

Symmetry and super-symmetry distribution for partitions

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Outline

- 1 Introduction
- 2 Main results
- 3 Super-Symmetry



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Partitions

- A **partition** $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of n , write $\lambda \vdash n$, if

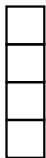
$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$$

$$|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_\ell = n.$$

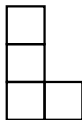
- $p(n)$ = the number of partitions of n .

Ferrers diagram

$(1,1,1,1)$



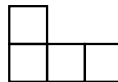
$(2,1,1)$



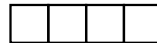
$(2,2)$



$(3,1)$

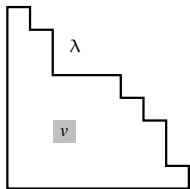


(4)



Pointed Partitions

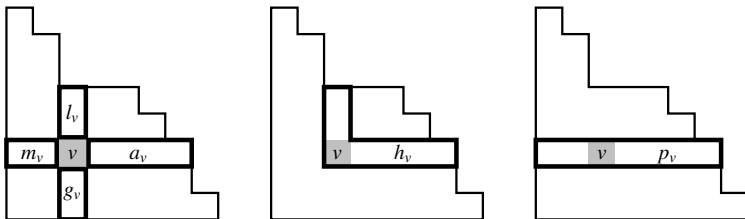
- A **pointed partition** (λ, v) of n if $\lambda \vdash n$ and v is a cell in the Ferrers diagram of λ .



- \mathcal{F}_n = the set of pointed partitions of n .

$$|\mathcal{F}_n| = p(n) \times n$$

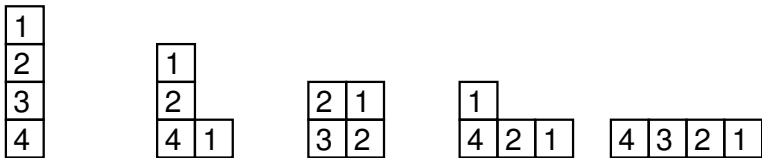
The arm, leg, coarm, coleg, hook, and part of a pointed partition



$$h_v = l_v + a_v + 1 \quad \text{and} \quad p_v = m_v + a_v + 1$$

The distribution of h_v

The distribution of h_v on \mathcal{F}_4 is given below:

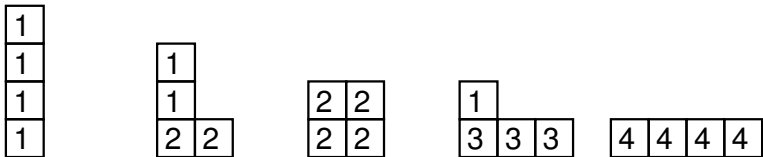


where h_v is written in a cell v .

h_v	1	2	3	4	Σ
$\#v$	7	6	3	4	20

The distribution of p_v

The distribution of p_v on \mathcal{F}_4 is given below:



where p_v is written in a cell v .

p_v	1	2	3	4	Σ
$\#v$	7	6	3	4	20

h_ν and p_ν are equidistributed

$$\sum_{(\lambda, \nu) \in \mathcal{F}_4} x^{h_\nu} = 7x + 6x^2 + 3x^3 + 4x^4.$$

$$\sum_{(\lambda, \nu) \in \mathcal{F}_4} x^{p_\nu} = 7x + 6x^2 + 3x^3 + 4x^4.$$

Theorem (Bessenrodt, 1998; Bacher-Manivel, 2001)

The hook length h_ν and the part length p_ν are equidistributed.

