Random partitions and representation theory of finite Chevalley groups

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If χ^V (respectively χ^λ) is the normalized character of G associated to V (resp., to V^λ), then

$$\chi^V = \sum_{\lambda \in \widehat{G}} \mathbb{P}_V[\lambda] \, \chi^\lambda.$$

The most studied case is when $G = \mathfrak{S}_n$ and $V = \mathbb{C}\mathfrak{S}_n$. Then, $\mathbb{P}_n$ is a probability measure on integer partitions of size n (the so-called **Plancherel measure**), and there is a law of large numbers and a central limit theorem for these spectral measures.

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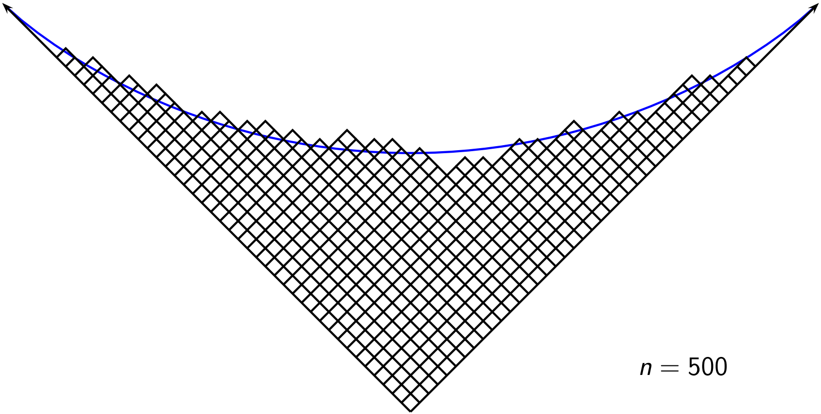

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Questions

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Spectral measure associated to a finite flag variety

We consider a finite (non-twisted) Chevalley group G (imagine: $\mathrm{G} = \mathrm{GL}(n, \mathbb{F}_q)$), and a Borel subgroup $\mathrm{B} \subset \mathrm{G}$ (imagine: $\mathrm{B} = \{$upper triangular matrices$\}$). The **flag variety** of G is the set of left cosets G/B, and G acts on this set, and therefore on the space $\mathbb{C}[\mathrm{G}/\mathrm{B}]$.

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The commutant algebra of G in $\mathrm{End}(V)$ is also $\mathbb{C}[\mathrm{B}\backslash\mathrm{G}/\mathrm{B}]$, and because of the Bruhat decomposition

$$\mathrm{G} = \bigsqcup_{w \in W} \mathrm{B}w\mathrm{B},$$

this algebra has a basis $(T_w = \mathrm{B}w\mathrm{B})_{w \in W}$ labelled by elements of the Weyl group.

Hecke algebra of a Coxeter group and Iwahori duality

Given a Coxeter group $W = \langle s \in S \mid \forall s, t \in S,\ (st)^{m_{st}} = 1 \rangle$, its **Hecke algebra** (with one parameter) is the $\mathbb{C}(q)$-algebra $\mathscr{H}(W) = \langle T_s,\ s \in S \rangle$, with

$$\forall s,\ (T_s - q)(T_s + 1) = 0 \ ;$$
$$\forall s \neq t,\ T_s T_t T_s T_t \cdots_{(m_{st} \text{ terms})} = T_t T_s T_t T_s \cdots_{(m_{st} \text{ terms})} .$$

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Theorem (Iwahori, 1964)

In the previous setting, $\mathbb{C}[\mathrm{B}\backslash\mathrm{G}/\mathrm{B}] = \mathscr{H}_q(W)$.

