Fusion process and representations of affine Hecke algebras

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joint work with V. Guizzi and M. Nazarov
We want to provide a detailed account of a combinatorial construction (after Cherednik) which exhibits the irreducible (finite dimensional) modules of the affine Hecke algebra of $GL_n$ as cyclic modules over the Hecke algebra of $GL_n$.

The main theorem we are going to discuss has been (roughly) stated by Cherednik without proof.
The affine Hecke algebra \( \tilde{H}_n \) (of \( GL_n \)) is the \( \mathbb{C}(q) \)-algebra with the following presentation.
Affine Hecke algebras: Bernstein presentation

Generators for $\tilde{H}_n$

$$T_1, \ldots, T_{n-1},$$
$$X_1^{\pm 1}, \ldots, X_n^{\pm 1}$$

Relations $\tilde{H}_n$

$$T_k T_{k+1} T_k = T_{k+1} T_k T_{k+1}, \quad 1 \leq k \leq n - 2,$$
$$T_k T_l = T_l T_k, \quad 1 < |k - l|,$$
$$(T_k - q)(T_k + 1) = 0, \quad 1 \leq k \leq n - 1,$$
$$X_k X_l = X_l X_k, \quad 1 \leq k, l \leq n,$$
$$X_l T_k = T_k X_l, \quad l \neq k, k + 1,$$
$$T_k X_k T_k = q X_{k+1}, \quad 1 \leq k \leq n - 1.$$
The “finite” Hecke algebra $\mathcal{H}_n$ is the subalgebra of $\tilde{\mathcal{H}}_n$ generated by the $T_i$; it has the following presentation, which shows it as a $q$ deformation of $\mathbb{C}[S_n]$. 
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$$(T_k - q)(T_k + 1) = 0.$$
Remarks

Recall the defining relations of $\tilde{H}_n$:

\[
T_k T_{k+1} T_k = T_{k+1} T_k T_{k+1},
\]

\[
T_k T_l = T_l T_k,
\]

\[
(T_k - q)(T_k + 1) = 0,
\]

\[
X_l T_k = T_k X_l,
\]

\[
X_k X_l = X_l X_k,
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\[
T_k X_k T_k = q X_{k+1}
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The $X_i^{\pm 1}$ generate a (maximal) commutative subalgebra $A_n$ of $\tilde{\mathcal{H}}_n$. 

$$\tilde{\mathcal{H}}_n = \mathcal{H}_n \otimes A_n$$ as vector spaces
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\[ T_k T_{k+1} T_k = T_{k+1} T_k T_{k+1}, \quad T_k T_l = T_l T_k, \]
\[ (T_k - q)(T_k + 1) = 0, \quad X_k X_l = X_l X_k, \]
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Theorem

The center of \( \tilde{\mathcal{H}}_n \) consists precisely of symmetric polynomials in \( X_1^{\pm 1}, \ldots, X_n^{\pm 1} \).
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$$\tilde{H}_n = H_n \otimes A_n \text{ as vector spaces}$$

**Theorem**

The center of $\tilde{H}_n$ consists precisely of symmetric polynomials in $X_1^\pm 1, \ldots, X_n^\pm 1$.

**Corollary**

Any irreducible $\tilde{H}_n$-module is finite dimensional.
Irreducible representations of $\tilde{H}_n$ after Zelevinsky-Rogawski

Technical reduction:
Can assume that the eigenvalues of all $X_i$ belong to $q\mathbb{Z}$.
Under this hypothesis, irreducible representations of $\tilde{H}_n$ are parametrized by multisegments:

$$M = \sum_{i \leq j} m_{ij} [i, j]$$

with $\sum_{i \leq j} m_{ij} (j - i + 1) = n$.

Zelevinsky classification sounds like: for a certain ordering of $M$, a suitable induced module built up starting from $M$ has a unique irreducible quotient. The irreducible $\tilde{H}_n$-modules obtained in this way are non-equivalent for different multisegments $M$ and form a complete set with all eigenvalues of $X_1, \ldots, X_n$ in $q\mathbb{Z}$.