$$\sum_{(\lambda, \nu) \in \mathcal{F}_n} x^{h_\nu} = \sum_{(\lambda, \nu) \in \mathcal{F}_n} x^{p_\nu}$$

h_ν and p_ν are equidistributed

$$\sum_{(\lambda, \nu) \in \mathcal{F}_4} x^{h_\nu} = 7x + 6x^2 + 3x^3 + 4x^4.$$

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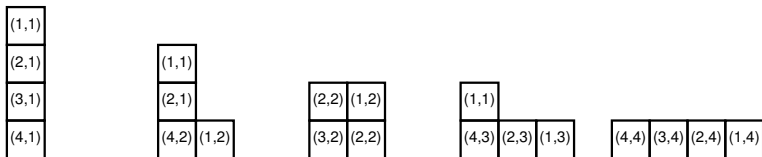
Theorem (Bessenrodt, 1998; Bacher-Manivel, 2001)

The hook length h_ν and the part length p_ν are equidistributed.

$$\sum_{(\lambda, \nu) \in \mathcal{F}_n} x^{h_\nu} = \sum_{(\lambda, \nu) \in \mathcal{F}_n} x^{p_\nu}$$

The joint distribution of h_v and p_v

The joint distribution of (h_v, p_v) on \mathcal{F}_4 is given below:



where (h_v, p_v) is written in a cell v .

$h_v \backslash p_v$	1	2	3	4	Σ
1	3	2	1	1	7
2	2	2	1	1	6
3	1	1	0	1	3
4	1	1	1	1	4
Σ	7	6	3	4	20

Theorem (Bessenrodt-Han, 2009)

The hook length h_v and the part length p_v are symmetric.

$$\sum_{(\lambda, \nu) \in \mathcal{F}_n} x^{h_\nu} y^{p_\nu} = \sum_{(\lambda, \nu) \in \mathcal{F}_n} x^{p_\nu} y^{h_\nu}.$$

$h_v \setminus p_v$	1	2	3	4	Σ
1	3	2	1	1	7
2	2	2	1	1	6
3	1	1	0	1	3
4	1	1	1	1	4
Σ	7	6	3	4	20

Theorem (Bessenrodt-Han, 2009)

The hook length h_v and the part length p_v are symmetric.

$$\sum_{(\lambda, \nu) \in \mathcal{F}_n} x^{h_\nu} y^{p_\nu} = \sum_{(\lambda, \nu) \in \mathcal{F}_n} x^{p_\nu} y^{h_\nu}.$$

Question

How to construct an involution on \mathcal{F}_n exchanging hook length and part length?

$a_v \backslash l_v$	0	1	2	3	Σ
0	7	3	1	1	12
1	3	1	1	0	5
2	1	1	0	0	2
3	1	0	0	0	1
Σ	12	5	2	1	20

Theorem (Bessenrodt, 1998; Bacher-Manivel, 2001)

The arm length a_v and the leg length l_v are super-symmetric.

$a_v \backslash l_v$	0	1	2	3	Σ
0	7	3	1	1	12
1	3	1	1	0	5
2	1	1	0	0	2
3	1	0	0	0	1
Σ	12	5	2	1	20

Theorem (Bessenrodt, 1998; Bacher-Manivel, 2001)

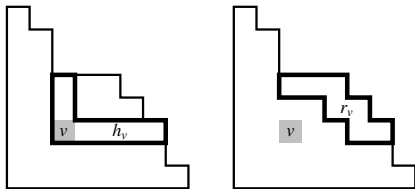
The arm length a_v and the leg length l_v are super-symmetric.

- $\mathcal{F}_n(\alpha, \beta)$ = the set of pointed partitions with arm length α and leg length β .

Question

How to construct a bijection from $\mathcal{F}_n(\alpha, \beta)$ to $\mathcal{F}_n(\alpha', \beta')$ where $\alpha + \beta = \alpha' + \beta'$?

- The **rim hook** R_v or $R_v(\lambda)$ is the contiguous border strip of λ connecting the rightmost and the uppermost cells of the hook H_v .



$$h_v = r_v = l_v + a_v + 1$$

- If λ be a partition, denote its **conjugate** by $\lambda' = (\lambda'_1, \lambda'_2, \dots)$, that is, λ'_i is the number of parts of λ that are $\geq i$.

