Surprising correlations in random orientations of graphs
(or what is special with $n = 27$)

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Edge percolation

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- $0 \leq p \leq 1$
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- Every edge exist with probability $p$ independently of other edges. This model is called **Edge percolation $E^p$**.

- Let $s, a \in V$ be two vertices of $G$. We define $P_{E^p(G)}(s \leftrightarrow a) :=$ probability that there is a path between $s$ and $a$. 
Bunkbed conjecture (BBC)
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$G \times K_2$ is called a bunkbed graph
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Conjecture (Kasteleyn ’85)

For any $G$ and $0 \leq p \leq 1$ and any vertices $s, a \in V$ we have

$$P(s_0 \leftrightarrow a_0) \geq P(s_0 \leftrightarrow a_1) \text{ in } G \times K_2$$
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**Theorem (L.’08)**

*BBC is true for all outerplanar graphs $G$.***
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**Theorem (L.’08)**

*BBC is true for all outerplanar graphs G.*

**Theorem (Leander ’09)**

*BBC is true for all wheels and subgraphs of wheels.*
Correlations

Given any graph $G = (V, E)$ and three vertices $s, a, b \in V$. 

\begin{center}
\includegraphics[width=0.5\textwidth]{diagram.png}
\end{center}
Correlations

Given any graph $G = (V, E)$ and three vertices $s, a, b \in V$.

Classical fact:

**Proposition**

The events $\{s \leftrightarrow a\}$ and $\{s \leftrightarrow b\}$ are positively correlated in $E^p$, i.e.

$$P_{E^p(G)}(s \leftrightarrow a | s \leftrightarrow b) \geq P_{E^p(G)}(s \leftrightarrow a)$$
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Note:

$$P(s \leftrightarrow a | s \leftrightarrow b) \geq P(s \leftrightarrow a) \iff P(s \leftrightarrow a, s \leftrightarrow b) \geq P(s \leftrightarrow a)P(s \leftrightarrow b)$$
Another correlation result in $E^p$

Given any graph $G = (V, E)$ and four vertices $s, t, a, b \in V$. 
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Given any graph $G = (V, E)$ and four vertices $s, t, a, b \in V$. Condition on $\{s \leftrightarrow t\}$
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Theorem (van den Berg & Kahn ’02)

For any $G$ the events $\{s \leftrightarrow a\}$ and $\{s \leftrightarrow b\}$ are positively correlated in $E^p$, also when we first condition on $\{s \leftrightarrow t\}$, i.e.

$$P_{E^p(G)}(s \leftrightarrow a | s \leftrightarrow b, s \leftrightarrow t) \geq P_{E^p(G)}(s \leftrightarrow a | s \leftrightarrow t)$$
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Proof.

Clever use of Ahlswede-Daykin’s inequality.
Random Orientations (O)

\[ G = (V, E) \text{ a graph} \]
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Every edge is independently given one of the two possible directions with equal probability.

Let \( s, a \in V \) be two vertices of \( G \). We define

\[ PO(G) (s \rightarrow a) := \text{probability that there is a path from } s \text{ to } a. \]
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We can extend classical fact:

**Proposition**

For any graph \( G \) the events \( \{ s \to a \} \) and \( \{ s \to b \} \) are positively correlated in model \( O \), i.e.

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Given any graph $G = (V, E)$ and three vertices $s, a, b \in V$.

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Follows from:

**Lemma (Mc Diarmid ’81)**

For any graph $G = (V, E)$ and $s, a \in V$ we have

$$P_{E^{1/2}(G)}(s \leftrightarrow a) = P_{O(G)}(s \rightarrow a).$$
Given any graph $G = (V, E)$ and three vertices $s, a, b \in V$.

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Define in model $O$ the \textbf{out-cluster} $\bar{C}_s(G) \subset V$ as the (random) set of all vertices $u$ for which there is a directed path from $s$ to $u$.