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Irreducible representations of $\tilde{H}_n$ after Cherednik

For each Cherednik diagram $\lambda$ we produce a pair $(E_\lambda, \chi_\lambda)$ where $E_\lambda \in H_n$ and $\chi_\lambda : A_n \to \mathbb{C}(q)$ is a character of the algebra $A_n$, such that:

(i) $E_\lambda$ is an eigenvector for $A_n$ inside $\text{Ind}_{H_n A_n} H_n \otimes A_n C\chi_\lambda$,

$\text{Ind}_{H_n A_n} H_n \otimes A_n C\chi_\lambda = (H_n \otimes A_n C\chi_\lambda) \otimes A_n C\chi_\lambda$.

(ii) the space $V_\lambda = H_n \cdot E_\lambda$ which is a $\tilde{H}_n$-module by (i), is irreducible.

The modules $V_\lambda$ are a complete set of inequivalent irreducible representations of $\tilde{H}_n$.

Proving the above step is non-trivial. However, it can be done by resorting to instances in which the results are known: indeed we use a version of a Drinfeld functor which reduces our problem to a statement about quantum affine algebras, known by the work of Akasaka-Kashiwara.
We use combinatorial objects which are in a bijection with Zelevinsky multisegments, and which we call Cherednik diagrams. They are certain subsets of $\mathbb{Z}^2$ similar to Young diagrams.
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$$\text{Ind}_{A_n}^{\tilde{\mathcal{H}}_n} \chi_\lambda = \tilde{\mathcal{H}}_n \otimes A_n \mathbb{C}_{\chi_\lambda} = (\mathcal{H}_n \otimes \mathbb{C}(q) A_n) \otimes A_n \mathbb{C}_{\chi_\lambda} = \mathcal{H}_n$$

2. the space $V_\lambda = \mathcal{H}_n \cdot E$ which is a $\tilde{\mathcal{H}}_n$-module by (i), is irreducible.
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- Proving the above step is non trivial. However, it can be done by resorting to instances in which the results are known: indeed we use a version of a Drinfeld functor which reduces our problem to a statement about quantum affine algebras, known by the work of Akasaka-Kashiwara.
The combinatorial definition of $E_\lambda$ and $\chi_\lambda$ will be the main theme of this lecture.
Remarks

One should recall that irreducible representations of affine Hecke algebras for any Weyl group have been classified by Kazhdan and Lusztig via equivariant K-theory of Steinberg varieties.

Their approach, like Zelevinsky's one, is non-constructive: irreducibles are described as quotients by a certain maximal submodule.

Our approach provides at least one vector in each irreducible! Indeed, when the action of $A^n$ is semisimple, one can write down combinatorially a basis for $V_\lambda$. These are precisely the cases in which $\lambda$ is a skew-shape.

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Cherednik’s diagrams

Given \((a_1, \ldots, a_r), (b_1, \ldots, b_r) \in \mathbb{Z}^r\) s.t. \(a_i \leq b_i \ \forall \ i\), we’ll consider subsets

\[
B = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq r, a_i \leq j \leq b_i\}.
\]

represented, with matrix-style coordinates, in the plane \(\mathbb{Z}^2\) by diagrams having boxes centered in

\[
(1, a_1), \ldots, (1, b_1), (2, a_2), \ldots, (2, b_2), \ldots, (r, a_r), \ldots, (r, b_r)
\]

**Definition**

The set \(B\) is a Cherednik diagram if for each \(i = 1, \ldots, r - 1\) either \(b_{i+1} \leq b_i\), or \(b_{i+1} = b_i + 1\) and \(a_{i+1} \leq a_i + 1\).
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\[(1, a_1), \ldots, (1, b_1), (2, a_2), \ldots, (2, b_2), \ldots \ldots (r, a_r), \ldots, (r, b_r)\]

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Let \(C_n\) denote the collection of Cherednik diagrams with \(n\) boxes.
The conditions can be rephrased as follows. Think of $\lambda$ as the juxtaposition of its rows $\lambda_1, \ldots, \lambda_r$, where $\lambda_i = [a_i, b_i]$.

