Loop operators and convolutions in the symmetric function Hopf algebra

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joint work with: Peter D. Jarvis & Ronald C. King

Plan of the talk

In Sym$[X]$ and $\mathcal{H}_{\sqcup}[X]$ (commutative Hopf algebras)

- Notational preliminaries (we use informally graphical calculus)
- What are a loop operator?
  - [inner|outer] loop operators
  - alternative forms and inverse loop operators
  - some relations between loop operators
- Hirota-Miwa change of variables (used for vertex operators)
- Forced Laplace pairings|expansions
  - the associative case
  - the 'inverse' case
  - 'undeformations' i.e. generalized straightenings (of Rota-Stein plethystic type Hopf algebras)

Sym$[X]$ definitions and graphical calculus

Sym$[X]$ Hopf algebra (self dual w.r.t. $(- \mid -)$)

Bases: $\{e_\mu\}_\mu$, $\{h_\mu\}_\mu$, $\{p_\mu\}_\mu$ multiplicative, $\{m_\mu\}_\mu$, $\{s_\mu\}_\mu$
General elements: $A, B, \ldots$; $\lambda, \mu, \nu, \ldots$ integer partitions
[outer|inner|products|coproducts]:

$$m : \mathrm{Sym} \otimes \mathrm{Sym} \to \mathrm{Sym} :: m(s_\mu \otimes s_\nu) = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda$$

$$\Delta : \mathrm{Sym} \to \mathrm{Sym} \otimes \mathrm{Sym} :: \Delta(s_\lambda) = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu \otimes s_\nu = s_{\lambda_{(1)}} \otimes s_{\lambda_{(2)}}$$

$c^\lambda_{\mu,\nu}$ Littlewood-Richardson coefficients

$$\star : \mathrm{Sym} \otimes \mathrm{Sym} \to \mathrm{Sym} :: \star(s_\mu \otimes s_\nu) = \sum_\lambda g^\lambda_{\mu,\nu} s_\lambda$$

$$\delta : \mathrm{Sym} \to \mathrm{Sym} \otimes \mathrm{Sym} :: \delta(s_\lambda) = \sum_{\mu,\nu} g^\lambda_{\mu,\nu} s_\mu \otimes s_\nu = s_{\lambda_{[1]}} \otimes s_{\lambda_{[2]}}$$

$g^\lambda_{\mu,\nu}$ Kronecker coefficients

Sym$[X]$ definitions and graphical calculus, cont.

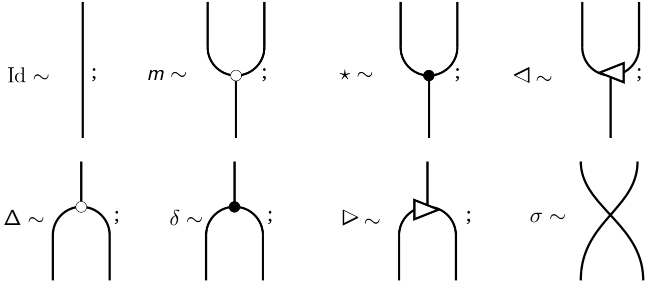
(co/)Plethysm: ($p^{\lambda}_{\mu,\nu}$ plethysm coeff. non-neg. integers)

$$\lhd : \mathsf{Sym} \otimes \mathsf{Sym} \to \mathsf{Sym} :: \lhd(s_\mu \otimes s_\nu) = (s_\mu[s_\nu]) = \sum_\lambda p^{\lambda}_{\mu,\nu} s_\lambda$$

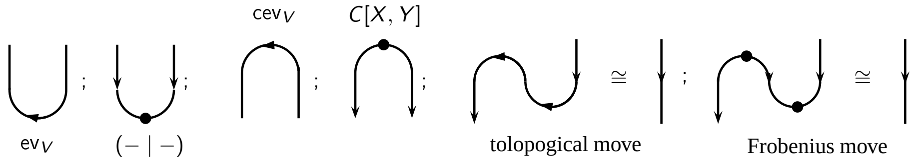
$$\left(\rhd : \mathsf{Sym}^+ \to \mathsf{Sym}^+ \otimes \mathsf{Sym}^+ :: \rhd(s_\lambda) = \sum_{\mu,\nu} p^{\lambda}_{\mu,\nu} s_\mu \otimes s_\nu \right)$$