Hence, as a $(\mathrm{G}, \mathscr{H}_q(W))$-bimodule, one has the decomposition

$$_{\mathrm{G}\curvearrowright}\mathbb{C}[\mathrm{G}/\mathrm{B}]_{\curvearrowleft\mathscr{H}_q(W)} = \bigoplus_{\lambda \in \widehat{W}} \left\{_{\mathrm{G}\curvearrowright} U^\lambda\right\} \otimes_{\mathbb{C}} \left\{ V^\lambda_{\curvearrowleft\mathscr{H}_q(W)} \right\}.$$

Consequences for the spectral measure

So, $\mathbb{P}_{\mathrm{G/B}}$ can be seen as a probability measure on $\widehat{W}$ (instead of $\widehat{\mathrm{G}}$). Moreover, the normalized trace of the action of an element $T_w = \mathrm{B}w\mathrm{B}$ on $\mathbb{C}[\mathrm{G/B}]$ is:

$$\tau(T_w) = \begin{cases} 1 & \text{if } w = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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Denote by χ_q^λ the normalized irreducible characters of the Hecke algebra $\mathscr{H}_q(W)$. The spectral measure $\mathbb{P}_{\mathrm{G/B}}$ can now be seen as the spectral measure of the regular trace of $\mathscr{H}_q(W)$:

$$\tau = \sum_{\lambda \in \widehat{W}} \mathbb{P}_{\mathrm{G/B}}[\lambda]\, \chi_q^\lambda.$$

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$$\forall i \in [\![1, n-1]\!]\,,\ (T_i - q)(T_i + 1) = 0\ ;$$
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The irreducible modules over $\mathscr{H}_q(\mathfrak{S}_n)$ are labelled by partitions of size n. The values of the irreducible characters can be encoded in the algebra of symmetric functions:

Theorem (Ram, 1991)

$$\forall \mu \in \mathfrak{Y}_n,\ q_\mu(X) = \frac{h_\mu(qX - X)}{(q-1)^{\ell(\mu)}} = \sum_{\lambda \in \mathfrak{Y}_n} \zeta_q^\lambda(T_\mu)\, s_\lambda(X).$$

q-Plancherel measures of type A

This provides an explicit formula for the **q-Plancherel measure of type A**:

$$\mathbb{P}^{\mathrm{A}}_{n,q}[\lambda] = \mathbb{P}_{\mathrm{GL}(n,\mathbb{F}_q)/\mathrm{B}(n,\mathbb{F}_q)}[\lambda] = (\dim \lambda)\; s_\lambda\left(\frac{[1-q]}{1-[q]}\right) = \frac{n!\; q^{n(\lambda)}}{\prod_{\Box\in\lambda} h(\Box)\,\{h(\Box)\}_q}.$$

where $n(\lambda) = \sum_{i=1}^{\ell(\lambda)}(i-1)\lambda_i$, and $h(\Box)$ is the hook length of a box $\Box$ in the Young diagram of λ. In particular, $\mathbb{P}^{\mathrm{A}}_{n,q}[\lambda] = \mathbb{P}^{\mathrm{A}}_{n,q^{-1}}[\lambda']$.

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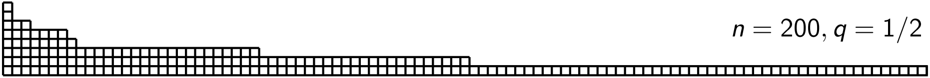

$n = 200, q = 1/2$

Asymptotic results (LLN and CLT)

Theorem (Féray-M., 2009)

As n goes to infinity, if λ is chosen randomly according to the q-Plancherel measure with $q \in]0,1[$, then

$$\frac{\lambda_i}{n} \to_{\mathbb{P}^{\mathrm{A}}_{\boldsymbol{n},\boldsymbol{q}}} (1-q)q^{i-1}\,.$$

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Set $X_{i,n} = \sqrt{n}\left(\frac{\lambda_i}{n} - (1-q)q^{i-1}\right)$; these are the normalized deviations of the first rows.

Theorem (Féray-M., 2009)

Under q-Plancherel measures $\mathbb{P}^{\mathrm{A}}_{n,q}$ with $q \in]0,1[$, the $X_{i,n}$'s converge in joint law towards centered gaussian variables $X_{i,\infty}$ with

$$\mathrm{cov}(X_{i,\infty}, X_{j,\infty}) = \delta_{ij}(1-q)q^{i-1} - (1-q)^2 q^{i+j-2}.$$

How the hell did we prove this?

