

- **Triple decomposition**

  $\Gamma \xrightarrow{\text{first}} (\Gamma'', \Gamma_1) \xrightarrow{\text{second}} (\Gamma_3, \Gamma_2, \Gamma_1)$

- **Void graph**

  $\Gamma \rightarrow (\emptyset, \Gamma) \quad \& \quad \Gamma \rightarrow (\Gamma, \emptyset)$

  unique decomposition

- **Symmetry**

  $(\Gamma'', \Gamma''') \in \langle \Gamma \rangle \implies (\Gamma''', \Gamma') \in \langle \Gamma \rangle$
Graph decomposition - Properties

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Let $\langle \Gamma \rangle$ denote the multi-set of all possible decompositions $\Gamma \rightsquigarrow (\Gamma|_L, \Gamma|_R)$ of the graph, i.e.

$$\langle \Gamma \rangle = \biguplus_{L+R=E_\Gamma} \{(\Gamma|_L, \Gamma|_R)\}$$

\textbf{Composition-decomposition compatibility}

$$(\Gamma_2, \Gamma_1) \leadsto (\langle \Gamma_2 \rangle \times \langle \Gamma_1 \rangle) \xrightarrow{\text{comp.}} \langle (\Gamma_2, \Gamma_1) \rangle \xrightarrow{\text{decomp.}} \Gamma_2 \blacktriangleleft \Gamma_1$$
Let \( \langle \Gamma \rangle \) denote the multi-set of all possible decompositions \( \Gamma \sim (\Gamma|_L, \Gamma|_R) \) of the graph, i.e.

\[
\langle \Gamma \rangle = \biguplus_{L+R=E_{\Gamma}} \{ (\Gamma|_L, \Gamma|_R) \}
\]

Composition-decomposition compatibility

Compatible with size

\[
d(\Gamma) = d(\Gamma|_L) + d(\Gamma|_R)
\]
Graph decomposition - Properties

Let $\langle \Gamma \rangle$ denote the multi-set of all possible decompositions $\Gamma \rightsquigarrow (\Gamma|_L, \Gamma|_R)$ of the graph, i.e.

$$\langle \Gamma \rangle = \biguplus_{L+R=E_{\Gamma}} \left\{ (\Gamma|_L, \Gamma|_R) \right\}$$

- Composition-decomposition compatibility

- Compatible with size

  $$d(\Gamma) = d(\Gamma|_L) + d(\Gamma|_R)$$

- Finiteness of multiple decompositions

  $$\{ \Gamma \rightsquigarrow (\Gamma_n, \ldots, \Gamma_1) : \Gamma_n, \ldots, \Gamma_1 \neq \emptyset \} = \emptyset$$
  for $n \geq N(\Gamma)$
We define $\mathcal{G}$ as a vector space over $\mathbb{K}$ spanned by the basis set consisting of all Heisenberg-Weyl diagrams, i.e.

$$\mathcal{G} = \left\{ \sum_i \alpha_i \, \Gamma_i : \alpha_i \in \mathbb{K}, \, \Gamma_i - \text{Heisenberg–Weyl graph} \right\}$$

Addition in $\mathcal{G}$ has the usual form:

$$\sum_i \alpha_i \, \Gamma_i + \sum_i \beta_i \, \Gamma_i = \sum_i (\alpha_i + \beta_i) \, \Gamma_i$$

What about multiplication?

$$\sum_i \alpha_i \, \Gamma_i \ast \sum_j \beta_j \, \Gamma_j = \sum_{i,j} \alpha_i \beta_j \, \Gamma_i \ast \Gamma_j$$

What about co-product, co-unit and antipode?
Multiplication of graphs

Definition

Multiplication of two graphs $\Gamma_2$ and $\Gamma_1$ in $\mathcal{G}$ is just a sum over all possible compositions:

$$\Gamma_2 \ast \Gamma_1 = \sum_{m \in \Gamma_2^- \leftrightarrow \Gamma_1^+} \Gamma_2^m \Diamond \Gamma_1$$

Proposition

Heisenberg-Weyl graphs form an associative algebra with unit $(\mathcal{G}, +, \ast, \emptyset)$. It is non-commutative!!
Co-product of graphs

**Definition**

Co-product \( \Delta : \mathcal{G} \rightarrow \mathcal{G} \otimes \mathcal{G} \) is defined on the basis as a sum over all possible decompositions:

\[
\Delta(\Gamma) = \sum_{L+R=E\Gamma} \Gamma|^L_L \otimes \Gamma|^R_R
\]