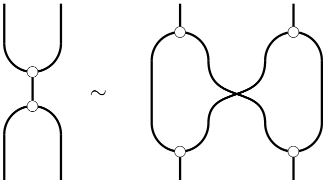
$\mathsf{Sym}^+ = \ker \epsilon^0$ augmentation ideal : $\mathsf{Sym} = \mathbb{Z} \cdot 1 + \mathsf{Sym}^+$

Graphical notation: (downward/pessimistic and left-handed oriented)

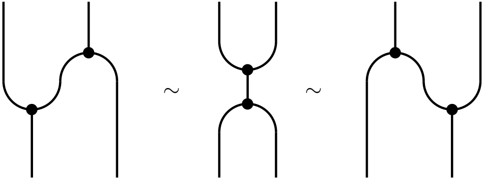


Graphical manipulations "moves"





Hopf algebra



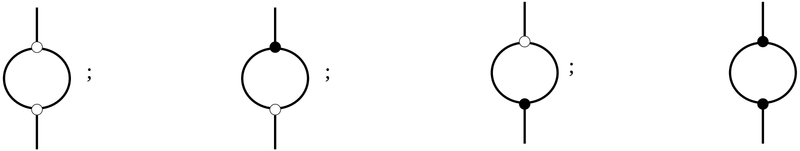
Frobenius algebra

Loop operators (convolution)

Def: loop operators:

$$
\begin{aligned}
\bigcirc_{oo}(A) &:= (m \circ \Delta)(A) = m \circ (\mathrm{Id} \otimes \mathrm{Id}) \circ \Delta(A) \\
\bigcirc_{io}(A) &:= (m \circ \delta)(A) = m \circ (\mathrm{Id} \otimes \mathrm{Id}) \circ \delta(A) \\
\bigcirc_{oi}(A) &:= (\star \circ \Delta)(A) = \star \circ (\mathrm{Id} \otimes \mathrm{Id}) \circ \Delta(A) \\
\bigcirc_{ii}(A) &:= (\star \circ \delta)(A) = \star \circ (\mathrm{Id} \otimes \mathrm{Id}) \circ \delta(A)
\end{aligned}
$$

Graphical notation: (these are convolutions of identity operators)



$\bigcirc_{oo}$ $\bigcirc_{io}$ $\bigcirc_{io}$ $\bigcirc_{ii}$

Alternative forms and inverses of loop operators:

'outer-outer' loop operator:

t-Specialization: $\quad (t \in \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \ldots)$

Def: $\epsilon^t(s_\mu[X]) := s_\mu[1^t] =: \dim_{s_\mu}(t) \quad (\epsilon^0 \text{ counit of outer coproduct})$

(Note: $\epsilon^{-t}(s_\mu[X]) = \epsilon^t(s_\mu[-X]) = \epsilon^t(\mathrm{S}(s_\mu)[X])$: $\mathrm{S}(A)$ antipode in Sym)

Thm: (alternative forms for outer-outer loop operator)

$$\begin{aligned}
[2](A)[X] &:= \bigcirc_{oo}(A)[X] = (m \circ \Delta)(A)[X] = (A_{(1)}A_{(2)})[X] \\
&= A[X+X] = A[2\cdot X] = A[2\cdot s_{(1)}][X] \\
&= \dim_{A_{[1]}}(2)A_{[2]}
\end{aligned}$$

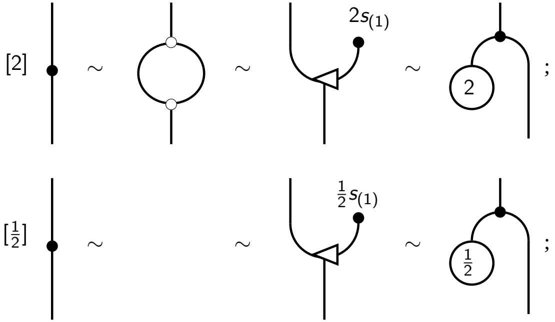
(on power sums: $[2](p_\mu) = 2^{\ell(\mu)}p_\mu$; relates to zonal sym. functions)

Plethysm allows to introduce the inverse outer-outer loop operator

$$[\tfrac{1}{2}](A) := \dim_{A_{[1]}}(\tfrac{1}{2})A_{[2]} \qquad \Leftrightarrow \qquad [\tfrac{1}{2}][2](A) = A = [2][\tfrac{1}{2}](A)$$

Graphical forms and inverses of loop operators:

in a suitable ring extension $\mathbb{Q}$, $\mathbb{Q}[q]$ etc.



