Skew Schur functions with interval support

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SLC-66

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Outline

- The Schur functions are a symmetric function basis.
- Given partitions $\mu \subseteq \lambda$,

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} s_{\nu},$$

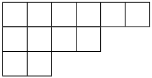
  where $c^{\lambda}_{\mu\nu}$ is the number of SSYT of shape λ/μ and content ν, satisfying the Littlewood-Richardson rule.
- $A := \lambda/\mu$. Let $r(A)$ denote the partition formed by the row lengths of A, and define c(A) similarly. The support of A, considered as a subposet of the dominance lattice, has a top element $r(A)'$ and a bottom element $c(A)$.

$$s_A = \sum_{c(A) \preceq \nu' \preceq r(A)'} c^{\lambda}_{\mu,\nu} s_{\nu},$$

- $\mathrm{supp}(A) = \{\nu' : c^{\lambda}_{\mu\nu} > 0\}$

- PROBLEM: We seek those shapes A whose support consists of the whole interval $[c(A), r(A)']$ in the dominance lattice.
- Particular case: to classify those multiplicity-free skew Schur function (i.e. skew Schur functions such that every Schur functions appears with either multiplicity 0 or 1) such that when written as a linear combination of Schur functions all partitions which lie in the mentioned interval have multiplicity 1.
- This problem is equivalent to the classification of skew characters of the symmetric group and to Schubert products which obey the same properties.

- Partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$,
  $n = \sum \lambda_i$ weight, $\ell(\lambda) = \ell$ length

  Example: $\lambda \quad = (6, 4, 2) \qquad =$ 

- If $\mu \subseteq \lambda$ then the skew diagram λ/μ is obtained by removing from the diagram of λ the boxes of μ

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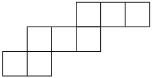
  Example: $\lambda/\mu = (6,4,2)/(3,1) =$  ribbon
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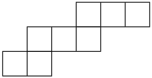
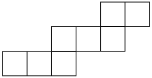
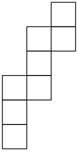
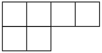
  Example: $\lambda/\mu = (6,4,2)/(3,1) =$  ribbon
- If $\mu \subseteq \lambda$ then the skew diagram λ/μ is obtained by removing from the diagram of λ the boxes of μ

$(\lambda/\mu)^\pi =$  , $(\lambda/\mu)' =$  , $\lambda^* =$ 

Dominance order on partitions λ, μ having the same weight: $\mu \preceq \lambda$ if

$$\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i$$

for $i = 1, 2, \ldots, \min\{\ell(\mu), \ell(\lambda)\}$.

- Semistandard Young tableau $T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 3 \\ \hline 3 & 3 & 4 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}$
  $\lambda = (4,3,1)$
  content$=(1,1,4,2)$ reading word $w(T) = 33214334$

- Semistandard Young tableau $\quad T = \begin{array}{cccc} & & \boxed{3} & \boxed{3} \\ \boxed{3} & \boxed{3} & \boxed{4} & \\ \boxed{4} & & & \end{array}$

  $\lambda/\mu = (4,3,1)/(2)$

  content=$(0,0,4,2)$ reading word $w(T) = 334334$

- Littlewood-Richardson Rule. A SSYT of shape λ/μ and content $\nu = (\nu_1, \nu_2, \ldots, \nu_t)'$ whose reading word is a shuffle of the words $12\cdots\nu_1, 12\cdots\nu_2, \ldots, 12\cdots\nu_t$ is said to be a LR tableau.
  $c^{\lambda}_{\mu\nu} = \#LR$ tableaux of shape λ/μ and content ν

Example

$T = \begin{array}{cccc} & & \boxed{1} & \boxed{1} \\ & \boxed{1} & \boxed{2} & \\ \boxed{2} & \boxed{2} & \boxed{3} & \\ \boxed{3} & \boxed{4} & & \end{array}$, with content $\nu = (432)'$, is a LR tableau since its reading word $w = 112132243$ is a shuffle of the words $1234, 123$ and 12.