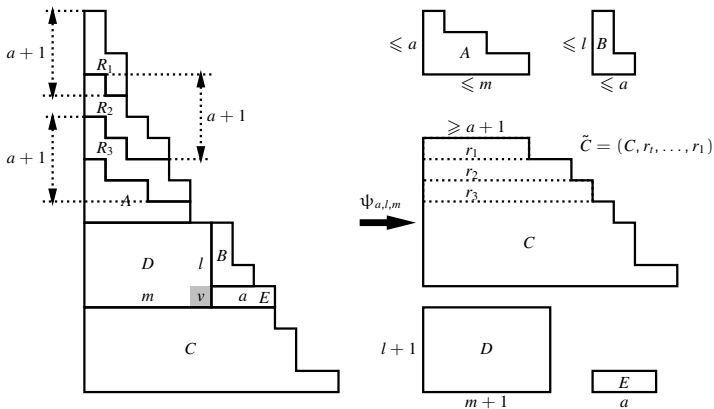
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Pointed partition to quintuple

We can construct the mapping $\psi_{a,l,m}$ and its inverse as follows:



- $\mathcal{F}_n(a, l, m)$ = the set of pointed partitions (λ, ν) of n such that $a_\nu = a$, $l_\nu = l$ and $m_\nu = m$.
- $\mathcal{Q}_n(a, l, m)$ = the set of quintuples (A, B, \tilde{C}, D, E) such that
 - $A \subset a \times m$ rectangle,
 - $B \subset l \times a$ rectangle,
 - \tilde{C} = a partition whose all parts are $\geq a + 1$,
 - $D = (l + 1) \times (m + 1)$ rectangle,
 - $E = 1 \times a$ rectangle, and

$$|A| + |B| + |\tilde{C}| + |D| + |E| = n.$$

- \mathcal{Q}_n = the set of such quintuples (A, B, \tilde{C}, D, E) .

Define the bijection ψ from \mathcal{F}_n to \mathcal{Q}_n by

$$\psi(\lambda, \nu) = \psi_{a,l,m}(\lambda, \nu) \quad \text{if } (\lambda, \nu) \in \mathcal{F}_n(a, l, m)$$

and the involution ρ on \mathcal{Q}_n by

$$\rho(A, B, \tilde{C}, D, E) = (B', A', \tilde{C}, D', E)$$

where X' is the conjugate of the partition X .

Theorem (S.-Zeng, 2009)

For all $n \geq 0$, the mapping

$$\varphi = \psi^{-1} \circ \rho \circ \psi$$

is an involution on \mathcal{F}_n such that if $\varphi : (\lambda, \nu) \mapsto (\mu, u)$ then

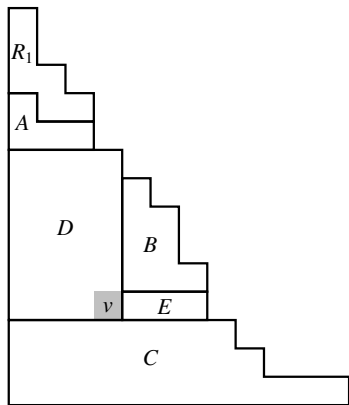
$$(a_\nu, l_\nu, m_\nu)(\lambda) = (a_u, m_u, l_u)(\mu). \quad (1)$$

In particular, the mapping φ also satisfies

$$(h_\nu, p_\nu)(\lambda) = (p_u, h_u)(\mu). \quad (2)$$

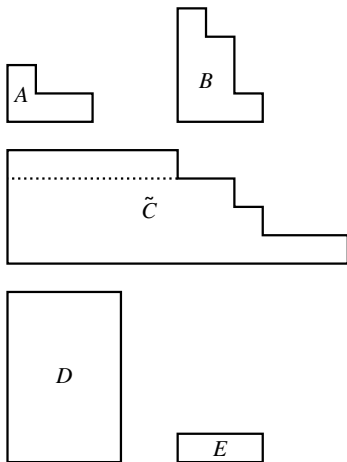
In other words, we have the following diagram:

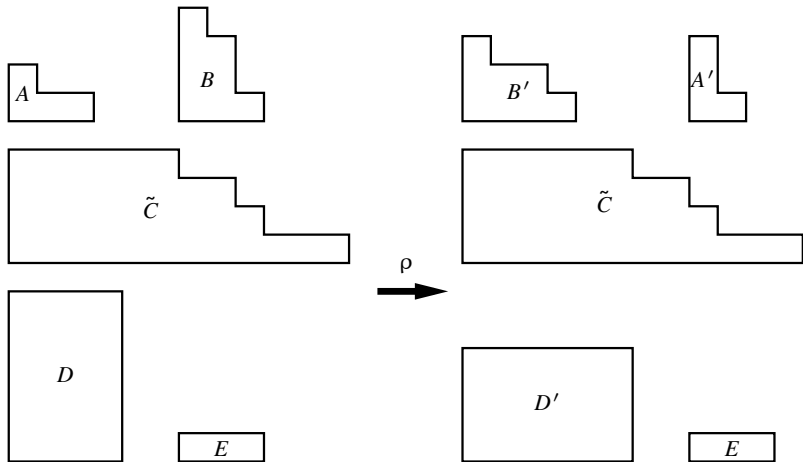
$$\begin{array}{ccc} \mathcal{F}_n(a, l, m) & \xrightarrow{\varphi} & \mathcal{F}_n(a, m, l) \\ \psi_{a,l,m} \downarrow & & \uparrow \psi_{a,m,l}^{-1} \\ \mathcal{Q}_n(a, l, m) & \xrightarrow{\rho} & \mathcal{Q}_n(a, m, l). \end{array}$$

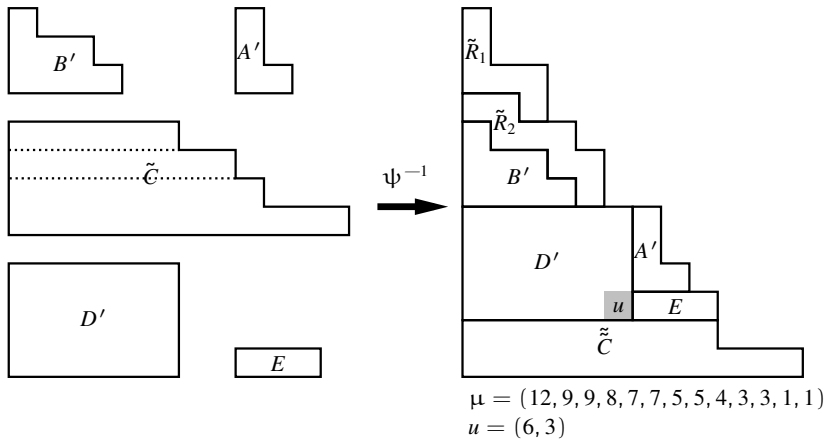


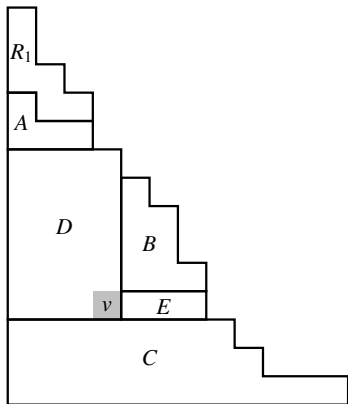
$$\lambda = (12, 9, 8, 7, 7, 6, 6, 5, 4, 3, 3, 2, 1, 1)$$