Let also $C_s(G) \subset V$ be the (random) \textbf{cluster} around $s$ in model $E^p$, i.e. all vertices $u$ for which there exists a path between $s$ and $u$. 
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**Lemma**

*For any graph $G = (V, E)$, $s \in U \subseteq V$ we have*

$$P_{E^{1/2}}(C_s = U) = P_{O}(\overrightarrow{C}_s = U)$$
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**Lemma**

For any graph $G = (V, E)$, $s \in U \subseteq V$ we have

$$P_{E^{1/2}}(C_s = U) = P_O(\overrightarrow{C}_s = U)$$

**Proof.**

We have the recursion

$$P_{E^p}(C_s(G) = U) = \sum_{W: s \in W \subseteq U \setminus v} P_{E^p}(C_s(G \setminus v) = W)(1 - q^r)P_{E^p}(C_v(G \setminus W) = U \setminus W).$$
Question:

Are the events \( \{s \to a\} \) and \( \{b \to s\} \) negatively correlated in any graph \( G \)?
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Answer (Sven Erick Alm): No, counterexample on 4 nodes. From now on everything is joint work with Alm.

**Theorem (Alm & L. ’09)**

In model O the events \( \{ s \to a \} \) and \( \{ b \to s \} \):

- are negatively correlated in \( K_3 \),
- are independent in \( K_4 \),
- are positively correlated in \( K_n \), \( n \geq 5 \),
- are negatively correlated in trees and cycles.
Random Orientation on $G(n, p)$

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Let $G(n, p)$ be the random graph obtained by edge percolation with probability $p$ on $K_n$. Then we give this random graph random orientation on the edges as in model $O$. 
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**Theorem (Alm & L.’09)**

*For fixed $p$, as $n \to \infty$ we have:*

- the events $\{s \to a\}$ and $\{b \to s\}$ are negatively correlated if $p < 1/2$,
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Random Orientation on $G(n, p)$

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**Proof.**

Identify main cases and then long tricky computations.
We also fixed $n$ and computed $P(s \rightarrow a)$ and $P(s \rightarrow a, b \rightarrow s)$ using exact recursions. With this we computed the value of critical $p$ as in the following table:

<table>
<thead>
<tr>
<th>$n$</th>
<th>critical $p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>0.729</td>
</tr>
<tr>
<td>6</td>
<td>0.276</td>
</tr>
<tr>
<td>7</td>
<td>0.152</td>
</tr>
<tr>
<td>8</td>
<td>0.107</td>
</tr>
<tr>
<td>9</td>
<td>0.082</td>
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<tr>
<td>10</td>
<td>0.067</td>
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<td>11</td>
<td>0.056</td>
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<td>12</td>
<td>0.049</td>
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<tr>
<td>13</td>
<td>0.043</td>
</tr>
<tr>
<td>14</td>
<td>0.038</td>
</tr>
<tr>
<td>15</td>
<td>0.035</td>
</tr>
<tr>
<td>16</td>
<td>0.032</td>
</tr>
</tbody>
</table>

Converges to $1/2$???
Recall:

**Theorem (Alm & L.’09)**

For fixed $p$, as $n \to \infty$ we have in model O of $G(n, p)$:

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In fact we proved

$$1 - \frac{P(b \not\to s)}{P(b \not\to s \mid s \not\to a)} \to \frac{2p - 1}{3}, \quad \text{as } n \to \infty$$
This is a plot of \(1 - \frac{P(b \rightarrow s)}{P(b \rightarrow s | s \rightarrow a)}\) for \(n = 10..24\).

What was wrong? We spent many days looking for an error.
Then I plotted $1 - \frac{P(b \rightarrow s)}{P(b \rightarrow s|s \rightarrow a)}$ for $n = 8..20$ and all $p$:

What would happen for larger $n$?
Plot of $1 - \frac{P(b\rightarrow s)}{P(b\rightarrow s | s\rightarrow a)}$ for $n = 12..30$ as a function of $p$:

Starting from $n = 27$ we get 3 critical values of $p$. 
Some open problems

Can one characterize in which graphs \( \{ s \to a \} \) and \( \{ b \to s \} \) are negatively (positively) correlated for all choices of \( a, b, s \in V \). Is this a monotone graph property?
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- Can one characterize in which graphs $\{s \rightarrow a\}$ and $\{b \rightarrow s\}$ are negatively (positively) correlated for all choices of $a, b, s \in V$. Is this a monotone graph property?
- Conjecture: For most graphs it will depend on the choice of $a, b, s \in V$. 
- Conjecture: If the degree of $s$ is 2, then we will have negative correlation.

Understand the three critical values of $p$ for fixed $n$ as $n \rightarrow \infty$. 

Correlations of other paths? 

Prove the Bunkbed Conjecture for all graphs!
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Linusson (KTH)
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