**Co-unit** \( \varepsilon : \mathcal{G} \rightarrow \mathbb{K} \) simply extracts the expansion coefficient standing at the void:

\[
\varepsilon(\Gamma) = \begin{cases} 
1 & \text{if } \Gamma = \emptyset, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proposition**

Heisenberg-Weyl graphs form a bi-algebra \((\mathcal{G}, +, *, \emptyset, \Delta, \varepsilon)\). It is co-commutative!!
Heisenberg - Weyl graphs form a bi-algebra. It is non-commutative and co-commutative.

Even more, it has a genuine Hopf algebra structure \((\mathcal{G}, +, *, \emptyset, \Delta, \varepsilon, S)\), with an antipode given by:

\[
S(\Gamma) = \sum_{A_n + \ldots + A_1 = E_{\Gamma}} (-1)^n \left[ \Gamma|_{A_n} * \cdots * \Gamma|_{A_1} \right]
\]

and \(S(\emptyset) = \emptyset\).

It is graded \(\mathcal{G} = \bigoplus_{n \in \mathbb{N}} \mathcal{G}_n\), \(\mathcal{G}_n = \text{Span}\{\Gamma : d(\Gamma) = n\}\)

\(* : \mathcal{G}_i \times \mathcal{G}_j \to \mathcal{G}_{i+j}\), \(\Delta : \mathcal{G}_k \to \bigoplus_{i+j=k} \mathcal{G}_i \otimes \mathcal{G}_j\)
We still need to provide mappings $\varphi : \mathcal{G} \rightarrow \mathcal{U}(\mathcal{L}_H)$ and $\bar{\varphi} : \mathcal{G} \rightarrow \mathcal{H}$ preserving (Hopf) algebraic structure of the Heisenberg-Weyl graphs $\mathcal{G}$. 
Model of the Heisenberg-Weyl algebra

Definition

We define a linear mapping \( \varphi : G \longrightarrow U(\mathcal{L}_H) \) which erases inner structure of a graph, given on the basis elements as:

\[
\varphi(\Gamma) = a^\dagger |\Gamma^+| a |\Gamma^-| e |\Gamma^0|
\]
Model of the Heisenberg-Weyl algebra

Definition
We define a linear mapping \( \varphi : \mathcal{G} \rightarrow \mathcal{U}(\mathcal{L}_\mathcal{H}) \) which erases inner structure of a graph, given on the basis elements as:

\[
\varphi (\Gamma) = a^{\dagger |\Gamma^+|} a^{\mid \Gamma^-\mid} e^{\mid \Gamma^0\mid}
\]
Model of the Heisenberg-Weyl algebra

**Definition**
We define a linear mapping \( \varphi : G \rightarrow \mathcal{U}(\mathcal{L}_H) \) which erases inner structure of a graph, given on the basis elements as:

\[
\varphi(\Gamma) = a^{\dagger}_{|\Gamma^+|} a_{|\Gamma^-|} e_{|\Gamma^0|}
\]

**Theorem**
Forgetful mapping \( \varphi : G \rightarrow \mathcal{U}(\mathcal{L}_H) \) is a Hopf algebra morphism.
**Model of the Heisenberg-Weyl algebra**

**Definition**
We define a linear mapping \( \varphi : \mathcal{G} \rightarrow \mathcal{U}(\mathcal{L}_\mathcal{H}) \) which erases inner structure of a graph, given on the basis elements as:

\[
\varphi(\Gamma) = a^\dagger_{|\Gamma^+|} a_{|\Gamma^-|} e_{|\Gamma^0|}
\]

Outgoing: 3

Ingoing: 4

Inner: 4

**Theorem**
Forgetful mapping \( \varphi : \mathcal{G} \rightarrow \mathcal{U}(\mathcal{L}_\mathcal{H}) \) is a Hopf algebra morphism.

