Maximal 0-1-fillings of moon polyominoes with restricted chain lengths and rc-graphs

Martin Rubey

March 8, 2011

Abstract

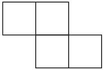
maximal 0-1 fillings of a moon polyomino M
with longest north-east chain having length k
(eg., k-triangulations, k-noncrossing partitions)

can be identified with

an interval in the poset generated by chute moves of
rc-graphs (also known as pipe dreams)
associated with a certain permutation $\omega(M, k)$

moon polyominoes

- a polyomino is a finite subset of $\mathbb{N}^2$, elements are called cells
- it is convex if for any two cells in a row or column all elements of $\mathbb{N}^2$ between them are also cells



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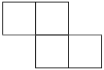 is *not* intersection-free.

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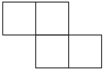
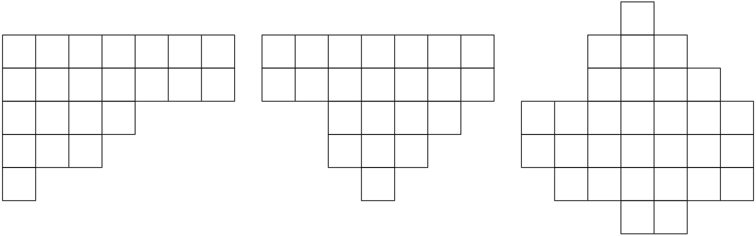
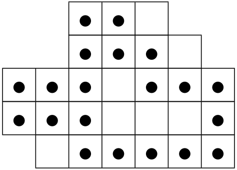
   is *not* intersection-free.

- a moon polyomino is a convex and intersection-free polyomino:



fillings and chains

Consider fillings of the cells of a moon polyomino with balls:



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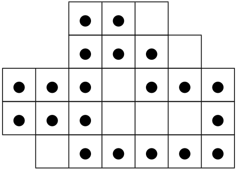
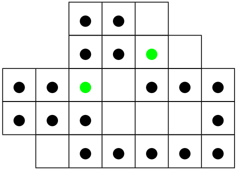
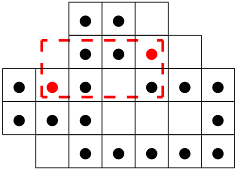


A north-east chain is a sequence of non-empty cells such that

- each entry is *strictly north-east* of its predecessor and
- the smallest rectangle containing all of them is *completely contained* in the polyomino.



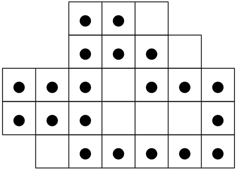
a chain of length 2



not a chain

fillings and chains

Let $\mathcal{F}_{01}^{ne}(M,k,r)$ be the set of 0-1 fillings of M with r balls and longest north-east chain having length k.
Let $\mathcal{F}_{01}^{ne}\big(M,k,(r_1,r_2,\dots)\big)$ be the subset of $\mathcal{F}_{01}^{ne}(M,k,\sum r_i)$ with r_i balls in row i.



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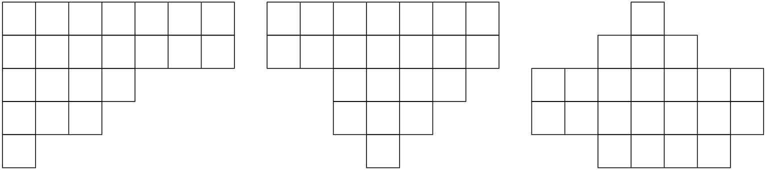
Theorem (2007)

Let M_1 and M_2 be moon polyominoes obtained from each other by rearranging columns. Then

$$\#\mathcal{F}_{01}^{ne}(M_1,k,(r_1,r_2,\dots)) = \#\mathcal{F}_{01}^{ne}(M_2,k,(r_1,r_2,\dots))$$

Corollary



all have the same number of 0-1 *fillings with r balls, for any k.*

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Let $\mathcal{F}_{01}^{ne}(M, k, (r_1, r_2, \dots))$ be the subset of $\mathcal{F}_{01}^{ne}(M, k, \sum r_i)$ with r_i balls in row i.

