Combinatorics of Affine Crystals and Affine Schubert Calculus

Anne Schilling

Department of Mathematics
University of California at Davis

Seminaire Lotharingien 66, Ellwangen, March 6-9, 2011

Outline

- **Lecture 1**: Crystal bases, energy function
- **Lecture 2**: Applications of affine crystals (Kirillov–Reshetikhin crystals, charge, Demazure crystals, nonsymmetric Macdonald polynomials)
- **Lecture 3**: k-Schur functions and affine Schubert calculus

Outline

Outline

Crystals

Affine crystals

KR crystals

Perfectness

Demazure crystals

Charge

Affine Schubert calculus

Overview

- Drinfeld and Jimbo $\sim$ 1984:
  independently introduced quantum groups $U_q(\mathfrak{g})$
- Kashiwara $\sim$ 1990:
  crystal bases, bases for $U_q(\mathfrak{g})$-modules as $q \to 0$
  combinatorial approach
- Lusztig $\sim$ 1990:
  canonical bases
  geometric approach

Overview

- Drinfeld and Jimbo $\sim$ 1984:
  independently introduced quantum groups $U_q(\mathfrak{g})$
- Kashiwara $\sim$ 1990:
  crystal bases, bases for $U_q(\mathfrak{g})$-modules as $q \to 0$
  combinatorial approach
- Lusztig $\sim$ 1990:
  canonical bases
  geometric approach

Overview

- Drinfeld and Jimbo $\sim$ 1984:
  independently introduced quantum groups $U_q(\mathfrak{g})$
- Kashiwara $\sim$ 1990:
  crystal bases, bases for $U_q(\mathfrak{g})$-modules as $q \to 0$
  combinatorial approach
- Lusztig $\sim$ 1990:
  canonical bases
  geometric approach

Applications in.....

representation theory
$\rightsquigarrow$ tensor product decomposition
solvable lattice models
$\rightsquigarrow$ one point functions
conformal field theory
$\rightsquigarrow$ characters
number theory
$\rightsquigarrow$ modular forms
Bethe Ansatz
$\rightsquigarrow$ fermionic formulas
combinatorics
$\rightsquigarrow$ tableaux combinatorics, charge
geometric representation theory
$\rightsquigarrow$ geometric crystals
topological invariant theory
$\rightsquigarrow$ knots and links

$U(\mathfrak{sl}_2)$

associative algebra over $\mathbb{C}$ with 1 generated by e, f, h

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with relations

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$$

$U_q(\mathfrak{sl}_2)$

associative algebra over $\mathbb{C}(q)$ with 1 generated by $e, f, t = q^h, t^{-1} = q^{-h}$ with relations

$$[e, f] = \frac{q^h - q^{-h}}{q - q^{-1}}$$

$$q^h e q^{-h} = q^2 e$$

$$q^h f q^{-h} = q^{-2} f$$

$U_q(\mathfrak{sl}_2)$ yields $U(\mathfrak{sl}_2)$ as $q \to 1$.

$U(\mathfrak{sl}_2)$

associative algebra over $\mathbb{C}$ with 1 generated by e, f, h

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with relations

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$$

$U_q(\mathfrak{sl}_2)$

associative algebra over $\mathbb{C}(q)$ with 1 generated by $e, f, t = q^h, t^{-1} = q^{-h}$ with relations

$$[e, f] = \frac{q^h - q^{-h}}{q - q^{-1}}$$
$$q^h e q^{-h} = q^2 e$$
$$q^h f q^{-h} = q^{-2} f$$

$U_q(\mathfrak{sl}_2)$ yields $U(\mathfrak{sl}_2)$ as $q \to 1$.

Finite dimensional representations of $U_q(\mathfrak{sl}_2)$

$V(\ell)$ is the $\ell + 1$ dimensional representation of $U_q(\mathfrak{sl}_2)$ with basis vectors $u_0, u_1, \ldots, u_\ell$

action of $U_q(\mathfrak{sl}_2)$

$$tu_k = q^{\ell-2k} u_k$$
$$eu_k = [\ell - k + 1] u_{k-1}$$
$$fu_k = [k+1] u_{k+1}$$

Note $u_k = f^{(k)} u_0 = e^{(\ell-k)} u_\ell$.

Notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad [n]! = [1][2][3]\cdots[n]$$
$$f^{(n)} = f^n/[n]! \qquad e^{(n)} = e^n/[n]!$$

Finite dimensional representations of $U_q(\mathfrak{sl}_2)$

$V(\ell)$ is the $\ell+1$ dimensional representation of $U_q(\mathfrak{sl}_2)$ with basis vectors $u_0, u_1, \ldots, u_\ell$

action of $U_q(\mathfrak{sl}_2)$

$$tu_k = q^{\ell-2k} u_k$$
$$eu_k = [\ell-k+1] u_{k-1}$$
$$fu_k = [k+1] u_{k+1}$$

Note $u_k = f^{(k)} u_0 = e^{(\ell-k)} u_\ell$.

Notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad [n]! = [1][2][3]\cdots[n]$$

$$f^{(n)} = f^n/[n]! \qquad e^{(n)} = e^n/[n]!$$

Finite dimensional representations of $U_q(\mathfrak{sl}_2)$

$V(\ell)$ is the $\ell+1$ dimensional representation of $U_q(\mathfrak{sl}_2)$ with basis vectors $u_0, u_1, \ldots, u_\ell$

action of $U_q(\mathfrak{sl}_2)$

$$\begin{aligned} tu_k &= q^{\ell-2k} u_k \\ eu_k &= [\ell-k+1] u_{k-1} \\ fu_k &= [k+1] u_{k+1} \end{aligned}$$

Note $u_k = f^{(k)} u_0 = e^{(\ell-k)} u_\ell$.

Notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad [n]! = [1][2][3]\cdots[n]$$

$$f^{(n)} = f^n/[n]! \qquad e^{(n)} = e^n/[n]!$$

Crystal graph $B(\ell)$ for $V(\ell)$

$B(1) = B(\square)$

$$u_0 \longrightarrow u_1$$

$$\boxed{1} \longrightarrow \boxed{2}$$

$B(3) = B(\square\square\square)$

$$u_0 \longrightarrow u_1 \longrightarrow u_2 \longrightarrow u_3$$

$$\boxed{1\,|\,1\,|\,1} \longrightarrow \boxed{1\,|\,1\,|\,2} \longrightarrow \boxed{1\,|\,2\,|\,2} \longrightarrow \boxed{2\,|\,2\,|\,2}$$

Crystal graph $B(\ell)$ for $V(\ell)$

$B(1) = B(\square)$

$$u_0 \longrightarrow u_1$$

$$\boxed{1} \longrightarrow \boxed{2}$$

$B(3) = B(\square\square\square)$

$$u_0 \longrightarrow u_1 \longrightarrow u_2 \longrightarrow u_3$$

$$\boxed{1|1|1} \longrightarrow \boxed{1|1|2} \longrightarrow \boxed{1|2|2} \longrightarrow \boxed{2|2|2}$$

Theorem by Lusztig

The representation theory of $\mathfrak{g}$ is the same as the representation theory of $U_q(\mathfrak{g})$.

M integrable highest weight module of $U(\mathfrak{g})$

$$M = \bigoplus_{\lambda \in P} M_\lambda$$

M^q integrable highest weight module of $U_q(\mathfrak{g})$

$$M^q = \bigoplus_{\lambda \in P} M^q_\lambda$$

Then $\dim_{\mathbb{C}(q)} M^q_\lambda = \dim_{\mathbb{C}} M_\lambda$.

character $\mathrm{ch} M^q = \sum_{\lambda \in P} (\dim_{\mathbb{C}(q)} M^q_\lambda) e^\lambda$ independent of q

Crystal idea: Take special point $q = 0$.

Theorem by Lusztig

The representation theory of $\mathfrak{g}$ is the same as the representation theory of $U_q(\mathfrak{g})$.

M integrable highest weight module of $U(\mathfrak{g})$

$$M = \bigoplus_{\lambda \in P} M_\lambda$$

M^q integrable highest weight module of $U_q(\mathfrak{g})$

$$M^q = \bigoplus_{\lambda \in P} M^q_\lambda$$

Then $\dim_{\mathbb{C}(q)} M^q_\lambda = \dim_{\mathbb{C}} M_\lambda$.

character $\mathrm{ch} M^q = \sum_{\lambda \in P} (\dim_{\mathbb{C}(q)} M^q_\lambda) e^\lambda$ independent of q

Crystal idea: Take special point $q = 0$.

Theorem by Lusztig

The representation theory of $\mathfrak{g}$ is the same as the representation theory of $U_q(\mathfrak{g})$.