**Note**
By additionally neglecting number of the inner edges \( \bar{\varphi}(\Gamma) = a^\dagger_{|\Gamma^+|} a_{|\Gamma^-|} \), we get an (AAU) algebra morphism \( \bar{\varphi} : \mathcal{G} \rightarrow \mathcal{H} \).
Morphism $\varphi : \mathcal{G} \to \mathcal{U}(\mathcal{L}_H)$ erases inner structure of a graph, and $\bar{\varphi} : \mathcal{G} \to \mathcal{H}$ erases inner structure of a graph & forgets number of its inner edges.
Morphism $\varphi: G \rightarrow U(\mathcal{L}_H)$ erases inner structure of a graph, and $\tilde{\varphi}: G \rightarrow \mathcal{H}$ erases inner structure of a graph & forgets number of its inner edges.
Algebraic "picture"

Morphism \( \varphi : \mathcal{G} \rightarrow \mathcal{U}(\mathcal{L}_\mathcal{H}) \) erases inner structure of a graph, and \( \bar{\varphi} : \mathcal{G} \rightarrow \mathcal{H} \) erases inner structure of a graph & forgets number of its inner edges.

Combinatorial algebra of Heisenberg-Weyl graphs

Enveloping algebra \( \mathcal{U}(\mathcal{L}_\mathcal{H}) \)

Heisenberg-Weyl algebra \( \mathcal{H} \)
Morphism $\varphi : \mathcal{G} \rightarrow \mathcal{U}(\mathcal{L}_\mathcal{H})$ erases inner structure of a graph,
and $\tilde{\varphi} : \mathcal{G} \rightarrow \mathcal{H}$ erases inner structure of a graph & forgets number of its inner edges.
We need to prove that \( \varphi : \mathcal{G} \rightarrow \mathcal{U}(\mathcal{L}_H) \) preserves product, i.e. \( \varphi(\Gamma_2 \ast \Gamma_1) = \varphi(\Gamma_2) \varphi(\Gamma_1) \)

\[
\Gamma_2 \ast \Gamma_1 = \sum m \in \Gamma_2^{-} \leftrightarrow \Gamma_1^{+} \Gamma_2^m \downarrow \Gamma_1
\]

\[
\Gamma_2^{-} \leftrightarrow \Gamma_1^{+} = \bigcup_i \Gamma_2^{-} \leftrightarrow \Gamma_1^{+}
\]
We need to prove that \( \varphi : \mathcal{G} \rightarrow \mathcal{U}(\mathcal{L}_H) \) preserves product, i.e. \( \varphi(\Gamma_2 \ast \Gamma_1) = \varphi(\Gamma_2) \varphi(\Gamma_1) \)

\[
\varphi(\Gamma_2 \ast \Gamma_1) = \sum_i \sum_{m \in \Gamma_2^- \bowtie_i \Gamma_1^+} \varphi(\Gamma_2^m \Gamma_1) \\
= \sum_i \sum_{m \in \Gamma_2^- \bowtie_i \Gamma_1^+} (a^\dagger) |\Gamma_2^+|-i a |\Gamma_2^-|+|\Gamma_1^-|-i e |\Gamma_2^0|+|\Gamma_1^0|+i \\
= \sum_i \left( \binom{|\Gamma_2^-|}{i} \binom{|\Gamma_1^+|}{i} \right) i! (a^\dagger) |\Gamma_2^+|-i a |\Gamma_2^-|+|\Gamma_1^-|-i e |\Gamma_2^0|+|\Gamma_1^0|+i \\
= \left( (a^\dagger) |\Gamma_2^+| a |\Gamma_2^-| e |\Gamma_2^0| \right) \left( (a^\dagger) |\Gamma_1^+| a |\Gamma_1^-| e |\Gamma_1^0| \right) = \varphi(\Gamma_2) \varphi(\Gamma_1)
\]
Algebraic "picture"

Structures preserved by morphisms

- $\varphi : \mathcal{G} \rightarrow U(L_H)$ (Hopf algebra)
- $\bar{\varphi} : \mathcal{G} \rightarrow \mathcal{H}$ (AAU)
More structured algebra of graphs can be seen as a combinatorial model of the Heisenberg-Weyl algebra. In this way, abstract algebraic structures $\mathcal{H}$ and $\mathcal{U}(\mathcal{L}_\mathcal{H})$ gain intuitive interpretation as a shadow of natural constructions on graphs in $\mathcal{G}$.
More structured algebra of graphs can be seen as a **combinatorial model** of the Heisenberg-Weyl algebra. In this way, abstract algebraic structures $\mathcal{H}$ and $\mathcal{U}(\mathcal{L}_\mathcal{H})$ gain intuitive interpretation as a shadow of natural constructions on graphs in $\mathcal{G}$.