1. For any $i$, if the final box of $\lambda_i$ lies on the diagonal $x + y = h$, then final box of $\lambda_{i+1}$ lies on the diagonal $x + y = k$, then $h \geq k$.

2. If $h = k$, then the length of $\lambda_{i+1}$ is greater or equal than the length of $\lambda_i$. 
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From the above remarks, it is clear that $C_n$ contains all ordinary and skew diagrams with $n$ boxes.
$C_5, C_6$ contains neither $\{[1, 2], [1, 4]\}$ nor $\{[1, 3], [3, 4]\}$
$C_7$ contains \{[1, 2], [0, 2], [1, 2]\}, \{[1, 2], [-1, 1], [0, 1]\}, \\{[1, 2], [2, 3], [0, 3]\}
The construction of $E_\lambda$ will be performed via a suitable version of the *fusion procedure*.
This method was initially used to reproduce the Young symmetrizers in the group ring of $S_n$.
Nazarov used it in his works on projective representations of the symmetric group and on their $q$-analogues.
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Roughly speaking, it consists in obtaining $E_\lambda$ as a limiting value of suitable rational functions, by using a mixture of combinatorics and noncommutative ring theory.
The basic functions

For any $x \in \mathbb{C}(q)$ put

$$\langle x \rangle = \frac{(1-q)}{(1-x)}.$$
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For any $x \in \mathbb{C}(q)$ put

$$\langle x \rangle = (1 - q)/(1 - x).$$

For each $k = 1, \ldots, n - 1$ define

$$\varphi_k(x_1, \ldots, x_n) = T_k + \langle x_{k+1}/x_k \rangle$$

$$= T_k + \frac{1 - q}{1 - x_k x_{k+1}}$$

with values in the algebra $\mathcal{H}_n$. 
The basic functions

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\]
\[
= T_k + \frac{1 - q}{1 - x_kx_{k+1}^{-1}}
\]
with values in the algebra \( \mathcal{H}_n \).

Using any reduced decomposition \( w = s_{k_1} \ldots s_{k_m} \in S_n \) define the rational function
\[
\varphi_w = \varphi_{k_1}(s_{k_1} \varphi_{k_2})(s_{k_1}s_{k_2} \varphi_{k_3}) \ldots (s_{k_1} \ldots s_{k_{m-1}} \varphi_{k_m}).
\]
For instance, if \( w = s_1 s_2 s_3 \) we have

\[
\varphi_w = \left( T_1 + \langle x_2 / x_1 \rangle \right) \left( T_2 + s_1 \langle x_3 / x_2 \rangle \right) \left( T_3 + s_1 s_2 \langle x_4 / x_3 \rangle \right)
\]

\[
= \left( T_1 + \langle x_2 / x_1 \rangle \right) \left( T_2 + \langle x_3 / x_2 \rangle \right) \left( T_3 + \langle x_4 / x_3 \rangle \right)
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\]
The function $\varphi_w$ does not depend on the choice of a reduced decomposition of $w$. 

Where $\beta_k = s_{i_1} \ldots s_{i_k-1} (\alpha_{i_k})$ for each $k = 1, \ldots, m$ and $\alpha_{i_k} = x_i x_{i+1}$ is a simple root written multiplicatively. Therefore, if we set $\varphi_i^k \beta_k = T_i^k + \langle \beta_k \rangle$ then we get $\varphi_w = \varphi_i^1 \beta_1 \varphi_i^2 \beta_2 \ldots \varphi_i^m \beta_m$. 