Alternative forms and inverses of loop operators:

'inner-inner' loop operator:

For power sums: $(\mu = (\mu_1, \mu_2, \ldots) = [1^{r_1} 2^{r_2} \ldots]$; $z_\mu = \prod_i i^{r_i} r_i!)$

$$\delta(p_\mu) = p_\mu \otimes p_\mu \text{ and } \star(p_\mu \otimes p_\nu) = z_\mu \delta_{\mu,\nu} p_\mu$$

Def: For power sums inner-inner loop operator and its inverse are:

$$\left.\begin{array}{l} \widehat{\mathsf{z}} : \mathrm{Sym} \to \mathrm{Sym} :: p_\mu \mapsto z_\mu p_\mu \\ \widehat{\overline{\mathsf{z}}} : \mathrm{Sym} \to \mathrm{Sym} :: p_\mu \mapsto \frac{1}{z_\mu} p_\mu \end{array}\right\} \quad \text{scaling by } z_\mu;\ z_\mu^{-1}$$

Def: zee-specialization

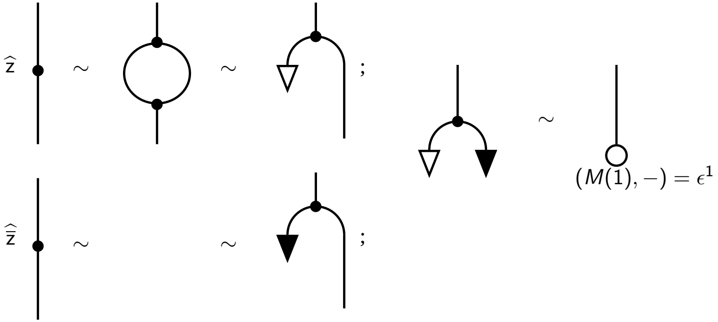
$$\mathfrak{z}, \overline{\mathfrak{z}} : \mathrm{Sym} \to \mathbb{Q} :: \begin{cases} \mathfrak{z} : p_\mu \mapsto z_\mu \ ; & \mathfrak{z} : s_\mu \mapsto \sum_\tau \chi^\mu(\tau) \\ \overline{\mathfrak{z}} : p_\mu \mapsto \frac{1}{z_\mu} \ ; & \overline{\mathfrak{z}} : s_\mu \mapsto \sum_\tau \frac{\chi^\mu(\tau)}{z_\tau^2} \end{cases}$$

Prop: $\mathfrak{z}$ and $\overline{\mathfrak{z}}$ are convolutive inverse 1-chains w.r.t. the inner convolution ($\widehat{\mathrm{Sym}} \ni M(z) = \sum_n h_n z^z$, $\epsilon^1 : \mathrm{Sym} \to \mathbb{Q}$):

$$\cdot_{\mathbb{Q}} \circ (\mathfrak{z} \otimes \overline{\mathfrak{z}}) \circ \delta(A) = \epsilon^1(A) \qquad (= (M(1), A) =: \epsilon^1(A))$$

Graphical forms and inverses of loop operators:

in a suitable ring extension $\mathbb{Q}$, $\mathbb{Q}[q]$ etc.



Thm: The inner-inner loop operator $\bigcirc_{ii}$ and its inverse have the following alternative forms:

$$\widehat{\mathsf{z}}(A) = \bigcirc_{ii}(A) = (\star \circ \delta)(A) = \mathfrak{z}(A_{[1]})A_{[2]}$$
$$\widehat{\bar{\mathsf{z}}}(A) = \bar{\mathfrak{z}}(A_{[1]})A_{[2]}$$

p-multiplicativity & Hirota-Miwa change of var.