When $(X_n)_{n\in\mathbb{N}}$ is a sequence of real or complex random variables, the limiting distribution of the X_n's is often determined by looking at the **moments** $\mathbb{E}[(X_n)^k]$.

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$$\mathbb{E}^{\mathbb{A}}_{n,q}[\chi^{\lambda}(\sigma)] \qquad \text{or} \qquad \mathbb{E}^{\mathbb{A}}_{n,q}[\chi^{\lambda}_q(T_\sigma)].$$

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The second expectation is simply $\tau(T_\sigma) = \delta_{\sigma=\mathrm{id}}$. Then, if σ is of type μ, by using the combinatorics of symmetric functions, one can show that

$$\mathbb{E}^{\mathbb{A}}_{n,q}[\chi^\lambda(\sigma)] = \prod_{i=1}^{\ell(\mu)} \frac{(1-q)^{\mu_i}}{1-q^{\mu_i}}.$$

Finally, one has to relate the values $\chi^\lambda(\sigma)$ to the geometry of λ.

A combinatorial interpretation with random permutations

If σ is a permutation of size n, its **descents** are the i's in $[\![1, n-1]\!]$ such that $\sigma(i) > \sigma(i+1)$. The **major index** of σ is the sum of its descents. For instance, if $\sigma = 25617384$, then $D(\sigma) = \{3, 5, 7\}$, $\mathrm{maj}(\sigma) = 3 + 5 + 7 = 15$.

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Proposition

If $\sigma \in \mathfrak{S}_n$, we set $\mathbb{Q}_{n,q}[\sigma] = \frac{q^{\mathrm{maj}(\sigma)}}{\{n!\}_q}$; this is a probability measure on $\mathfrak{S}_n$, and for any partition λ,

$$\mathbb{P}^{\mathrm{A}}_{n,q}[\lambda] = \sum_{\sigma \,\mid\, \Lambda(\sigma)=\lambda} \mathbb{Q}_{n,q}[\sigma].$$

In particular, $\ell(\sigma) \simeq_{\mathbb{Q}_{n,q}} (1-q)n + \sqrt{q(1-q)}\, n^{1/2}\, \mathcal{N}(0,1) + o(n^{1/2})$.

What happens when $\mathrm{G} = \mathrm{Sp}(2n, \mathbb{F}_q)$?

Suppose that $\mathrm{G} = \mathrm{Sp}(2n, \mathbb{F}_q)$. Then, $W = (\mathbb{Z}/2\mathbb{Z}) \wr \mathfrak{S}_n = \mathfrak{H}_n$ and $\mathscr{H}_q(\mathfrak{H}_n)$ is the algebra generated by elements $T_0, T_1, \ldots, T_{n-1}$ with

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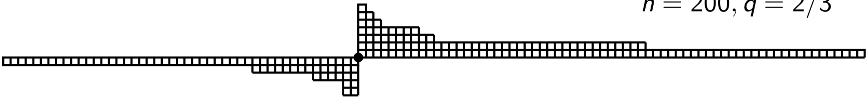
The irreducible modules over $\mathscr{H}_q(\mathfrak{H}_n)$ are labelled by pairs $\Lambda = (\lambda^{(1)}, \lambda^{(2)})$ of partitions such that $|\lambda^{(1)}| + |\lambda^{(2)}| = n$. There is an explicit hook length formula for the **q-Plancherel measure of type B**

$$\mathbb{P}^{\mathrm{B}}_{n,q} = \mathbb{P}_{\mathrm{Sp}(2n,\mathbb{F}_q)/\mathrm{BSp}(2n,\mathbb{F}_q)}.$$

Again, there is a symmetry: $\mathbb{P}^{\mathrm{B}}_{n,q}[\lambda^{(1)}, \lambda^{(2)}] = \mathbb{P}^{\mathrm{B}}_{n,q^{-1}}[\lambda^{(2)\prime}, \lambda^{(1)\prime}]$.

