Lattice Green functions in higher dimensions

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Talk outline

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- Define Lattice Green functions
- Review known results in 2d, plus new result for triangular LGF coefficients
- Review 3d results including Watson integrals
- Connection with Ramanujan-type formulae for $1/\pi$.
- Calabi-Yau ODEs
- Extending results for LGF to higher dimensions.
- Constant term formulae
- Connection with logarithmic Mahler measures

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Lattice Green functions

- For a regular lattice, the probability that a walker starting at the origin will be at position $\vec{l}$ after n steps has PGF known as the Lattice Green Function (LGF).
- It is
$$P(\vec{l}; z) = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\exp(-i\vec{l}.\vec{k}) d^d \vec{k}}{1 - z\Lambda(\vec{k})}.$$
- $[z^n]P(\vec{l}; z)$ is the probability that a walker starting at the origin will be at $\vec{l}$ after n steps.
- $\Lambda(\vec{k})$ is the *structure function* of the walk, and is given by the discrete FT of the individual step probabilities.
- For example, for the d-dimensional hypercubic lattice, it is $\Lambda(\vec{k}) = \frac{1}{d}(\cos k_1 + \cos k_2 + \cdots \cos k_d)$.
- Of greatest interest are $P(\vec{0}; z)$ and $P(\vec{0}; 1)$.

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Summary of key points

- We consider LGFs on a range of lattices in dimension $d \geq 2$.
- In dimension $d = 2$, 3, and 4 the LGFs satisfy a d-dimensional Fuchsian ODE with regular singular points at the origin all having exponent zero. (MUM)
- In two dimensions the LGF is given by a complete elliptic integral of the first kind (or a ${}_2F_1$ hypergeometric function).
- In three dimensions the LGF is given by the square of a complete elliptic integral of the first kind.
- In four dimensions, the results have been obtained because the ODEs are in a special class, known as Calabi-Yau ODEs. A catalogue of 4d CY ODEs that have been solved has been given by Almkvist, Zudilin and van Straten.
- We think that they can be expressed as an integral of the square of a complete elliptic integral of the first kind. (So far only explicitly demonstrated for 4-d s.c and b.c.c. lattices).

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Summary of key points

- In all cases, it turns out to be both possible and useful to express the series coefficients as sums of products of binomial coefficients. The coefficients are just the number of *n*-step returns to the origin.
- In 5d all but the 5d f.c.c. lattice satisfies a fifth order Fuchsian ODE with regular singular points at the origin.
- We have some strange connections with number theory.
- The 2d LGF ODEs naturally satisfy an equivalent recurrence relation. For certain lattices these are the same as those found by Zagier as an Apery recurrence, which in this case gives Dirichlet functions.
- There are a bunch of old (Ramanujan), newer (Borwein and Borwein) and very new (Rogers) formulae for $1/\pi$. They give sums for $1/\pi$ that involve products of binomial coefficients. We find that these coefficients are precisely those of the 3d LGFs for a variety of lattices.

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Summary of key points

- We find that the d-dimensional hypercubic LGF and the (d+1) dimensional hyper-diamond lattice are the Abel transform of each other!
- For all lattices in all dimensions, we can express the LGF by a constant-term formulae. (This means, take a Laurent polynomial, then the coefficient of the LGF is given by the constant term of the Laurent polynomial raised to the n^{th} power).
- Laurent polynomials have measure called the (logarithmic) Mahler measure. The Laurent polynomials that arise in the study of LGFs have another life, and results have been found for them, giving rise to new hypergeometric identities, which give new results for 3d LGFs.

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One dimension. The linear chain

- $$P_{lc}(\vec{0}; z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dk}{1 - z\cos k} = \frac{1}{\sqrt{1-z^2}} = \sum_{k\geq 0} \binom{2k}{k} \left(\frac{z}{2}\right)^{2k}$$

Two dimensions

- The probability of return to the origin is $1 - 1/P(\vec{0};1)$.
- $$P(\vec{0};1) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{dk_1\, dk_2}{1-\Lambda(\vec{k})}$$
- Since $P(\vec{0};1)$ diverges for two-dimensional lattices, this leads to the well-known result that the probability of return in two dimensions is certain.