$$v = (4, 4)$$



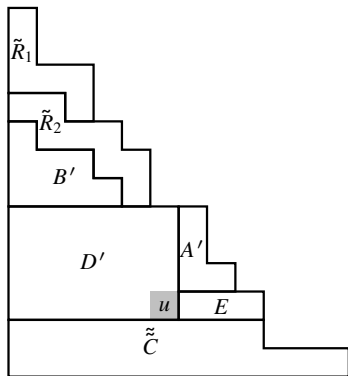






$$\lambda = (12, 9, 8, 7, 7, 6, 6, 5, 4, 3, 3, 2, 1, 1)$$

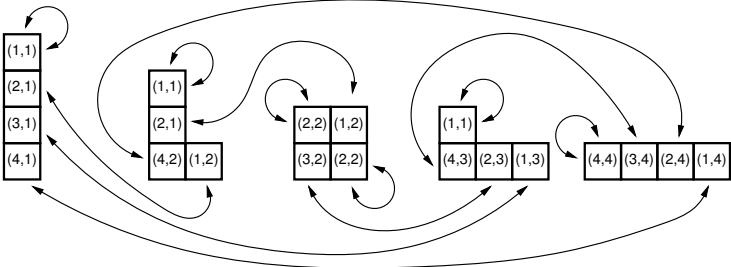
$$v = (4, 4)$$



$$\mu = (12, 9, 9, 8, 7, 7, 5, 5, 4, 3, 3, 1, 1)$$

$$u = (6, 3)$$

For example, the bijection φ on \mathcal{F}_4 is illustrated below:



We derive immediately the following result of Bessenrodt and Han [BH09, Theorem 3].

Corollary (Bessenrodt-Han, 2009)

The triple statistic (a_ν, l_ν, m_ν) has the same distribution as (a_ν, m_ν, l_ν) . In other words,

$$Q_n(x, y, z) = Q_n(x, z, y)$$

where

$$Q_n(x, y, z) = \sum_{(\lambda, \nu) \in \mathcal{F}_n} x^{a_\nu} y^{l_\nu} z^{m_\nu}.$$

is the generating function for (a_ν, l_ν, m_ν) .

For nonnegative integers m and n ,

- q -ascending factorial

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$$

- q -binomial coefficient

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_m (q; q)_{n-m}} \quad \text{for } 0 \leq m \leq n.$$

It is easy (see [And98, Chapter 3]) to see that

$$A(q) = \begin{bmatrix} m+a \\ a \end{bmatrix}_q,$$

$$B(q) = \begin{bmatrix} l+a \\ a \end{bmatrix}_q,$$

$$\tilde{C}(q) = \frac{1}{(q^{a+1}; q)_\infty},$$

$$D(q) = q^{(m+1)(l+1)},$$

$$E(q) = q^a.$$

Let $f_n(a, l, m)$ be the cardinality of $\mathcal{F}_n(a, l, m)$. We can apply the bijection φ to give a different proof of Bessenrodt and Han's formula [BH09, Theorem 2] for $\sum_{n \geq 0} f_n(a, l, m)q^n$.

Corollary (Bessenrodt-Han, 2009)

The generating function of $f_n(a, l, m)$ is given by the following formula:

$$\sum_{n \geq 0} f_n(a, l, m)q^n = \frac{1}{(q^{a+1}; q)_\infty} \begin{bmatrix} m+a \\ a \end{bmatrix}_q \begin{bmatrix} l+a \\ a \end{bmatrix}_q q^{(m+1)(l+1)+a}.$$

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A polynomial $P(x, y)$ in two variables x and y is **super-symmetric** if

$$[x^\alpha y^\beta]P(x, y) = [x^{\alpha'} y^{\beta'}]P(x, y)$$

when $\alpha + \beta = \alpha' + \beta'$.