Def: μ is relatively prime to ν ($\mu \not| \nu$) iff $\gcd(\mu, \nu) = (0)$

Prop: $\mathfrak{z}$ (and $\bar{\mathfrak{z}}$) is p-multiplicative and not a homomorphism

(hom = complete multiplicative in the sense of number theory)

$$\mathfrak{z}(p_\mu p_\nu) = \mathfrak{z}(p_\mu)\mathfrak{z}(p_\nu)\binom{\mu \cup \nu}{\mu, \nu} \quad ; \quad \mathfrak{w}(p_\mu, p_\nu) := \binom{\mu \cup \nu}{\mu, \nu} = 1 \text{ iff } \mu \not| \nu$$

(That is: $\mathfrak{z} : \mathrm{Sym} \to \mathbb{Q}$ is not a 1-cocycle, $\mathfrak{w} = \partial\mathfrak{z}$ is a non-trivial 2-cocycle)

Hirota-Miwa change of variables [Miw82]

Let $\{\mathfrak{p}^*_\mu\}_\mu$ basis of $\mathrm{Hom}({}_{\mathbb{Q}}\mathrm{Sym}, \mathbb{Q})$ (gr. dual), s.t. $\mathfrak{p}^*_\mu(p_\nu) = \delta_{\mu,\nu}$

Def: $\gamma : \mathrm{Sym} \to \mathrm{Sym} :: p_n \mapsto \frac{1}{n} p_n$ (our $p_\mu \mapsto \bar{\mathfrak{z}}(p_\mu)p_\mu = (\bar{\mathfrak{z}} \otimes \mathrm{Id}) \circ \delta(p_\mu)$)

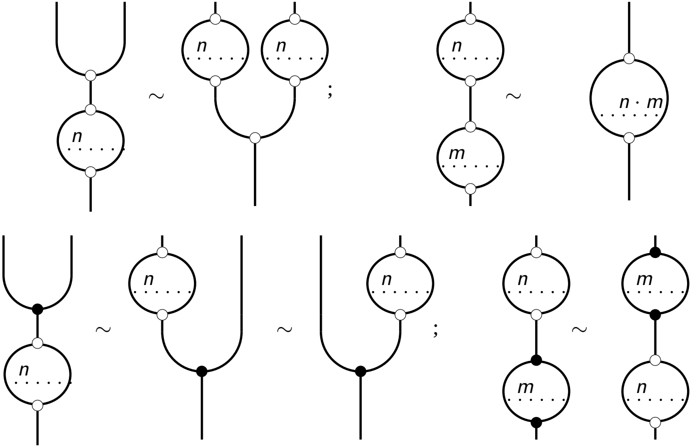
We get two identifications ${}_{\mathbb{Q}}\mathrm{Sym} \to {}_{\mathbb{Q}}\mathrm{Sym}^*$ related by $\widehat{\overline{\mathrm{z}}}, \widehat{\mathrm{z}}$:

$$(A \mid B)_{\widehat{\overline{\mathrm{z}}}} := \epsilon^1 \circ \star \circ (\widehat{\overline{\mathrm{z}}} \otimes 1)(A \otimes B) = (\widehat{\overline{\mathrm{z}}}(A) \mid B) = \mathrm{ev}(A^* \otimes B)$$

$$\begin{tikzcd} \mathrm{Sym}[X] \arrow[rr, bend left, "(-)^*"] \arrow[rr, bend right, "(A|-)"'] \arrow[rr, phantom, "\widehat{\overline{\mathrm{z}}} \Uparrow \; \Downarrow \widehat{\mathrm{z}}"] & & \mathrm{Sym}[X]^* \end{tikzcd}$$

Some relations between loop operators. . .

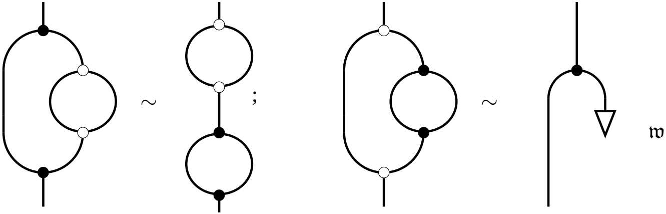
Thm: The n-fold outer-outer loop operator $[n] = m^{n-1} \circ \Delta^{n-1}$ acts group like; The iterated n-fold outer-outer loop operator behaves multiplicative; The outer-outer loop operator maps left/right on inner products; $\bigcirc_{oo}$ and $\bigcirc_{ii}$ commute:



Some relations between loop operators. . . cont.