2d LGFs

- 2d LGFs are given by 2d integrals, and satisfy 2nd order linear Fuchsian ODEs.
- The expansion coefficients give # of returns to the origin.
- $$P(\vec{0}; z) = \sum_{n \geq 0} a_n \left(\frac{z}{q}\right)^n$$
- here q is the *co-ordination number* of the lattice. Thus $q = 3, 4, 6$ for the honeycomb, square, triangular lattices.

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Coefficients

- For the honeycomb lattice,

$$a_{2n} = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2j}{j}.$$

- For the square lattice,

$$a_{2n} = \binom{2n}{n}^2.$$

- For the triangular lattice,

$$a_n = \sum_{j=0}^{n} \binom{n}{j} (-3)^{k-j} b_j,$$

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Three dimensions

- In $d = 3$ for $z = 1$, $P(\vec{0}; 1)$ gives the famous *Watson integrals.*
- Encountered by van Peype, a student of Kramers, who solved the b.c.c. case, but Watson did the s.c and f.c.c cases.
- The structure functions are:

$$\Lambda(\vec{k}) = \frac{1}{3}(\cos k_1 + \cos k_2 + \cos k_3). \text{ sc}$$

$$\Lambda(\vec{k}) = (\cos k_1 \cos k_2 \cos k_3). \text{ bcc}$$

$$\Lambda(\vec{k}) = \frac{1}{3}(\cos k_1 \cos k_2 + \cos k_2 \cos k_3 + \cos k_1 \cos k_3). \text{ fcc.}$$

Watson integrals

- $$P(\vec{0};1) = \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{dk_1\, dk_2\, dk_3}{1 - \Lambda(\vec{k})}$$
- $$P(\vec{0};1)_{sc} = \frac{1}{32\pi^3}(\sqrt{3}-1)[\Gamma(1/24)\Gamma(11/24)]^2 \approx 1.516386$$
- $$P(\vec{0};1)_{bcc} = \frac{1}{2^{14/3}\pi^4}[\Gamma(1/4)]^4 \approx 1.3932039$$
- $$P(\vec{0};1)_{fcc} = \frac{9}{4\pi^3}[\Gamma(1/3)]^6 \approx 1.344661$$

Simple cubic lattice Green function

- $$P(\vec{0};z) = \frac{1}{(\pi)^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\frac{dk_1\,dk_2\,dk_3}{1-\frac{z}{3}(\cos k_1+\cos k_2+\cos k_3)}$$
- Joyce (1998) showed that this could be expressed as
$$P(\vec{0};z) = \frac{1-9\xi^4}{(1-\xi)^3(1+3\xi)}\left[\frac{2}{\pi}K(k_1)\right]^2;$$
- where
$$k_1^2 = \frac{16\xi^3}{(1-\xi)^3(1+3\xi)};$$
$$\xi = (1+\sqrt{1-z^2})^{-1/2}(1-\sqrt{1-z^2/9})^{1/2}$$

Body-centred cubic lattice Green function

- $$P(\vec{0}; z) = \frac{1}{(\pi)^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{dk_1\, dk_2\, dk_3}{1 - z(\cos k_1 \cos k_2 \cos k_3)}.$$
- Maradudin *et al.* (1960) showed that this could be expressed as
$$P(\vec{0}; z) = \left[\frac{2}{\pi} K(k_2)\right]^2$$
- where
$$k_2^2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - z^2}.$$

Face-centred cubic lattice Green function

- $$P(\vec{0}; z) = \frac{1}{(\pi)^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{dk_1\, dk_2\, dk_3}{1 - \frac{z}{3}(c_1 c_2 + c_1 c_3 + c_2 c_3)}$$

  where $c_i = \cos k_i$.
- Joyce (1998) showed that this could be expressed as

$$P(\vec{0}; z) = \frac{(1+3\xi^2)^2}{(1-\xi)^3(1+3\xi)} \left[\frac{2}{\pi} K(k_3)\right]^2;$$

- where

$$k_3^2 = \frac{16\xi^3}{(1-\xi)^3(1+3\xi)};$$

$$\xi = (1+\sqrt{1-z})^{-1}(-1+\sqrt{1+3z}).$$

- We have for the coefficients of the 3d lattices:
- For the diamond lattice,

$$a_{2n} = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2j}{j} \binom{2n-2j}{n-j}.$$

- For the simple cubic lattice,

$$a_{2n} = \binom{2n}{n} \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2j}{j}.$$

- For the body-centred cubic lattice,

$$a_{2n} = \binom{2n}{n}^3,$$

- while for the face-centred cubic lattice, (Bailey et al.)

$$a_n = \sum_{j=0}^{n} \binom{n}{j} (-4)^{n-j} b_j, \text{ where } b_j = a_{2j}(diam).$$

Connections with number theory

Recently, Rogers obtained new Ramanujan-type formulae for $1/\pi$ in terms of sums of products of binomial coefficients. I recognised these as coefficients of the LGF of the diamond and simple cubic lattices. We have, as corollaries, the following surprising formulae for $1/\pi$.