M integrable highest weight module of $U(\mathfrak{g})$

$$M = \bigoplus_{\lambda \in P} M_\lambda$$

M^q integrable highest weight module of $U_q(\mathfrak{g})$

$$M^q = \bigoplus_{\lambda \in P} M^q_\lambda$$

Then $\dim_{\mathbb{C}(q)} M^q_\lambda = \dim_{\mathbb{C}} M_\lambda$.

character $\mathrm{ch} M^q = \sum_{\lambda \in P} (\dim_{\mathbb{C}(q)} M^q_\lambda) e^\lambda$ independent of q

Crystal idea: Take special point $q = 0$.

Theorem by Lusztig

The representation theory of $\mathfrak{g}$ is the same as the representation theory of $U_q(\mathfrak{g})$.
M integrable highest weight module of $U(\mathfrak{g})$

$$M = \bigoplus_{\lambda \in P} M_\lambda$$

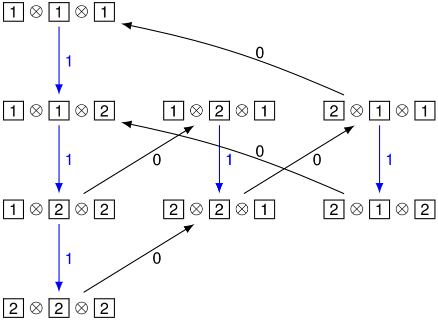
M^q integrable highest weight module of $U_q(\mathfrak{g})$

$$M^q = \bigoplus_{\lambda \in P} M^q_\lambda$$

Then $\dim_{\mathbb{C}(q)} M^q_\lambda = \dim_{\mathbb{C}} M_\lambda$.

character $\mathrm{ch} M^q = \sum_{\lambda \in P} (\dim_{\mathbb{C}(q)} M^q_\lambda) e^\lambda$ independent of q
Crystal idea: Take special point $q = 0$.

Crystal graph



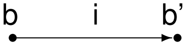
Axiomatic Crystals

A $U_q(\mathfrak{g})$-crystal is a nonempty set B with maps

$$\mathrm{wt}\colon B \to P$$
$$e_i, f_i\colon B \to B \cup \{\emptyset\} \quad \text{for all } i \in I$$

satisfying

$$f_i(b) = b' \Leftrightarrow e_i(b') = b \qquad \text{if } b, b' \in B$$
$$\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i \qquad \text{if } f_i(b) \in B$$
$$\langle h_i\,, \mathrm{wt}(b)\rangle = \varphi_i(b) - \varepsilon_i(b)$$

Write 

for $b' = f_i(b)$

Tensor products

Definition

B, B' crystals

$B \otimes B'$ is $B \times B'$ as sets with

$$\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b')$$

$$f_i(b \otimes b') = \begin{cases} f_i(b) \otimes b' & \text{if } \varepsilon_i(b) \geq \varphi_i(b') \\ b \otimes f_i(b') & \text{otherwise} \end{cases}$$

$$\underset{\underbrace{- - -}_{\varphi_i(b)}}{b} \quad \otimes \quad \underset{\underbrace{+ + + +}_{\varepsilon_i(b')}}{b'}$$

Tensor products

Definition

B, B' crystals

$B \otimes B'$ is $B \times B'$ as sets with

$$\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b')$$

$$f_i(b \otimes b') = \begin{cases} f_i(b) \otimes b' & \text{if } \varepsilon_i(b) \geq \varphi_i(b') \\ b \otimes f_i(b') & \text{otherwise} \end{cases}$$

$$\overset{b}{\underbrace{- - -}_{\varphi_i(b)} \; \underbrace{+ + +}_{\varepsilon_i(b)}} \quad \otimes \quad \overset{b'}{\underbrace{- -}_{\varphi_i(b')} \; \underbrace{+ + + +}_{\varepsilon_i(b')}}$$

Tensor products

Definition

B, B' crystals

$B \otimes B'$ is $B \times B'$ as sets with

$$\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b')$$

$$f_i(b \otimes b') = \begin{cases} f_i(b) \otimes b' & \text{if } \varepsilon_i(b) \geq \varphi_i(b') \\ b \otimes f_i(b') & \text{otherwise} \end{cases}$$

$$\overset{b}{\underbrace{- - -}_{\varphi_i(b)} \; \underbrace{+ + +}_{\varepsilon_i(b)}} \quad \otimes \quad \overset{b'}{\underbrace{- -}_{\varphi_i(b')} \; \underbrace{+ + + +}_{\varepsilon_i(b')}}$$