- $\mathrm{supp}(\lambda/\mu) = \{\nu' : c^{\lambda}_{\mu\nu} > 0\}$ support of $s_{\lambda/\mu}$ (or λ/μ)

Littlewood- Richardson coefficients satisfy a number of symmetry properties, including:

- $c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$ and $c^{\lambda'}_{\mu'\nu'} = c^{\lambda}_{\mu\nu}$;

It is useful to note that

- $s_{\lambda/\mu} = s_{(\lambda/\mu)^{\pi}}$ and $s_{\lambda/\mu} = s_{\widehat{\lambda}/\widehat{\mu}}$

where $\widehat{\lambda}/\widehat{\mu}$ is the skew diagram obtained from λ/μ by deleting any empty rows and any empty columns.

$$A = \lambda/\mu = \ydiagram{2+2,1+2,3,2}\,, \quad T(c(A)) = \begin{ytableau} \none & \none & 1 & 1 \\ \none & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 \end{ytableau}\,, \quad T(r(A)') = \begin{ytableau} \none & \none & 1 & 1 \\ \none & 2 & 2 \\ 1 & 3 & 3 \\ 4 & 4 \end{ytableau}$$

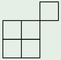
$$c(A) = (3,3,2,1) \preceq r(A)' = (4,4,1), \text{ with } r(A) = (3,2,2,2)$$

- $\mathrm{supp}(A) \subseteq [c(A), r(A)']$ with $c(A), r(A)' \in \mathrm{supp}(A)$ and $c^{\lambda}_{\mu r(A)} = c^{\lambda}_{\mu c(A)'} = 1$.
- $[c(A), r(A)']$ is called the Schur interval of s_A

Proposition

Let A be a skew diagram with two or more connected components. If there is a component containing a two by two block of boxes, then the support of A is not the entire Schur interval.

Example

The support of $A =$  is not the entire Schur interval.
$[c(A), r(A)'] = \{c(A) = 221, \xi = 311, r(A)' = 32\}$ with $\xi \notin \mathrm{supp}(A) = \{c(A), r(A)'\}$.

Corollary

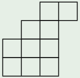
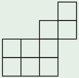
If A is a skew diagram with two or more components and the support of A is the entire Schur interval, then the connected components of A are ribbon shapes.

Corollary

Let A be a skew diagram such that $\ell(c(A)) > \ell(r(A)') = s$ (equivalently, it has no block of maximal width), and the strip V_s is a column of A of length greater than, or equal to 2. Then, the support of A is not $[c(A), r(A)']$.

Example

The supports of the skew diagrams

$A =$  ; $B =$ 

are strictly contained in the Schur interval $[c(A), r(A)']$. In the first case, $c(A) = (4,3,2,1) \preceq r(A)' = (4,4,2)$, and in the second, $c(A) = (3,2,2,2) \preceq r(A)' = (4,3,2)$.

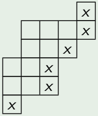
Proposition

Let A be a connected skew diagram such that

$$c(A)=(w_1,\ldots,w_r)\preceq\sigma^1=(n_1)\cup c(A)^1=(n_1,\overline{w}_2,\ldots,\overline{w}_\ell,w_{\ell+1},\ldots,w_r)\preceq r(A)'=(n_1,\ldots,n_s)$$

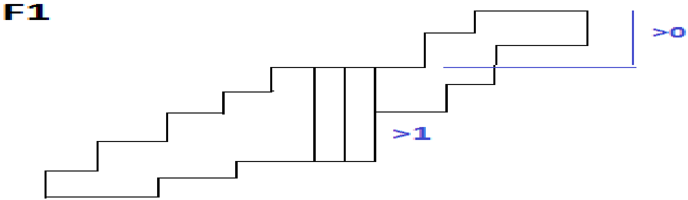
for some $3 \leq \ell \leq r$ such that $\overline{w}_k \leq w_k$ for $k = 1, \ldots, \ell$ and $0 < \overline{w}_\ell < w_\ell$. Moreover, assume the existence of two integers $2 \leq i < j \leq \ell$ such that $\overline{w}_i \geq \overline{w}_j + 2$ and $w_j > \overline{w}_j$. Then the support of A is not the entire Schur interval.

Example

$A =$ 

$$c(A) = (4, 4, 3, 2, 2)$$
$$r(A)' = (6, 4, 4, 1)$$
$$\sigma^1 = (6, \underline{4}, 2, 2, \underline{1})$$

$\mathrm{supp}(A) \subsetneq [c(A), r(A)']$, $\xi = (5, 3, 3, 2, 2) \notin \mathrm{supp}(A)$ but $c(a) \preceq \xi \preceq r(A)'$



Corollary

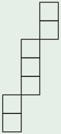
If, up to a π-rotation and/or conjugation, λ/μ is an $F1$ configuration then $\mathrm{supp}(\lambda/\mu) \subsetneq [c(A), r(A)']$.