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n \frac{(3n+1)}{32^n} a_{2n}^{(d)}.$$

$$\frac{8\sqrt{3}}{3\pi} = \sum_{n=0}^{\infty} \frac{(5n+1)}{64^n} a_{2n}^{(d)}.$$

$$\frac{9+5\sqrt{3}}{\pi} = \sum_{n=0}^{\infty} (6n+3-\sqrt{3}) \left(\frac{3\sqrt{3}-5}{4}\right)^n a_{2n}^{(d)}.$$

$$\frac{2(64+29\sqrt{3})}{\pi} = \sum_{n=0}^{\infty} (520n+159-48\sqrt{3}) \left(\frac{80\sqrt{3}-139}{484}\right)^n a_{2n}^{(sc)}.$$

One of Ramanujan's 17 formulae for $1/\pi$ can also be expressed in terms of the number of returns on the b.c.c. lattice:

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(6n+1)}{256^n} a_{2n}^{(bcc)}.$$

There is also the remarkable formula due to the Borwein brothers which allows one to calculate any binary digit of $1/\pi$:

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \frac{(42n+5)}{4096^n} a_{2n}^{(bcc)}.$$

The general form of a Ramanujan type series for $1/\pi$ is:

$$\alpha f(z_0) + \beta\theta f(z_0) = \frac{1}{\pi}, \;\; \theta = z\frac{\mathrm{d}}{\mathrm{d}z}$$

where α, β and z_0 are algebraic numbers and $f(z)$ is the analytic solution around the origin of a particularly "nice" third order Fuchsian linear ODE. This ODE must be the (symmetric) square of a second order ODE, a situation first discussed by Appell more than a century ago.

The three dimensional LGFs for all four common 3-d lattices satisfy the Appell property, so in that sense it is not surprising that they are *candidates* for a Ramanujan type formula. What is more surprising is that they actually do occur in this way.

We point out that there exists another family of formulae for $1/\pi^2$, called the Ramanujan-Guillera formulae. We can also construct known formulae for $1/\pi^2$ from the LGF ODE coefficients.

Apéry limits

The celebrated proof by Apéry (1978) that $\zeta(3)$ is irrational was a consequence of the properties of the recursion

$$(n+2)^3 a_{n+2} - (2n+3)(17n^2+51n+39)a_n + (n+1)^3 a_n = 0.$$

Let $a_{-1} = 0, \; a_0 = 1$, and let the sequence $\{b_n\}$ satisfy the same recursion, but with $b_0 = 0, \; b_1 = 1$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\zeta(3)}{6}.$$

We refer to such limits as *Apéry type limits*.
Surprisingly, some of the ODEs given above for the 2d lattice Green functions correspond to recursions with interesting Apéry limits. For example, the honeycomb lattice ODE was treated by Zagier, who showed that the recursion gave the Apéry limit $L(2, \chi_3)/2$.

What are Calabi-Yau ODEs?

- A class of ODE that are pivotal in string theory. Here we consider only 4$^{\text{th}}$ order ODEs, (corresponding to the case of Calabi-Yau threefolds).
- Consider ODE's of the form

$$y^{(s)} + a_{s-1}(z)y^{(s-1)} + \cdots + a_1(z)y' + a_0(z)y(z) = 0,$$

where $\{a_i\}$ are meromorphic fns. of z. If $z = 0$ is a regular singular point, we can write

$$a_{s-j}(z) = z^{-j}\tilde{a}_{s-j}(z) \;\; j = 1, \ldots, s,$$

where $\tilde{a}_{s-j}(z)$ are analytic at $z = 0$.