Tensor products

Definition

B, B' crystals

$B \otimes B'$ is $B \times B'$ as sets with

$$\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b')$$

$$f_i(b \otimes b') = \begin{cases} f_i(b) \otimes b' & \text{if } \varepsilon_i(b) \geq \varphi_i(b') \\ b \otimes f_i(b') & \text{otherwise} \end{cases}$$

$$\overset{b}{\underbrace{- - -}_{\varphi_i(b)} \; \underbrace{+++}_{\varepsilon_i(b)}} \quad \otimes \quad \overset{b'}{\underbrace{--}_{\varphi_i(b')} \; \underbrace{++++}_{\varepsilon_i(b')}}$$

Combinatorial rule for crystal operators in type A

1. Consider letters i and $i+1$ in row reading word of the tableau
2. "Bracket" pairs of the form $(i+1, i)$
3. Change last unbracketed i to an $i+1$

Combinatorial rule for crystal operators in type A

1. Consider letters i and $i + 1$ in row reading word of the tableau
2. “Bracket” pairs of the form $(i + 1, i)$
3. Change last unbracketed i to an $i + 1$

3	3	4	5	
2	2	3	4	
1	1	2	2	3

$\xrightarrow{2}$

Combinatorial rule for crystal operators in type A

1. Consider letters i and $i+1$ in row reading word of the tableau
2. "Bracket" pairs of the form $(i+1, i)$
3. Change last unbracketed i to an $i+1$

3	3	4	5	
2	2	3	4	
1	1	2	2	3

$\xrightarrow{2}$

3 3 4 5 2 2 3 4 1 1 2 2 3

Combinatorial rule for crystal operators in type A

1. Consider letters i and $i+1$ in row reading word of the tableau
2. “Bracket” pairs of the form $(i+1, i)$
3. Change last unbracketed i to an $i+1$

3	3	4	5	
2	2	3	4	
1	1	2	2	3

$\xrightarrow{2}$

3 3 4 5 2 2 3 4 1 1 2 2 3

Combinatorial rule for crystal operators in type A

1. Consider letters i and $i+1$ in row reading word of the tableau
2. "Bracket" pairs of the form $(i+1, i)$
3. Change last unbracketed i to an $i+1$

3	3	4	5	
2	2	3	4	
1	1	2	2	3

$\xrightarrow{2}$

3	3	4	5	2	2	3	4	1	1	2	2	3
(	(	·	·	)	)	(	·	·	·	)		

Combinatorial rule for crystal operators in type A

1. Consider letters i and $i+1$ in row reading word of the tableau
2. “Bracket” pairs of the form $(i+1, i)$
3. Change last unbracketed i to an $i+1$

$$\begin{array}{|c|c|c|c|c} \hline 3 & 3 & 4 & 5 \\ \cline{1-4} 2 & 2 & 3 & 4 \\ \hline 1 & 1 & 2 & 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \xrightarrow{2} \begin{array}{|c|c|c|c|c} \hline 3 & 3 & 4 & 5 \\ \cline{1-4} 2 & 2 & 3 & 4 \\ \hline 1 & 1 & 2 & 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

$$\begin{array}{ccccccccccccc} 3 & 3 & 4 & 5 & 2 & 2 & 3 & 4 & 1 & 1 & 2 & 3 & 3 \\ (& (& \cdot & \cdot &) &) & (& \cdot & \cdot & \cdot &) & & \end{array}$$

Kashiwara–Nakashima tableaux

embed $B(\lambda) \hookrightarrow B(\lambda^t_1) \otimes \cdots \otimes B(\lambda^t_{\lambda_1}) \hookrightarrow B(\square)^{\otimes|\lambda|}$

Type A_r: $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{r+1}$

Example

Type A_3

$$\begin{array}{|c|c|c|}\hline 4 & & \\ \hline 3 & 4 & \\ \hline 1 & 1 & 2 \\ \hline\end{array} \mapsto \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 1 \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 4 \\ \hline 1 \\ \hline\end{array} \otimes \boxed{2} \mapsto \boxed{4} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{4} \otimes \boxed{1} \otimes \boxed{2}$$