$s_{(1^n)} = \sum_{i_1 < \ldots < i_n} x_{i_1} \ldots x_{i_n} = e_n$ elementary symmetric function
$\mu = (\mu_1, \ldots, \mu_l)$ partition; $A = (1^{\mu_1}) \oplus \cdots \oplus (1^{\mu_l})$; the Schur interval of A is $[\mu, (|\mu|)]$.
$s_A = e_{\mu_1} e_{\mu_2} \ldots e_{\mu_l} = \sum_\lambda K_{\lambda,\mu} s_{\lambda'}$, $K_{\lambda,\mu} \neq 0$ iff $\mu \preceq \lambda$ iff $\lambda \in [\mu, (|\mu|)]$.

Proposition

Skew Schur functions whose shapes are strips made either of columns or rows always attain the full interval.

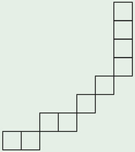
Example

$A =$  , $supp(A) = [(3,2,2),(7)]$

Proposition

Let $A = \oplus_{i=1}^{s}(1^{c_i}) \bigoplus \oplus_{i=1}^{b}(1) \bigoplus \oplus_{i=1}^{k}(l_{k-i+1})$ with $c_1 \geq \cdots \geq c_s > 1$, $s \geq 1$, $b \geq 0$, and $l_1 \geq \cdots \geq l_k > 1$, $k \geq 1$. Then $\mathrm{supp}(A) = [c(A), r(A)']$ only if $c_1 \leq b + \sum_{i\geq 2} c_i + 1$ and $l_1 \leq b + \sum_{i\geq 2} l_i + 1$.

Example

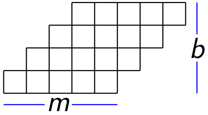
$A =$  , $supp(A) \subsetneq [(4,1^6),(8,2)]$

$(4,1^6) \preceq (5,5) \preceq (8,2)$ but $(5,5) \notin \mathrm{supp}(A)$

Theorem (F.Rodrigues,M.M.Torres,'10)

If $A = (1^c) \bigoplus \oplus_{i=1}^{b}(1) \bigoplus (l)$ with $b \geq 0$ and $c, l \geq 2$, then $\mathrm{supp}(A) = [c(A), r(A)']$ if and only if $b + l > (l-1)c$.

m-diagonal strip

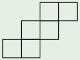


$$c(A) = (m^q, (m-1)^2, \ldots, 2^2, 1^2) \preceq r(A)' = (b^m)$$

Theorem

Skew Schur functions whose shapes are m-diagonal strips ($m \geq 2$) always attain the full interval.

Example

$A =$  2-diagonal strip $c(A) = (2^2, 1^1) \preceq r(A)' = (3^2)$,

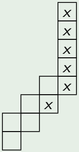
$\mathrm{supp}(A) = \{c(A), 31^3, 2^3, 321, r(A)'\} = [c(A), r(A)']$.

Theorem

Let $r = (r_1, \dots, r_s)$ be a ribbon with all column lengths greater than one (except possibly the first and last column). Then, $\mathrm{supp}(r) \subsetneq [c(A), r(A)']$ if and only if we can partition $[s]$ into three sets S, B (possibly empty), and $\{k\}$ such that $\mathcal{I}(S) = p \geq 1$ and

$$r_\ell,\ r_k + (p-1) \geq \sum_{q \in S} r_q - (p-1) \quad \text{and} \quad r_\ell \geq r_k \quad \text{for all} \quad \ell \in B.$$

Example

$r = (5, 2, 2, 2) =$  $, B = \emptyset, S = \{2, 3, 4\}, k = 1,\ p = 2;$

$c(A) = (4, 2^3) \preceq (6, 5) \preceq r(A)' = (8, 3)$ but $(6, 5) \notin \mathrm{supp}(r)$

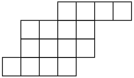
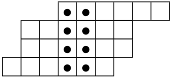
Lemma

- (A. 99') $c(\lambda/\mu) = r(\lambda/\mu)'$ if and only if $\lambda/\mu = \nu$ or $\lambda/\mu = \nu^\pi$.
- (C. Bessenrodt and A. Kleshchev 99'; van Willigenburg 04') $s_{\lambda/\mu} = s_\nu$ if and only if $\lambda/\mu = \nu$ or $\lambda/\mu = \nu^\pi$.