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where $\{a_i\}$ are meromorphic fns. of z. If $z = 0$ is a regular singular point, we can write

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where $\tilde{a}_{s-j}(z)$ are analytic at $z = 0$.

Maximal unipotent monodromy MUM

- Then the roots of the indicial equation

$$\lambda(\lambda-1)\cdots(\lambda-s+1)+\tilde{a}_{s-1}(0)\lambda(\lambda-1)\cdots(\lambda-s+2)+$$

$$+\cdots+\tilde{a}_1(0)\lambda+\tilde{a}_0(0)=0$$

determine the exponents of the ODE at the origin.

- The DE is said to be of Maximal Unipotent Monodromy (MUM) if *all* the exponents at 0 are zero.
- Consider a 4^{th} order ODE which is MUM:

$$y^{(4)}+a_3(z)y^{(3)}+a_2(z)y''+a_1(z)y'+a_0(z)y=0.$$

It has four solutions, $y_0,\ y_1,\ y_2,\ y_3$.

- The C-Y condition is $a_1=\frac{1}{2}a_2a_3-\frac{1}{8}a_3^3+a_2'-\frac{3}{4}a_3a_3'-\frac{1}{2}a_3''$.

Maximal unipotent monodromy MUM-II

- Being MUM means that:

$$y_0 = 1 + \sum_{n\geq 1} a_n z^n;$$

$$y_1 = y_0 \log z + \sum_{n\geq 1} b_n z^n;$$

$$y_2 = \frac{1}{2} y_0 \log^2 z + \left(\sum_{n\geq 1} b_n z^n\right) \log z + \sum_{n\geq 1} c_n z^n;$$

$$y_3 = \frac{1}{6} y_0 \log^3 z + \frac{1}{2}\left(\sum_{n\geq 1} b_n z^n\right) \log^2 z + \left(\sum_{n\geq 1} c_n z^n\right) \log z$$

$$+ \sum_{n\geq 1} d_n z^n;$$

Yukawa coupling and Instanton numbers

- Define $q = \exp(y_1/y_0) = \sum_{n \geq 1} t^n z^n$, (the inverse function $z = z(q) = \sum u_n q^n$ is the *mirror map* in C-Y language).
- Then the Yukawa coupling $K(q)$ is given by

$$K(q) = \left(q \frac{d}{dq}\right)^2 \left(\frac{y_2}{y_0}\right) = 1 + \sum_{k=1}^{\infty} \frac{n_k q^k}{1 - q^k}.$$

- n_k are called *instanton numbers,* and two C-Y equations are considered equivalent if they have the same instanton numbers.
- Usually (exception later), $N_k = N_0 n_k / k^3$ are integers, where N_0 is a small integer—usually 1, 2 or 3.
- We don't know the combinatorial significance of the coefficients of $K(q)$ or of the modified instanton numbers N_k.

Almkvist and Zudilin list

- Almkvist and Zudilin have catalogued a large number of 4$^{\text{th}}$ order C.Y. ODEs by ordering them in terms of increasing *degree*, k.
- We use the operator $\theta = z\frac{d}{dz}$, and write the 4$^{\text{th}}$ order ODE as $\mathcal{D}f(z) = 0$.
- Here $\mathcal{D} = \theta^4 + zP_1(\theta) + z^2P_2(\theta)\ldots + z^kP_k(\theta)$ where P_l, $l = 1\ldots k$ are polynomials of degree 4 in θ.
- Thus ODEs of 1st degree take the form $[\theta^4 + zP_1(\theta)]f(z) = 0$.
- Almkvist et al found exactly 14 such ODEs. All can be solved. Their solutions have coefficients expressed as finite sums of products of binomial coefficients.

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Hyper body-centred cubic lattice

- $$P(\vec{0}; z) = \frac{1}{(\pi)^4} \int_0^{\pi} \cdots \int_0^{\pi} \frac{dk_1\, dk_2\, dk_3\, dk_4}{1 - z(\cos k_1 \cos k_2 \cos k_3 \cos k_4)}.$$

- $$P(\vec{0}; z) = \frac{1}{\pi^4} \sum_{n=0}^{\infty} z^n \left(\int_0^{\pi} \cos^n k dk \right)^4$$

- $$= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_n (\frac{1}{2})_n (\frac{1}{2})_n}{(1)_n (1)_n (1)_n n!} z^{2n}$$

- $$= {}_4F_3(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1, 1; z^2) = \sum_{n=0}^{\infty} \binom{2n}{n}^4 \left(\frac{z}{16}\right)^{2n}$$

Hyper body-centred cubic lattice-cont.