Thm: An outer-outer loop operator inside an inner-inner loop operator is equivalent to their composition; An inner-inner loop operator inside and outer-outer loop operator is equivalent to the linear form (as outer convolution of 1-chains : $\mathfrak{w} := \cdot_{\mathbb{Q}} \circ (\epsilon^1 \otimes \mathfrak{z}) \circ \Delta$)

$$\mathfrak{w} : \mathrm{Sym} \to \mathbb{Z} :: p_\lambda \mapsto \sum_{\mu\cup\nu=\lambda} \binom{\lambda}{\mu, \nu} z_\nu = \sum_{\mu\cup\nu=\lambda} \prod_i ((r_i + s_i) \uparrow (1^{s_i}))i^{s_i}$$

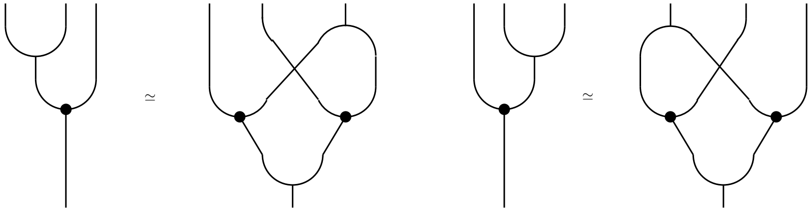


Forced Laplace expansions

In Hopf algebra deformation theory we encounter the following two Laplace expansion laws (straightenings)

$$i) \qquad (AB) \star C = (A \star C_{(1)})(B \star C_{(2)})$$

$$ii) \qquad A \star (BC) = (A_{(1)} \star B)(A_{(2)} \star C)$$



Left Laplace expansion   Right Laplace expansion

Questions:

i) Can we have a Laplace expansion for m or $\star$ with itself?

ii) Can we have a Laplace expansion with m and $\star$ interchanged?

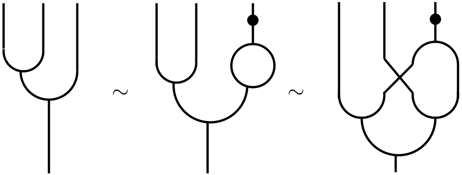
an easy check shows NO and NO! But...

Forced Laplace expansions: the associative case

Thm: The outer and inner products obtain a *modified Laplace expansion* by introducing the resp. inverse loop operators $[\frac{1}{2}], \widehat{\overline{\mathsf{z}}}$

$$
\begin{aligned}
(AB)C &= (A([\frac{1}{2}](C))_{(1)})\,(B([\frac{1}{2}](C))_{(2)}) \\
A(BC) &= ((([\frac{1}{2}](A))_{(1)}B)\,((([\frac{1}{2}](A))_{(2)}C) \\
(A\star B)\star C &= (A\star(\widehat{\overline{\mathsf{z}}}(C))_{(1)})\star(B\star(\widehat{\overline{\mathsf{z}}}(C))_{(2)}) \\
A\star(B\star C) &= ((\widehat{\overline{\mathsf{z}}}(A))_{(1)}\star B)\star((\widehat{\overline{\mathsf{z}}}(A))_{(2)}\star C)
\end{aligned}
$$

Proof: (undecorated $\cong$ any assoc. com. prod.; $\bullet$ $\cong$ resp. inverse loop)



Forced Laplace expansions: the inverse case

Thm: The inverse Laplace expansion between inner and outer product is given by

$$(A * B)C = \widehat{\widehat{\mathsf{z}}}((\widehat{\mathsf{z}}(A)C_{[1]}) * (BC_{[2]})) = \widehat{\widehat{\mathsf{z}}}((AC_{[1]}) * (\widehat{\mathsf{z}}(B)C_{[2]}))$$

$$A(B * C) = \widehat{\widehat{\mathsf{z}}}((A_{[1]}B) * (A_{[2]}\widehat{\mathsf{z}}(C))) = \widehat{\widehat{\mathsf{z}}}((A_{[1]}\widehat{\mathsf{z}}(B)) * (A_{[2]}C))$$

Application to Rota-Stein's plethystic Hopf algebras

Idea [Rota-Stein'94]: Recover Sym from $\mathcal{H}_{\sqcup}$

Def: $\mathcal{H}_{\sqcup}$ is the cofree cogenerated Hopf algebra on the module spanned by the monomial sym. fun. $\{m_\mu\}_\mu$ with structure maps

$$m_\mu \sqcup m_\nu = \binom{\mu \cup \nu}{\mu, \nu} m_{\mu\cup\nu} \qquad \Delta_{\sqcup}(m_\lambda) = m_{\lambda_{(1)}} \otimes m_{\lambda_{(2)}} \quad (\Delta_{\sqcup} \equiv \Delta)$$

$$\mathsf{S}_{\sqcup}(m_\mu) = (-1)^{\ell(\mu)} m_\mu \qquad \text{shows } m_n\text{'s are primitive}$$