Asymptotic results for q-Plancherel measures of type B



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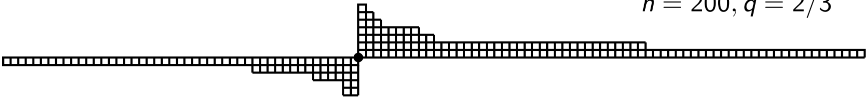

Theorem (M., 2010)

If $\Lambda = (\lambda^{(1)}, \lambda^{(2)})$ *is chosen randomly according to* $\mathbb{P}^{\mathrm{B}}_{n,q}$ *and* $\lambda = \lambda^{(1)} \sqcup \lambda^{(2)}$*, then* λ *has the same asymptotics as in type A (LLN and CLT).*

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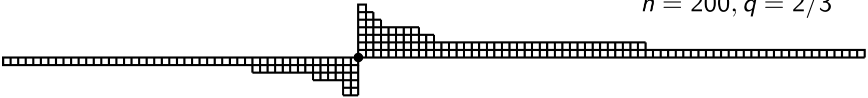

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Conjecture (M., 2010)

As n goes to infinity, the $(2k+1)$-th part of λ falls in $\lambda^{(1)}$ with probability

$$c_{k,n} \to \frac{1}{2}\left(1 + \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\,\frac{(q;q^2)_k}{(q^2;q^2)_k}\,q^k\right).$$

For the $(2k+2)$-th part, the probability converges to $1/2$.

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1. One can replace the regular trace of $\mathscr{H}_q(\mathfrak{S}_n)$ by a more general trace, for instance a Jones-Ocneanu trace. One obtains a (q, t)-deformation of the usual Plancherel measure, with a LLN and a CLT. More generally, one can do this for a Markov trace of an Ariki-Koike algebra.

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2. In type A, one can do much better: for (almost) any irreducible trace of the infinite symmetric group or the infinite Hecke algebra, the spectral measures corresponding to the restrictions of the trace to $\mathfrak{S}_n$ or $\mathscr{H}_q(\mathfrak{S}_n)$ satisfy a LLN (Kerov-Vershik, 1981) and a CLT (M., 2011).

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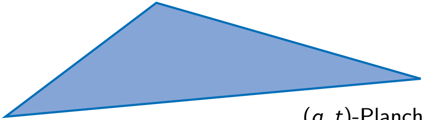
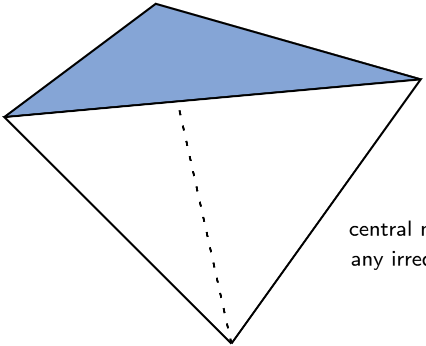
3. In the setting of finite Chevalley groups, one can look at spectral measures of modules $\mathrm{R}_{\mathrm{L}}^{\mathrm{G}}(\theta)$ obtained by parabolic induction from a cuspidal character of a Levi subgroup. There is an analog of Iwahori's duality in this setting, and the spectral measure charges in fact the set of irreducibles of a Coxeter group; most of the arguments can be reused.

q-Plancherel measure, $q \in]0, 1[$
regular trace of an Hecke algebra



(q,t)-Plancherel measure, $(q,t)\in]0,1[^2$
Jones-Ocneanu trace of an Hecke algebra



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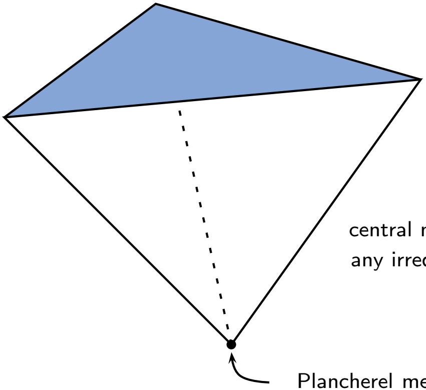

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Plancherel measures $\rightarrow$ singularity
random matrix theory!