- This admits to no further simplification, or special values at $z = 1$, though the series is rapidly convergent, so we can evaluate the integral at $z = 1$, giving $P(\vec{0}; 1) \approx 1.118636$.
- It is the C-Y ODE #3 on the list of Almkvist *et al.*
- With $\theta = z\frac{d}{dz}$, the LGF satisfies $\mathcal{D}P(\vec{0}; 16z) = 0$, where

$$\mathcal{D} = \theta^4 - 256z(\theta + \frac{1}{2})^4$$

- Because of the simple structure, we can also consider higher dimensions. For example, with $d = 5$,

$$P(\vec{0}; z) = {}_5F_4(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1, 1, 1; z^2)$$

$$= \sum_{n=0}^{\infty} \binom{2n}{n}^5 \left(\frac{z}{32}\right)^{2n}$$

which satisfies a 5^{th} order Fuchsian ODE.

Hyper cubic lattice: Glasser and Guttmann 1994

- $$P(\vec{0};z) = \frac{1}{(\pi)^4}\int_0^{\pi}\cdots\int_0^{\pi}\frac{dk_1\,dk_2\,dk_3\,dk_4}{1-\frac{z}{4}(c_1+c_2+c_3+c_4)},$$

  where $c_i = \cos k_i$.
- Use the identity $\frac{1}{\lambda} = \int_0^{\infty}\exp(-\lambda t)dt$,
- $$\begin{aligned} P(\vec{0};z) &= \frac{1}{(\pi)^4}\int_0^{\infty}\int_0^{\pi}\cdots\int_0^{\pi} e^{-t}\prod_{j=1}^{4} e^{(ztc_j/4)}dtd\vec{k} \\ &= \int_0^{\infty} e^{-t} I_0^4\left(\frac{zt}{4}\right) = \sum_{n=0}^{\infty} a_n z^n, \end{aligned}$$

  since $I_0(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}d\theta.$
- $$a_n = \left(\frac{(\frac{1}{2})_n}{n!}\right)^3 {}_4F_3(-\frac{n}{2}, 1-\frac{n}{2}, -n, \frac{1}{2}; \frac{1}{2}-n, \frac{1}{2}-n, 1; 1)$$

Hyper cubic lattice

- With $\theta = z\frac{d}{dz}$, the LGF satisfies $\mathcal{D}P(\vec{0};z) = 0$, where

$$\begin{aligned}\mathcal{D} &= \theta^4 - 4z(2\theta+1)^2(5\theta^2+5\theta+2) + \\ &+ 2^8 z^2(\theta+1)^2(2\theta+1)(2\theta+3)\end{aligned}$$

- This is equation 16 on the list of Almkvist et al., who give

$$\begin{aligned}a_n &= \binom{2n}{n}\sum_{j+k+l+m=n}\left(\frac{n!}{j!k!l!m!}\right)^2 \\ &= \binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}.\end{aligned}$$

- Almkvist (2007) gives a further 15 distinct expressions for a_n, all involving single or double sums of products of binomial coefficients.

d-dimensional generalisations

- From the work of Glasser and Montaldi and Guttmann and Prellberg, we can write for the d-dimensional hyper-cubic LGF

$$[(2dz)^{2n}]P_d(\vec{0};z) = \binom{2n}{n} \sum_{k_1+k_2+\ldots k_d=n} \left(\frac{n!}{k_1!k_2!\ldots k_d!} \right)^2$$

- The 5 dimensional LGFs satisfies a 5th degree ODE, which can be "pulled back" to a degree 4 ODE, that is also C-Y. This is also the case for the 5d bcc. All the ODEs are in A & Z's list.