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Theorem (2010, following Serrano and Stump)

Let S_1 and S_2 be stack polyominoes obtained from each other by rearranging columns and fix k. Then, for $\sum r_i$ maximal,

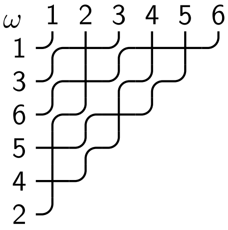
$$\mathcal{F}_{01}^{ne}(S_1, k, (r_1, r_2, \dots)) \xleftrightarrow{bij} \mathcal{F}_{01}^{ne}(S_2, k, (r_1, r_2, \dots))$$

rc-graphs

A pipe dream for a permutation ω is a filling of $\mathbb{N}^2$ with

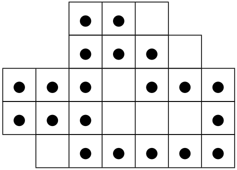
- elbow joints (╯╭) and
- a finite number of crosses (┼), such that
- a pipe entering from above in column i exits left in row $\omega^{-1}(i)$

If every pair of pipes crosses at most once the pipe dream is reduced (or an rc-graph, ‘reduced word compatible sequence graph’).



pipe dreams and fillings

We can associate a pipe dream with a filling of a moon polyomino M:



We will see that this pipe dream is reduced when the filling is maximal.

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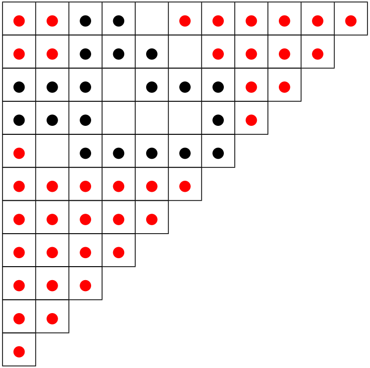

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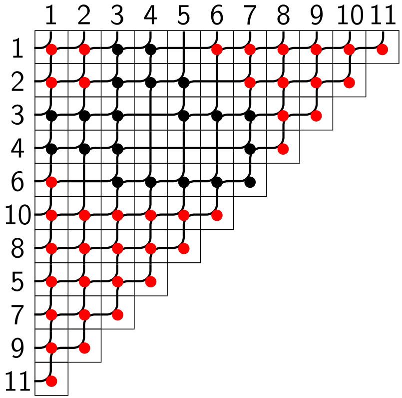

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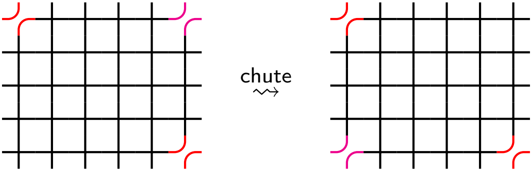
chute moves

A (generalised) chute move is a (local) modification of a reduced pipe dream of the form



Generalised chute moves preserve the permutation associated with a reduced pipe dream!

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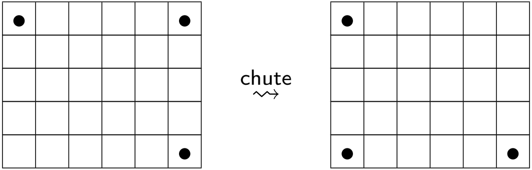


Generalised chute moves preserve the permutation associated with a reduced pipe dream!
Generalised chute moves in moon polyominoes preserve the length of the longest north-east chain!

chute moves

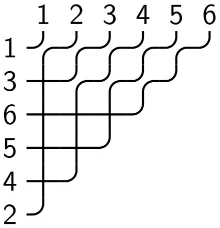
Conjecture

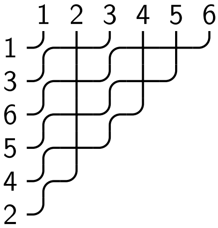
The poset of reduced pipe dreams associated with a permutation ω *generated by* generalised *chute moves is a* lattice *with bottom element having crosses at*

$$D_{bot}(\omega) = \big\{(i, c) : c \leq \#\{j : j > i, \omega_j < \omega_i\}\big\}$$

and top element having crosses at

$$D_{top}(\omega) = \big\{(c, j) : c \leq \#\{i : i < \omega_j^{-1}, \omega_i > j\}\big\}.$$

$D_{bot}(\omega) =$ 

$D_{top}(\omega) =$ 

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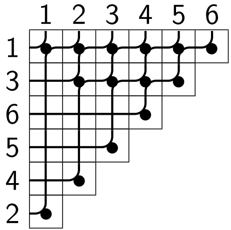

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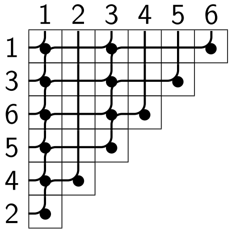