Forced Laplace expansions: 'undeformations'

Def: Rota-Stein Laplace pairing on $\mathcal{H}_\sqcup$ primitives:

(1) $\langle 1, 1\rangle_{Lp} = 1$,
(2) $\langle m_\mu, m_\nu\rangle_{Lp} = 0$ if $\ell(\mu) \neq \ell(\nu)$,
(3) $\langle m_{[k^r]}, m_{[l^s]}\rangle_{Lp} = \delta_{r,s}\, m_{[(k+l)^r]} = \delta_{r,s} m_{(k+l,\ldots,k+l)}$
(4) $\langle m_\mu \sqcup m_\nu, m_\lambda\rangle_{Lp} = \langle m_\mu, m_{\lambda_{(1)}}\rangle_{Lp} \sqcup \langle m_\nu, m_{\lambda_{(2)}}\rangle_{Lp}$
(5) $\langle m_\mu, m_\nu \sqcup m_\lambda\rangle_{Lp} = \langle m_{\mu_{(1)}}, m_\nu\rangle_{Lp} \sqcup \langle m_{\mu_{(2)}}, m_\lambda\rangle_{Lp}$,

and extend by linearity.

Def. [(modified) circle product]: Using the R-S Laplace pairing the deformed product $\circ : \mathcal{H}_\sqcup \otimes \mathcal{H}_\sqcup \to \mathcal{H}_\sqcup$ is given by

$$m_\mu \circ m_\nu = \langle m_{\mu_{(1)}}, m_{\nu_{(1)}}\rangle_{Lp} \sqcup m_{\mu_{(2)}} \sqcup m_{\nu_{(2)}} \qquad \text{circle}$$

$$m_\mu \blacklozenge m_\nu = \langle m_{\mu_{(1)}}, m_{\nu_{(1)}}\rangle_{Lp} \sqcup m_{\mu_{(2)}} \sqcup [\frac{1}{2}](m_{\nu_{(2)}}) \qquad \text{modified}$$

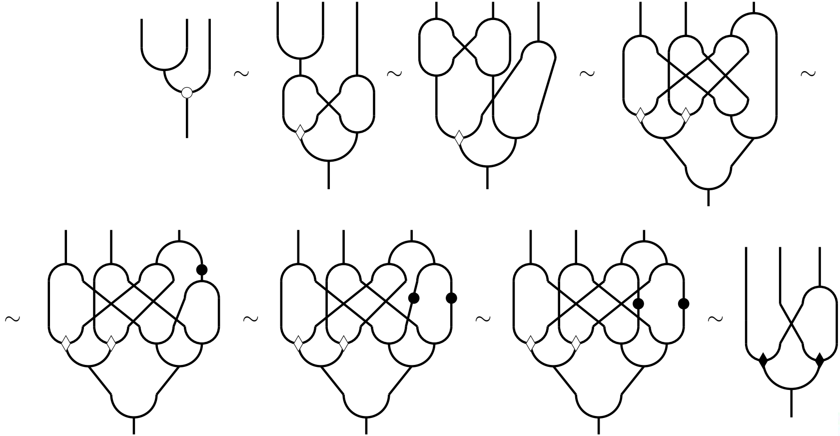
Prop: $\blacklozenge$ is noncommutative and nonassociative.

Thm. [R-S]: $(\mathcal{H}_\sqcup, \circ, \Delta; S_\circ) \simeq \mathrm{Sym}$

Forced Laplace expansions: 'undeformations' cont.

Thm: $(\circ, \blacklozenge)$ forms a bi-modified Laplace expansion
$(A \sqcup B) \circ C = (A \blacklozenge C_{(1)}) \sqcup (B \blacklozenge C_{(2)}) \quad A \circ (B \sqcup C) = (A_{(1)} \blacklozenge B) \sqcup (A_{(2)} \blacklozenge C)$

Proof: $(\diamond \cong \langle - \mid - \rangle_{Lp}$; no decoration $\cong \sqcup)$



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