Hyper face-centred cubic lattice

- $$P(\vec{0};z) = \frac{1}{(\pi)^4}\int_0^\pi \cdots \int_0^\pi \frac{dk_1\,dk_2\,dk_3\,dk_4}{1-\frac{z}{6}\lambda},$$
  where $\lambda = (c_1c_2 + c_1c_3 + c_1c_4 + c_2c_3 + c_2c_4 + c_3c_4)$
- Set $a = 1 - \frac{z}{6}(c_2c_3 + c_2c_4 + c_3c_4);\quad b = \frac{z}{6}(c_2 + c_3 + c_4)$.
- Then the integrand is $[a - b\cos k_1]^{-1}$. Use
- $$\frac{1}{\pi}\int_0^\pi \frac{d\theta}{a - b\cos\theta} = \frac{1}{\sqrt{a^2-b^2}} = \frac{1}{\sqrt{(a+b)(a-b)}}$$
  to eliminate k_1.
- Next write $(a+b)(a-b) = e(c-\cos k_2)(d-\cos k_2)$, where c, d, e are independent of k_2, and use
  $$\int_0^\pi \frac{d\theta}{\sqrt{(c-\cos\theta)(d-\cos\theta)}} = \frac{2K(k)}{\sqrt{(c-1)(d+1)}}$$
  to eliminate k_2, where $k^2 = \frac{2(c-d)}{(c-1)(d+1)}$.

Hyper face-centred cubic lattice

- We are left with a two-dimensional integral, which was expanded as a power series in z and integrated term-by-term in Maple. We got to 40 terms in a few hours, then searched for an ODE.
- With $\theta = z\frac{d}{dz}$, the LGF satisfies $\mathcal{D}P(\vec{0};z) = 0$, where

$$
\begin{aligned}
\mathcal{D} \;=\;& \theta^4 + z(39\theta^4 - 30\theta^3 - 19\theta^2 - 4\theta) \\
+\;& 2z^2(16\theta^4 - 1070\theta^3 - 1057\theta^2 - 676\theta - 192) \\
-\;& 36z^3(171\theta^3 + 566\theta^2 + 600\theta + 316)(3\theta + 2) \\
-\;& 2^5 3^3 z^4(+384\theta^4 + 1542\theta^3 + 2635\theta^2 + 2173\theta + 702) \\
-\;& 2^6 3^3 z^5(1393\theta^3 + 5571\theta^2 + 8378\theta + 4584)(1 + \theta) \\
-\;& 2^{10} 3^5 z^6(31\theta^2 + 105\theta + 98)(1 + \theta)(\theta + 2) \\
-\;& 2^{12} 3^7 z^7(\theta + 1)(\theta + 2)^2(\theta + 3).
\end{aligned}
$$

Hyper face-centred cubic lattice

- This is a 4th order, degree 7 Calabi-Yau ODE with regular singular points at 0, 1/24, -1/3, -1/4, -1/8, -1/12, and ∞.
- It is new, and one of only 3 known C-Y ODEs of degree 7 (Almkvist, private communication).
- We do not yet have a nice expression for the series coefficients in terms of binomial coefficients. Only

$$
\begin{aligned}
a_n \;=\; & \sum_{i+j+k+l+m=n} \binom{2i}{i}\binom{2j}{j}\binom{2k}{k}\binom{l+m}{m}\binom{2(n-i-j-k)}{n-i-j-k} \\
& \binom{n}{2(n-i-j-k)}\binom{2(n-i-j-k)}{l+m}\binom{2(i+j+k)-n}{n-2i-l-m} \\
& \binom{4i+2j+2k+l+m-2n}{2i+j+m-n}
\end{aligned}
$$

The mirror map for the differential equation gives Yukawa coupling $K(q)$ whose instanton numbers, N_k, are $3; -4; 64; -253; 4292; -25608; 442008; -3202512; 56565002; \ldots$

Hyper diamond lattice

The structure function is:

- $\Lambda = 2\cos(k_2 - k_3) + 2\cos(k_2 - k_4) + 2\cos(k_3 - k_4) + 4\cos k_1(\cos k_1 + \cos k_2 + \cos k_3 + \cos k_4) + 3.$
- Then I (and D Broadhurst) find that $P(\vec{0}; z)$ is given by the square of the 5-d multinomial coeffs:

$$\sum a_{2n} z^n = \sum_{i+j+k+l+m=n} \left(\frac{n!}{i!j!k!l!m!} \right)^2 (z/5)^n \ (n \ even).$$

- ODE given by G & P in 1993 paper, and is simply related to the gen. fn. for 5d staircase polygons! It is $\theta^4 - z(35\theta^4 + 70\theta^3 + 63\theta^2 + 28\theta + 5) + z^2(\theta + 1)^2(259\theta^2 + 518\theta + 285) - 225z^3(\theta + 1)^2(\theta + 2)^2,$ and number 34 on the Almkvist et al. list.
- Higher dimensional generalisations are obvious. They have been studied by Broadhurst who finds that $n_k/k^{\mathbf{2}}$ are integers (rather than $N_d n_k/k^3$). Tested for $D < 10$, $k < 100$.