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The above proposition is better understood if we write $\varphi_w$ as

$$
\varphi_w = (\varphi_{i_1})(s_{i_1}\varphi_{i_2})(s_{i_1}s_{i_2}\varphi_{i_3})\ldots(s_{i_1}\ldots s_{i_{m-1}}\varphi_{i_m})
= (T_{i_1} + \langle \beta_1 \rangle)(T_{i_2} + \langle \beta_2 \rangle)(T_{i_3} + \langle \beta_3 \rangle)\ldots(T_{i_m} + \langle \beta_m \rangle)
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where $\beta_k = s_{i_1}\ldots s_{i_{k-1}}(\alpha_{i_k})$ for each $k = 1, \ldots, m$ and $\alpha_i = x_i x_{i+1}^{-1}$ is a simple root written multiplicatively. Therefore, if we set

$$
\varphi_{\beta_k}^{i_k} = T_{i_k} + \langle \beta_k \rangle
$$

then we get

$$
\varphi_w = \varphi_{\beta_1}^{i_1} \varphi_{\beta_2}^{i_2} \ldots \varphi_{\beta_m}^{i_m}
$$
\[ \varphi_w = \varphi_{\beta_1}^{i_1} \varphi_{\beta_2}^{i_2} \cdots \varphi_{\beta_m}^{i_m} \]

Therefore, when we change the reduced expression for \( w \), the upper indices change according to the braid relations, whereas the lower roots change according to the “moves” that regulate the convex orders on \( N(w) = \{ \alpha \in \Delta^+ \mid w^{-1}(\alpha) \in -\Delta^+ \} \).

In this case these moves are

\[ \alpha, \alpha + \beta, \beta \leftrightarrow \beta, \alpha + \beta, \alpha \]
\[ \alpha, \beta \leftrightarrow \beta, \alpha \text{ if } \alpha \perp \beta \]
Let $\lambda \in \mathbb{C}$ and $\Lambda$ be the row filling of $\lambda$. For every box $(i, j)$ of $\lambda$, the difference $j - i$ is called the \textit{content} of this box.

For $k = 1, \ldots, n$, denote by $c_k$ the content of the box which is filled with the number $k$ in $\Lambda$.

Set $\delta_{\lambda}(x_1, \ldots, x_n) = \prod (1 - x_l x_k)$, the product being taken over all pairs $(k, l)$ such that $k < l$ while in $\Lambda$ the numbers $k, l$ occur in the leftmost boxes of two parallel rows of $\lambda$.

Let $w_0 \in S_n$ be the element of maximal length.
Let $\lambda \in C_n$ and $\Lambda$ be the row filling of $\lambda$. For every box $(i, j)$ of $\lambda$, the difference $j - i$ is called the content of this box. For $k = 1, \ldots, n$, denote by $c_k$ the content of the box which is filled with the number $k$ in $\Lambda$. Set $\delta_\lambda(x_1, \ldots, x_n) = \prod (1 - x_l x_k)$, the product being taken over all pairs $(k, l)$ such that $k < l$ while in $\Lambda$ the numbers $k$, $l$ occur in the leftmost boxes of two parallel rows of $\lambda$. Let $w_0 \in S_n$ be the element of maximal length.
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\[
\delta_\lambda(x_1, \ldots, x_n) = \prod \left( 1 - \frac{x_l}{x_k} \right),
\]

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Main Theorem: statement

\[ \mathcal{F}_\lambda = \{(x_1, \ldots, x_n) \in \mathbb{C}(q)^n \mid x_k q^{c_l} = q^{c_k} x_l \text{ whenever } k \text{ and } l \text{ are in the same row of } \Lambda\} \]
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Theorem

For any \( \lambda \in \mathcal{C}_n \) the restriction of the rational function \( \delta_\lambda \varphi_{w_0} \),

\[
(\delta_\lambda \varphi_{w_0})(x_1, \ldots, x_n) = \prod \left(1 - \frac{x_l}{x_k}\right) \prod (T_{i_s} + \langle \beta_s \rangle)
\]

to the subspace \( \mathcal{F}_\lambda \) is regular and non-zero at the point 
\((x_1, \ldots, x_n) = (q^{c_1}, \ldots, q^{c_n})\).