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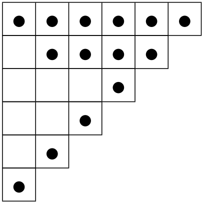
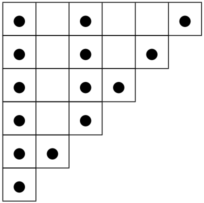
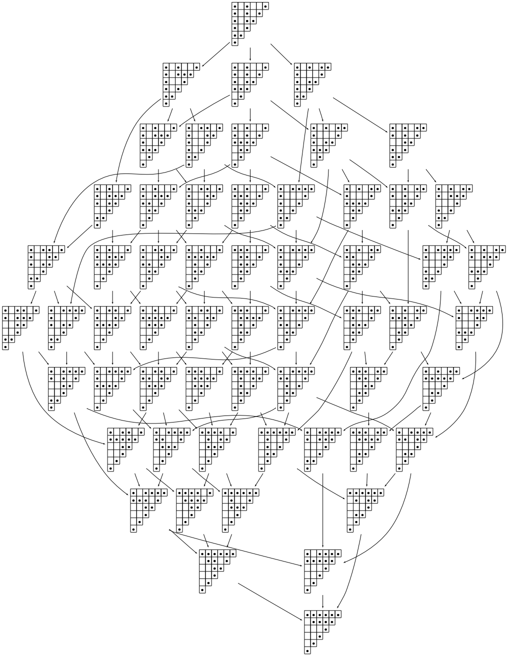
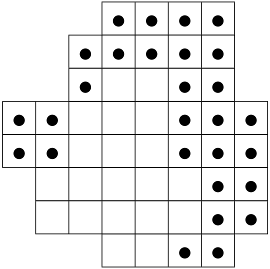
$D_{bot}(\omega) =$  $D_{top}(\omega) =$ 



rc-graphs and fillings

Theorem (2010, see also Serrano and Stump)

The set of maximal fillings $\mathcal{F}_{01}^{ne}(M, k, r_{max})$ *is an interval in the poset of reduced pipe dreams with minimal element* $D_{bot}(M, k)$ *and maximal element* $D_{top}(M, k)$*.*



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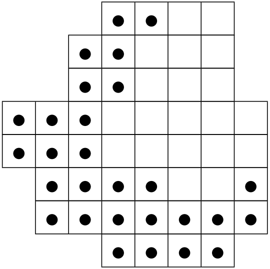

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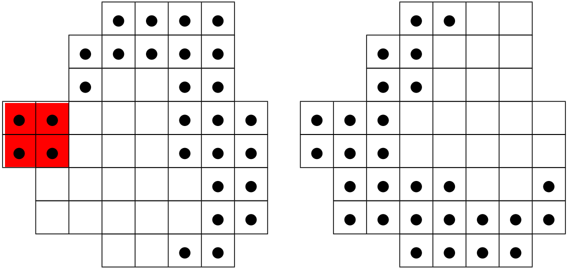

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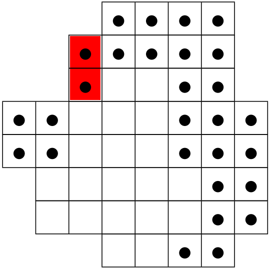

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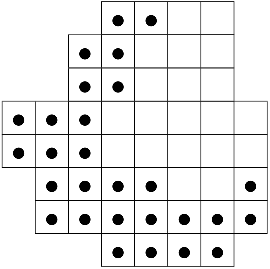

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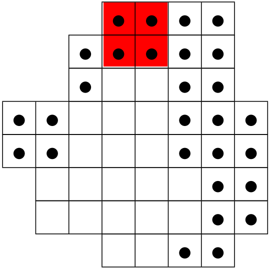

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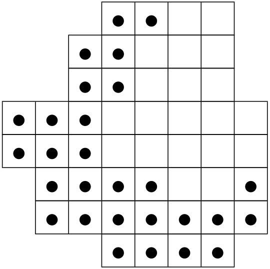

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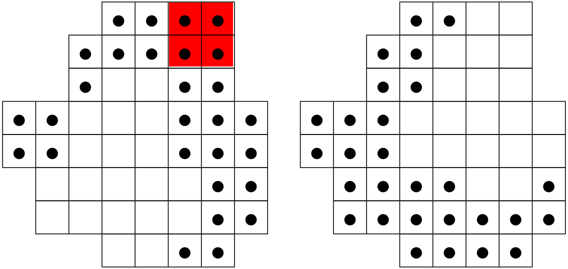

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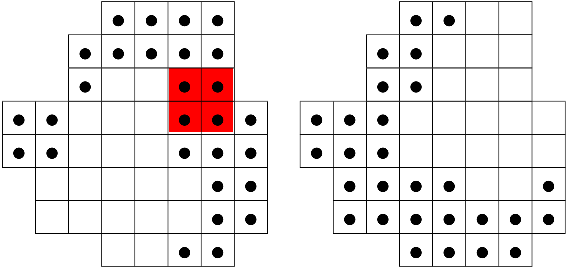

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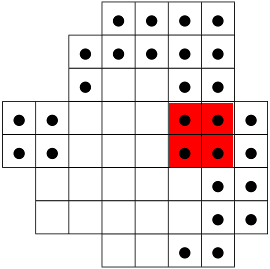