Theorem (Bessenrodt, 1998; Bacher-Manivel, 2001; Bessenrodt-Han, 2009)

The generating function for the pointed partitions of \mathcal{F}_n by the two joint statistics arm length and coarm length (resp. leg length) is super-symmetric. In other words, the polynomial

$$\sum_{(\lambda, \nu) \in \mathcal{F}_n} x^{a_\nu} y^{m_\nu} \quad (\text{resp.} \quad \sum_{(\lambda, \nu) \in \mathcal{F}_n} x^{a_\nu} y^{l_\nu})$$

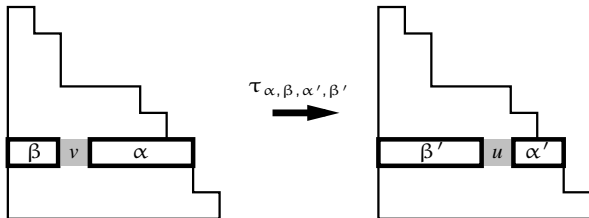
is super-symmetric.

Note that the above two polynomials are actually equal due to the corollary for the polynomial Q_n . [▶ Go](#)

- $\mathcal{F}_n(a, *, m)$ = the set of pointed partitions (λ, ν) of n such that $a_\nu = a$ and $m_\nu = m$.
- $\mathcal{F}_n(a, l, *)$ = the set of pointed partitions (λ, ν) of n such that $a_\nu = a$ and $l_\nu = l$.

$$\tau_{\alpha, \beta, \alpha', \beta'} : \mathcal{F}_n(\alpha, *, \beta) \rightarrow \mathcal{F}_n(\alpha', *, \beta')$$

It is easy to give a combinatorial proof of the super-symmetry of the first polynomial $\sum_{(\lambda, v) \in \mathcal{F}_n} x^{\alpha_v} y^{m_v}$.



$$\zeta_{\alpha,\beta,\alpha',\beta'} : \mathcal{F}_n(\alpha, \beta, *) \rightarrow \mathcal{F}_n(\alpha', \beta', *)$$

We can prove bijectively the super-symmetry of the polynomial $\sum_{(\lambda,\nu) \in \mathcal{F}_n} x^{a_\nu} y^{l_\nu}$. The bijection $\zeta_{\alpha,\beta,\alpha',\beta'}$ can be defined by

$$\begin{array}{ccc} \mathcal{F}_n(\alpha, \beta, *) & \xrightarrow{\zeta_{\alpha,\beta,\alpha',\beta'}} & \mathcal{F}_n(\alpha', \beta', *) \\ \varphi \downarrow & & \uparrow \varphi \\ \mathcal{F}_n(\alpha, *, \beta) & \xrightarrow{\tau_{\alpha,\beta,\alpha',\beta'}} & \mathcal{F}_n(\alpha', *, \beta'). \end{array}$$

Theorem (S.-Zeng, 2009)

If $\alpha + \beta = \alpha' + \beta'$, the mapping

$$\zeta_{\alpha, \beta, \alpha', \beta'} = \varphi \circ \tau_{\alpha, \beta, \alpha', \beta'} \circ \varphi$$





is a bijection from $\mathcal{F}_n(\alpha, \beta, *)$ to $\mathcal{F}_n(\alpha', \beta', *)$.

This theorem yields that the generating function of \mathcal{F}_n by the bivariate joint distribution of arm length and leg length is super-symmetric.

Summary

- 1 h_v and p_v are symmetric. \leftarrow the involution φ .
- 2 l_v and m_v are symmetric. \leftarrow the involution φ .
- 3 a_v and m_v are super-symmetric. \leftarrow the bijection $\tau_{\alpha, \beta, \alpha', \beta'}$.
- 4 a_v and l_v are super-symmetric. \leftarrow the bijection $\zeta_{\alpha, \beta, \alpha', \beta'}$.

References

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-  Christine Bessenrodt, *On hooks of Young diagrams*, Ann. Comb. **2** (1998), no. 2, 103–110.
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Thank you for your attention.

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