Lemma

(A. '99) Let v, w be partitions of length ℓ.
(a) If $\lambda/\mu = v^\pi \bullet w$ and $\beta/\alpha = [v + (x^n)]^\pi \bullet w$, then $b \in \mathrm{supp}(\lambda/\mu)$ if and only if $b \cup (n^x) \in \mathrm{supp}(\beta/\alpha)$.
(b) If $\lambda/\mu = (v^\pi \bullet w)'$ and $\beta/\alpha = ([v + (x^n)]^\pi \bullet w)'$, then $b \in \mathrm{supp}(\lambda/\mu)$ if and only if $b + (x^n) \in \mathrm{supp}(\beta/\alpha)$.

$v = (3, 2, 2, 0), w = (4, 2, 2, 1)$ with $\ell = 4$

$v^\pi \bullet w =$  , $[v + (2^4)]^\pi \bullet w =$ 

Theorem (Gutschwager '06, Thomas and Yong '05)

The basic skew Schur function $s_{\lambda/\mu}$ is multiplicity-free if and only if one or more of the following is true:

*R*0 μ or λ^* is the zero partition 0;

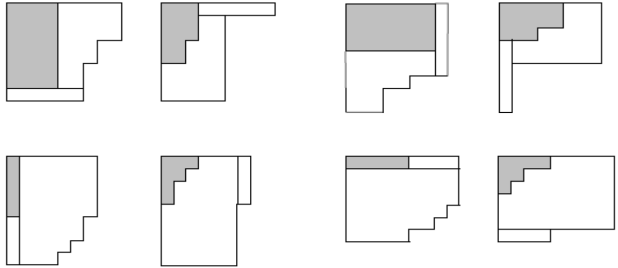
*R*1 μ or λ^* is a rectangle of m^n-shortness 1;

*R*2 μ is a rectangle of m^n-shortness 2 and λ^* is a fat hook;

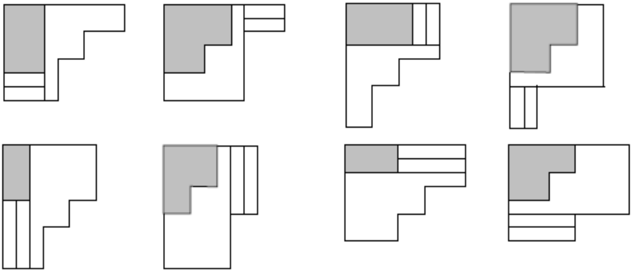
*R*3 μ is a rectangle and λ^* is a fat hook of m^n-shortness 1;

*R*4 μ and λ^* are rectangles.

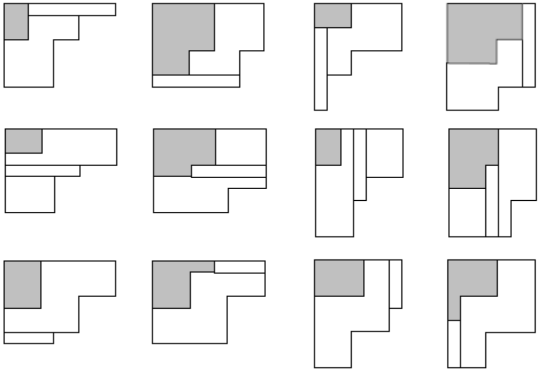
R1



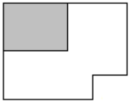
R2



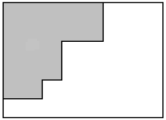
R3



R4



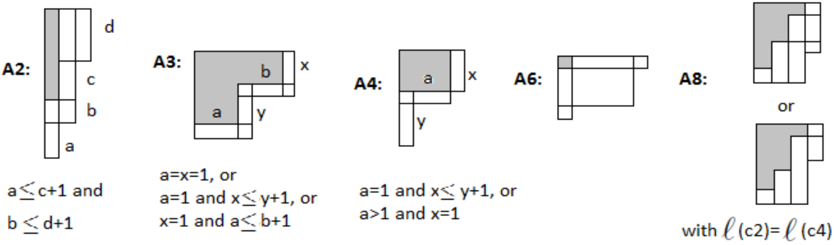
R0

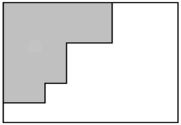


Theorem

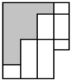
The basic skew Schur function $s_{\lambda/\mu}$ is multiplicity-free and its support is the entire Schur interval $[\mathbf{w}, \mathbf{n}]$ if and only if, up to a block of maximal width or maximal length, and up to a π-rotation and/or conjugation, one or more of the following is true:

- (i) μ or λ^* is the zero partition 0.
- (ii) λ/μ is a two column or a two row diagram.
- (iii) λ/μ is an $A2, A3, A4, A6$ or $A8$ configuration.

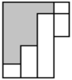




A8:

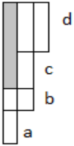


or



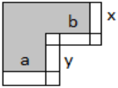
with $\ell(c2)=\ell(c4)$

A2:



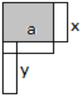
$a \leq c+1$ and $b \leq d+1$

A3:



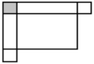
$a=x=1$, or
$a=1$ and $x \leq y+1$, or
$x=1$ and $a \leq b+1$

A4:



$a=1$ and $x \leq y+1$, or
$a>1$ and $x=1$

A6:

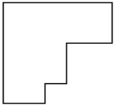


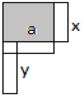
Corollary

The Schur function product $s_\mu s_\nu$ is multiplicity–free and its support is the entire Schur interval if and only if one or more of the following is true:

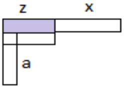
- (a) μ or ν is the zero partition.
- (b) μ and ν are both rows or both columns.
- (c) $\mu = (1^x)$ is a one–column rectangle and $\nu = (a, 1^y)$ is a hook such that either $a = 2$ and $1 \leq x \leq y + 1$, or $a \geq 3$ and $x = 1$ (or vice versa).
- (c') $\mu = (x)$ is a one–row rectangle and $\nu = (z, 1^a)$ is a hook such that either $a = 1$ and $1 \leq x \leq z$, or $a \geq 2$ and $x = 1$ (or vice versa).

The Schur function product $s_\mu s_\nu$ has all LR coefficients positive if and only if one of the conditions above or one of the following is true: $\mu = (r_1, 1^{r_2})$ and $\nu = (s_1, 1^{s_2})$ are hooks such that $s_2 = r_2 = 1$, and either $r_1 = s_1 \geq 2$ or $r_1 = 2,\ \ s_1 = r_1 + 1$ (or vice versa).

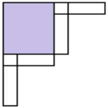




$a=2$ and $1 \leq x \leq y+1$, or
$3 \leq a$ and $x=1$



$a=1$ and $1 \leq x \leq z$, or
$2 \leq a$ and $x=1$



- O. Azenhas, The admissible interval for the invariant factors of a product of matrices, Linear and Multilinear Algebra 46 (1999), 51–99.
- D. Dou, R. Tang, R. King, A hive model determination of multiplicity-free Schur function products and skew Schur functions, arXiv:0901.0186v1 [math.CO].
- C. Gutschwager, On multiplicity-free skew characters and the Schubert Calculus, arXiv:1002.1610v1 [math.CO].
- P. McNamara, Necessary conditions for Schur-positivity, J Algebr Comb 28 (2008), 495–507.
- J.R. Stembridge, Multiplicity-free products of Schur functions, Ann. Combin. 5 (2001), 113-121.
- S. van Willigenburg, Equality of Schur and skew Schur functions, Ann. Comb. 9 (2005), 355–362 .
- H. Thomas and A. Yong, Multiplicity-free Schubert calculus, Canad. Math. Bull. 53 (2010), 171–186.

- I. Zaballa. Increasing and Decreasing Littlewood-Richardson Sequences and Duality, preprint, University of Basque Country, 1996.
- C. Bessenrodt, A. Kleshchev, On Kronecker products of complexes representations of the symmetric and alternating groups, Pacific J. Math. 190 (1999) 201-223.
- O. Azenhas, A. Conflitti, R. Mamede, Multiplicity-free skew Schur functions with interval support, arXiv:1009.4170v1.