Its value \( E_\lambda \) is a cyclic generator for \( V_\lambda \) as a submodule of the \( \tilde{\mathcal{H}}_n \)-module obtained by inducing the character of \( A_n \)

\[
\chi_\lambda(X_k) = q^{c_k}
\]
Comments on the proof

Recall that if $\beta = x_i x_{i+1} x_{i+2} \cdots x_{j-1} x_j$, we have

$$\phi_k \beta = T_k + 1 - q^{1-x_i x_{i+1} x_{i+2} \cdots x_{j-1} x_j}.$$ 

This function has a simple pole at our point $(q_1, \ldots, q_n)$ precisely when $c_i = c_j$. The root $\beta$ (which we identify combinatorially with the pair $(i, j)$, $i < j$) will be called singular if $c_i = c_j$.

The basic tool to deal with singular roots is the following Lemma:

Suppose that the pair $(i, j)$ is singular. Set $x = (q_1, \ldots, q_n)$.

(a) If $i, i+1$ belong to the same row of $\Lambda$ then

$$\phi_k \beta, \phi_{k+1} \beta, \phi_{k+1} \beta | F_\lambda(x) = (1 + T_k)(T_{k+1} - q T_k + 1).$$

(b) If $j-1, j$ belong to the same row of $\Lambda$ then

$$\phi_k \beta, \phi_{k+1} \beta, \phi_{k+1} \beta | F_\lambda(x) = (T_{k+1} - q T_k + 1)(1 + T_k).$$
Comments on the proof

Recall that if $\beta = x_i x_j^{-1}$, we have

$$\varphi^k_\beta = T_k + \frac{1 - q}{1 - x_i x_j^{-1}}.$$

This function has a simple pole at our point $(q^{c_1}, \ldots, q^{c_n})$ precisely when $c_i = c_j$. The root $\beta$ (which we identify combinatorially with the pair $(i, j)$, $i < j$) will be called \textbf{singular} if $c_i = c_j$. 
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The basic tool to deal with singular roots is the following

**Lemma**

*Suppose that the pair $(i, j)$ is singular. Set $x = (q^{c_1}, \ldots, q^{c_n})$.*

**(a)** If $i, i + 1$ belong to the same row of $\Lambda$ then

$$\left. \left( \phi^{k}_{i, i+1} \phi^{k+1}_{i, j} \phi^{k}_{i+1, j} \right) \right|_{\mathcal{F}_\lambda(x)} = (1 + T_k)(T_{k+1} T_k - q T_{k+1} - q).$$

**(b)** If $j - 1, j$ belong to the same row of $\Lambda$ then

$$\left. \left( \phi^{k}_{i, j-1} \phi^{k+1}_{i, j} \phi^{k}_{j-1, j} \right) \right|_{\mathcal{F}_\lambda(x)} = (T_k T_{k+1} - q T_{k+1} - q)(1 + T_k).$$
Strategy

Choose a suitable convex ordering on $\Delta^+$ which makes the easiest possible to apply the above lemma.

Caveat: we should perform this procedure for each singular pair, so we should check some kind of compatibility conditions. Moreover, not all singularities are removed in this way: here the correction factor $\delta_\lambda$ plays its role.
Strategy

Choose a suitable convex ordering on $\Delta^+$ which makes the easiest possible to apply the above lemma. Indeed, a singular root is always decomposable, i.e. of the form $\alpha + \beta$. We will introduce an ordering such that for any singular $\alpha + \beta$, we have either

$$\alpha, \alpha + \beta \ldots \beta \quad \text{or} \quad \alpha \ldots \alpha + \beta, \beta$$
Strategy

Choose a suitable convex ordering on \( \Delta^+ \) which makes the easiest possible to apply the above lemma. Indeed, a singular root is always decomposable, i.e. of the form \( \alpha + \beta \). We will introduce an ordering such that for any singular \( \alpha + \beta \), we have either

\[
\alpha, \alpha + \beta, \ldots, \beta \quad \text{or} \quad \alpha, \ldots, \alpha + \beta, \beta
\]

Using the local moves on the ordering, we will achieve

\[
\alpha, \alpha + \beta, \gamma_1, \ldots, \gamma_k, \beta \rightarrow \alpha, \alpha + \beta, \beta, \gamma'_1, \ldots, \gamma'_k
\]

so that we can use the lemma.