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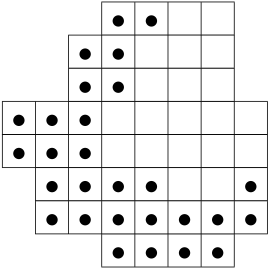

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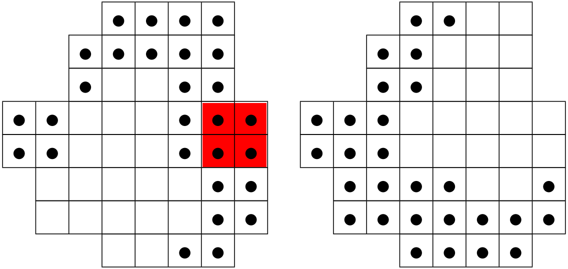

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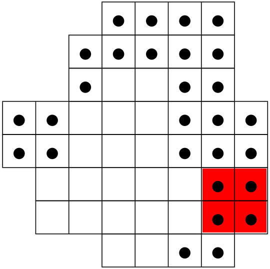
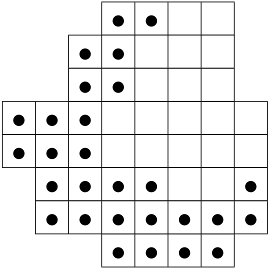


$D_{bot}(M, 2)$



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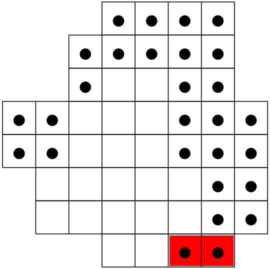

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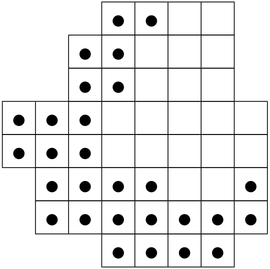

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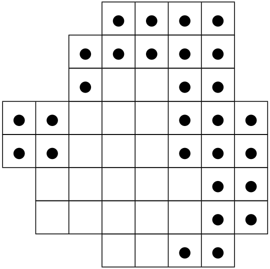

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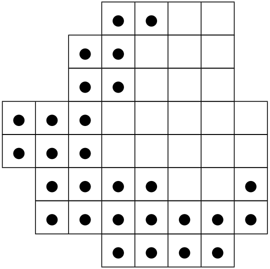
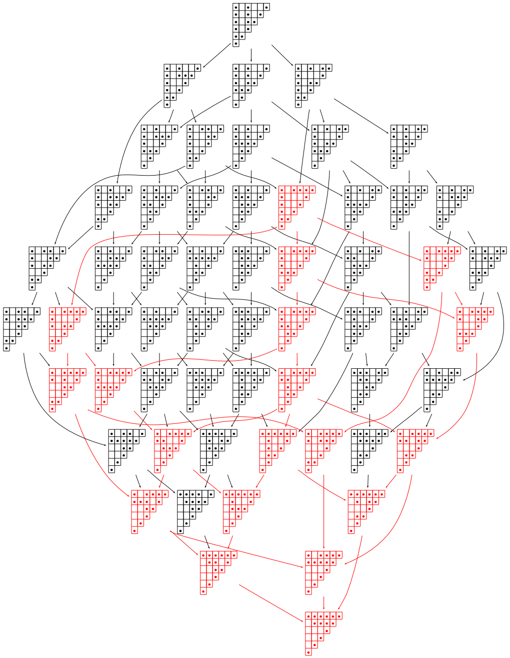


$D_{top}(M, 2)$

Proof.

Show that $D_{top}(M, 2)$ is the only filling that does not admit a generalised chute move. Details are tricky. □



Corollary

Let S_1 and S_2 be stack polyominoes obtained from each other by rearranging columns and fix k. Then, for $\sum r_i$ maximal,

$$\mathcal{F}_{01}^{ne}(S_1, k, (r_1, r_2, \dots)) \xleftrightarrow{bij} \mathcal{F}_{01}^{ne}(S_2, k, (r_1, r_2, \dots))$$

Proof.

Follow Woo (2004) and Serrano and Stump (2010):

- Notice that the permutation ω associated with a stack polyomino (indeed: any moon polyomino) is vexillary.
- Apply the Edelman-Greene correspondence to the reduced factorisation of ω given by an rc-graph associated with a filling to obtain a pair of tableaux (P, Q).
- The P-tableau determines the shape of the stack polyomino, and is independent of the filling, the Q-tableau determines the filling.

□

Open problems

- Prove the lattice property – does it have consequences?
- Find bijective proof of invariance for moon polyominoes.
- Generalise to non-maximal fillings.
- Find rc-graphs for other Dynkin types.