Other LGFs

An “obvious" structure function in 4d is

$$\Lambda = c_1 c_2 c_3 + c_1 c_2 c_4 + c_2 c_3 c_4 + c_1 c_3 c_4$$

It is satisfied by an 8^{th} order ODE of degree 16, which is not MUM (D. Broadhurst)
D. Broadhurst has also calculated the LGF for the 5d hyper-FCC. The structure function is

$$\Lambda = c_1(c_2 + c_3 + c_4 + c_5) + c_2(c_3 + c_4 + c_5) + c_3(c_4 + c_5) + c_4 c_5$$

Then in an heroic calculation, Broadhurst showed that the underlying ODE is of order 6 and degree 13, and is not MUM.

Connection between the hyper-diamond and hyper-cubic lattices

- The d-dim. hyper-diam. LGF has coeffs given by the sum of the squares of the $(d+1)$-dim. multinomial coefficients.
- The d-dim. hyper-cub. lattice has LGF coeffs given by $\binom{2n}{n}$ times the sum of the squares of the d-dim. multinomial coeffs.
- Thus there appears to be a simple relationship between the LGFs of the two lattices.
- As it happens, this has been formalised in another context by Guttmann and Prellberg and by Glasser and Montaldi.
- G & M: The g.f. for the d-dim. multinomial coeffs is,

$$Z_d(x^2) = \sum_{k_1,\ldots,k_d=0}^{\infty} \binom{k_1+\ldots+k_d}{k_1,\ldots,k_d}^2 x^{2(k_1+\ldots+k_d)}.$$

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$$P_d(z) = \frac{1}{\pi^d} \int_0^\infty \cdots \int_0^\infty \frac{d\theta_1 \cdots d\theta_d}{1 - \frac{z}{d}[\cos(\theta_1) + \cdots + \cos(\theta_d)]}.$$

- These are related through an Abel transform.

$$P_d(z) = \frac{2}{\pi} \int_0^1 \frac{Z_d(t^2 z^2/d^2)}{\sqrt{1-t^2}} dt$$

and the inverse

$$Z_d(x^2) = \frac{d}{dx} \left(x \int_0^1 \frac{t P_d(dxt)}{\sqrt{1-t^2}} dt \right).$$

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Constant term (CT) identities

- An alternative way to view the expansion for the no. of returns of a r.w. after n steps, is to ask for a CT formulation.
- The CT enumerates the no. of distinct n-step returns by generating all possible sums of lattice basis vectors that sum to zero. It is also obvious that the value is an integer – something less obvious from the integral formulation.
- The idea is to write down a function which when raised to the n^{th} power has a CT equal to the no of n-step returns.
- For the d-dim. h.c. and h.b.c.c. lattices, only even n contributes, (returns must have an even no. of steps).
- For the triang. and f.c.c lattices, all values of n are used.
- For bipartite lattices, such as the honey. and diam. lattices, all values of n are used, but the result is the number of $2n$ step returns.

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- The idea is to write down a function which when raised to the n^{th} power has a CT equal to the no of n-step returns.
- For the d-dim. h.c. and h.b.c.c. lattices, only even n contributes, (returns must have an even no. of steps).
- For the triang. and f.c.c lattices, all values of n are used.
- For bipartite lattices, such as the honey. and diam. lattices, all values of n are used, but the result is the number of $2n$ step returns.