Caveat: we should perform this procedure for each singular pair, so we should check some kind of compatibility conditions. Moreover, not all singularities are removed in this way: here the correction factor \( \delta^{\lambda} \) plays its role.
Choose a suitable convex ordering on $\Delta^+$ which makes the easiest possible to apply the above lemma. Indeed, a singular root is always decomposable, i.e. of the form $\alpha + \beta$. We will introduce an ordering such that for any singular $\alpha + \beta$, we have either

$$\alpha, \alpha + \beta, \ldots, \beta \quad \text{or} \quad \alpha, \ldots, \alpha + \beta, \beta$$

Using the local moves on the ordering, we will achieve

$$\alpha, \alpha + \beta, \gamma_1, \ldots, \gamma_k, \beta \rightarrow \alpha, \alpha + \beta, \beta, \gamma'_1, \ldots, \gamma'_k$$

so that we can use the lemma.

**Caveat**: we should perform this procedure for each singular pair, so we should check some kind of compatibility conditions. Moreover, not all singularities are removed in this way: here the correction factor $\delta_\lambda$ plays its role.
Final example

\[ \lambda = \begin{array}{cccc}
\end{array} \]

\[ \Lambda = \begin{array}{cc}
1 & 2 \\
3 & 4 \\
5 & 6 \\
\end{array} \]

\[ \mathcal{F}_\lambda = \{ x_2 = q x_1, x_4 = q x_3, x_6 = q x_5 \} \]
Final example

\[ \lambda = \begin{array}{c|c|c} & & \\
& & \\
\end{array} \]

\[ \Lambda = \begin{array}{c|c|c}
1 & 2 & \\
3 & 4 & \\
5 & 6 & \\
\end{array} \]

\[ \mathcal{F}_\lambda = \{ x_2 = q x_1, x_4 = q x_3, x_6 = q x_5 \} \]

\[ \Delta^+_{11}: (1, 2), \]

\[ \Delta^+_{12}: (1, 3), (1, 4), (2, 3), (2, 4), \]

\[ \Delta^+_{22}: (3, 4), \]

\[ \Delta^+_{13}: (1, 5), (2, 5), (1, 6), (2, 6), \]

\[ \Delta^+_{23}: (3, 5), (4, 5), (3, 6), (4, 6), \]

\[ \Delta^+_{33}: (5, 6). \]
Final example

\[ \lambda = \begin{array}{c}
\square \\
\square \\
\square \\
\square \\
\square \\
\end{array} \quad \Lambda = \begin{array}{ccc}
1 & 2 \\
3 & 4 \\
5 & 6 \\
\end{array} \quad \mathcal{F}_\lambda = \{ x_2 = q x_1, x_4 = q x_3, x_6 = q x_5 \} \]

\[ \Delta^+_{11} : (1, 2), \]
\[ \Delta^+_{12} : (1, 3), (1, 4), (2, 3), (2, 4), \]
\[ \Delta^+_{22} : (3, 4), \]
\[ \Delta^+_{13} : (1, 5), (2, 5), (1, 6), (2, 6), \]
\[ \Delta^+_{23} : (3, 5), (4, 5), (3, 6), (4, 6), \]
\[ \Delta^+_{33} : (5, 6). \]

\[ w_0 = s_1 s_2 s_3 s_1 s_2 s_1 s_4 s_3 s_5 s_4 s_2 s_1 s_3 s_2 s_1 \]
In the above display \((i,j)\) is a shortcut for \(\varphi_{i,j}\). We shall start desingularizing from the rightmost singular pair in the “red” piece. So we move \((3,4)\) next to \((3,6), (4,6)\), which are already adjacent thanks to the special choice of the order. Then we apply the Lemma evaluating the three factors restricted to \(F_\lambda\). Here the main technical point is that \(\varphi_{3,4}\), upon restriction to \(F_\lambda\), is a constant function, hence it is unaffected by evaluation! Therefore we can take back \(\varphi_{3,4}\) to its original position. The next display should suggest that a careful choice of the order on pairs to desingularize allows us to perform the previous procedure for any singular pair (up to those which are canceled by the correction factor \(\delta_\lambda\)).
The moves

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)
The moves

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)
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(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)
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(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (1, 5), (3, 4), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)
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(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (1, 5), (2, 5), (3, 4), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)
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(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (1, 5), (2, 5), (1, 6), (3, 4), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)
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\[(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)\]

\[(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)\]

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(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 2, 6), (4, 5), (3, 5), (3, 4), (3, 4, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 2, 6), (4, 5), (3, 5), (3, 4), (3, 4, 6), (5, 6)
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(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (4, 5), (3, 5), (3, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (4, 5), (3, 5), (3, 4), (3, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (4, 5), (3, 5), (3, 4), (3, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (4, 5), (3, 5), (3, 4), (3, 6), (5, 6)
The moves

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (4, 5), (3, 5), (3, 4), (3, 4, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (4, 5), (3, 5), (3, 4), (3, 4, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 2, 6), (4, 5), (3, 5), (3, 4), (3, 4, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3, 4), (3, 4), (1, 5), (2, 5), (1, 2, 6), (4, 5), (3, 5), (3, 4), (3, 4, 6), (5, 6)
The moves

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 6), (4, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (3, 5), (4, 5), (3, 4), (3, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 6), (2, 6), (4, 5), (3, 5), (3, 4), (3, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (1, 2, 6), (4, 5), (3, 5), (3, 4), (3, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3, 4), (3, 4), (1, 5), (2, 5), (1, 2, 6), (4, 5), (3, 5), (3, 4), (3, 6), (5, 6)

(1, 2), (1, 3), (1, 4), (2, 3, 4), (3, 4), (1, 5), (2, 5), (1, 2, 6), (4, 5), (3, 5), (3, 4), (3, 6), (5, 6)
The final outcome

\[ E_\lambda = (\partial_\lambda \varphi_0)|_{F_\lambda}(1, q, 1, q, q^{-1}, 1) = \]

\[ (T_1 + 1)(1 - q)(T_3 + 1)(T_1 T_2 - q T_2 - q)(T_1 + 1)(T_4 - q) \]

\[ (T_3 - q^2(q + 1)^{-1})(T_5 T_4 - q T_5 - q)(T_2 - q) \]

\[ (T_1 - q^2(q + 1)^{-1})(T_3 T_2 - q T_3 - q)(T_1 + 1). \]
The final outcome

\[ E_{\lambda} = (\partial_{\lambda} \varphi_0)|_{F_{\lambda}}(1, q, 1, q, q^{-1}, 1) = \]

\[ (T_1 + 1)(1 - q)(T_3 + 1)(T_1 T_2 - q T_2 - q)(T_1 + 1)(T_4 - q) \]

\[ (T_3 - q^2(q + 1)^{-1})(T_5 T_4 - q T_5 - q)(T_2 - q) \]

\[ (T_1 - q^2(q + 1)^{-1})(T_3 T_2 - q T_3 - q)(T_1 + 1). \]

\[ E_{\lambda} \] is nonzero! Indeed:
The final outcome

\[ E_\lambda = (\partial_\lambda \varphi_0)|_{F_\lambda} (1, q, 1, q, q^{-1}, 1) = \]
\[ (T_1 + 1)(1 - q)(T_3 + 1)(T_1 T_2 - q T_2 - q)(T_1 + 1)(T_4 - q) \]
\[ (T_3 - q^2 (q + 1)^{-1})(T_5 T_4 - q T_5 - q)(T_2 - q) \]
\[ (T_1 - q^2 (q + 1)^{-1})(T_3 T_2 - q T_3 - q)(T_1 + 1). \]

\[ E_\lambda \text{ is nonzero! Indeed:} \]

\[ E_\lambda = q (q^2 - 1) T_1 T_3 T_4 T_3 T_5 T_4 T_2 T_1 T_3 T_2 T_1 + \sum_{\ell(w)<11} a_w T_w \]