Constant term (CT) identities

- For example, for the square lattice the required expression is
$$f(x,y) = (x + \frac{1}{x})(y + \frac{1}{y}).$$
Then $\mathrm{CT} f(x,y)^{2n} = \binom{2n}{n}^2$.
- Alternatively, if
$$g(x,y) = (x + \frac{1}{x} + y + \frac{1}{y}),$$
then $\mathrm{CT} g(x,y)^{2n} = \binom{2n}{n}^2$.
- For the triangular lattice the relevant function is
$$h(x,y) = (x + \frac{1}{x} + y + \frac{1}{y} + \frac{y}{x} + \frac{x}{y}).$$
- For the honeycomb lattice one has
$$u(x,y) = (1 + x + y)(1 + 1/x + 1/y)$$

CT formulations and Mahler measure

- For the diamond, s.c, b.c.c, f.c.c. lattices:

$$(1/x + x + z(y + 1/y))(x + 1/x + (y + 1/y)/z) \quad \text{diamond}$$

$$(x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z}) \quad \text{s.c.}$$

$$(x + \frac{1}{x})(y + \frac{1}{y})(z + \frac{1}{z}) \quad \text{b.c.c.}$$

$$(x + \frac{1}{x})(y + \frac{1}{y}) + (x + \frac{1}{x})(z + \frac{1}{z}) + (z + \frac{1}{z})(y + \frac{1}{y}) \quad \text{f.c..c.}$$

For the four dimensional diamond, s.c, b.c.c, f.c.c. lattices:

$$(1/x + x + zy + z/y + w/x)(x + 1/x + y/z + 1/yz + x/w) \quad \text{4d d}$$

$$(x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + w + \frac{1}{w}) \quad \text{4d s.c.}$$

$$(x + \frac{1}{x})(y + \frac{1}{y})(z + \frac{1}{z})(w + \frac{1}{w}) \quad \text{4d b.c..c.}$$

These then give CT formulae for the C-Y ODEs 34, 16, 3

CT formulations and Mahler measure

There is also a connection between the CT formulation of LGFs and the Mahler measure of a Laurent polynomial F.
For the L.p $F(x, y, z)$ the logarithmic Mahler measure $\mathbf{m}(F)$,

$$\mathbf{m}(F) = \frac{1}{(2\pi i)^3} \oint \oint \oint_{|x|=|y|=|z|=1} \log|F(x, y, z)| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} \frac{\mathrm{d}z}{z}$$

- Boyd considers the lMm of the Laurent polynomial

$$(x + \frac{1}{x} + y + \frac{1}{y} + 1)$$

which apart from 1, is the CT kernel for the sq. lattice LGF.

CT formulations and Mahler measure

- Boyd showed that

$$\mathbf{m}(x+\frac{1}{x}+y+\frac{1}{y}+1)=\frac{15}{(2\pi)^2}L(E,2)=L'(E,0),$$

  where $L(E,s)$ is the L-function of the elliptic curve E of conductor 15 that is the proj. closure of $(x+\frac{1}{x}+y+\frac{1}{y}+1)$.
- Rogers considers two three-variable Laurent polynomials, and studies

$$h(u):=\mathbf{m}(x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}+u),$$

  and

$$g(u):=\mathbf{m}\left(4-u+(y+\frac{1}{y})(z+\frac{1}{z})+(x+\frac{1}{x})(y+\frac{1}{y})+(x+\frac{1}{x})(\right.$$

  which are clearly related to the CT kernel of the simple cubic and face-centred cubic lattices respectively.

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CT formulations and Mahler measure

- By considering these and other L.ps, and their lMm, Rogers found a number of new identities for the hypergeometric function ${}_5F_4$, and new formulae for $1/\pi$, some of which we have mentioned above.
- Other results of Rogers are relevant here. E.g: with a trivial change of variable, Thm 3.1 of Rogers can be written

$$P(\vec{0}, z)_{diamond} = \frac{1}{(1 - z^2/4)} {}_3F_2\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}; 1, 1; \frac{27z^4}{64(1 - z^2/4)^3}\right).$$

- Given the known relationship between the diamond and f.c.c. LGFs, we can also write

$$P(\vec{0}, z)_{fcc} = {}_3F_2\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}; 1, 1; \frac{z^2(3+z)}{4}\right). \qquad (1)$$

Alternative expressions were earlier given by Joyce in terms of different ${}_3F_2$ h.g.f.

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Conclusion

- More detail can be found in *Lattice Green's functions in all dimensions* J. Phys. A: Math. Theor. 43 (2010) 305205.
- We have found explicit LGFs for the common lattices for all dimensionality for all but the f.c.c. lattice, for which we have the solutions for $d \leq 5$.
- We have given constant term formulae for LGFs for all the lattices in all dimensions.
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- We have made some curious connections with number-theoretical concepts, and with Mahler measures.
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