- strictly increasing in columns
- weakly increasing in rows

Kashiwara–Nakashima tableaux

embed $B(\lambda) \hookrightarrow B(\lambda_1^t) \otimes \cdots \otimes B(\lambda_{\lambda_1}^t) \hookrightarrow B(\boxed{})^{\otimes |\lambda|}$

Type A_r: $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{r+1}$

Example

Type A_3

$$\begin{array}{|c|c|c|}\hline 4 \\ \cline{1-1} 3 & 4 \\ \cline{1-2} 1 & 1 & 2 \\ \hline \end{array} \mapsto \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 1 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 4 \\ \hline 1 \\ \hline \end{array} \otimes \boxed{2} \mapsto \boxed{4} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{4} \otimes \boxed{1} \otimes \boxed{2}$$

- strictly increasing in columns
- weakly increasing in rows

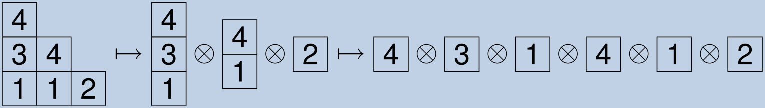
Kashiwara–Nakashima tableaux

embed $B(\lambda) \hookrightarrow B(\lambda_1^t) \otimes \cdots \otimes B(\lambda_{\lambda_1}^t) \hookrightarrow B(\square)^{\otimes |\lambda|}$

Type A_r:

Example

Type A_3



- strictly increasing in columns
- weakly increasing in rows

Kashiwara–Nakashima tableaux

embed $B(\lambda) \hookrightarrow B(\lambda_1^t) \otimes \cdots \otimes B(\lambda_{\lambda_1}^t) \hookrightarrow B(\square)^{\otimes |\lambda|}$

Type C_r: $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{-r} \xrightarrow{r-1} \cdots \xrightarrow{1} \boxed{-1}$

Example

Type C_3 $\quad \begin{array}{|c|c|} \hline \bar{3} & \\ \hline 3 & \bar{1} \\ \hline 1 & 2 \\ \hline \end{array} \mapsto \begin{array}{|c|} \hline \bar{3} \\ \hline 3 \\ \hline 1 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \bar{1} \\ \hline 2 \\ \hline \end{array} \mapsto \boxed{\bar{3}} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{\bar{1}} \otimes \boxed{2}$

- alphabet $[\bar{r}] := \{1 < 2 < \ldots < r < \bar{r} < \overline{r-1} < \ldots < \bar{1}\}$
- strictly increasing in columns
- for column $b = b(k) \ldots b(1)$ there is no pair $(z, \bar{z})$ s.t.:

$$z = b(p)\,, \qquad \bar{z} = b(q)\,, \qquad q - p \le k - z\,.$$

- more complicated rules for rows

Kashiwara–Nakashima tableaux

embed $B(\lambda) \hookrightarrow B(\lambda_1^t) \otimes \cdots \otimes B(\lambda_{\lambda_1}^t) \hookrightarrow B(\square)^{\otimes|\lambda|}$

Type C_r: $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{-r} \xrightarrow{r-1} \cdots \xrightarrow{1} \boxed{-1}$

Example

Type C_3 $\begin{array}{|c|c|} \hline \bar{3} & \\ \hline 3 & \bar{1} \\ \hline 1 & 2 \\ \hline \end{array} \mapsto \begin{array}{|c|} \hline \bar{3} \\ \hline 3 \\ \hline 1 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \bar{1} \\ \hline 2 \\ \hline \end{array} \mapsto \boxed{\bar{3}} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{\bar{1}} \otimes \boxed{2}$

- alphabet $[\bar{r}] := \{1 < 2 < \ldots < r < \bar{r} < \overline{r-1} < \ldots < \bar{1}\}$
- strictly increasing in columns
- for column $b = b(k) \ldots b(1)$ there is no pair $(z, \bar{z})$ s.t.:

$$z = b(p), \qquad \bar{z} = b(q), \qquad q - p \leq k - z.$$

- more complicated rules for rows

Kashiwara–Nakashima tableaux

embed $B(\lambda) \hookrightarrow B(\lambda^t_1) \otimes \cdots \otimes B(\lambda^t_{\lambda_1}) \hookrightarrow B(\square)^{\otimes|\lambda|}$

Type C_r: $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{-r} \xrightarrow{r-1} \cdots \xrightarrow{1} \boxed{-1}$

Example

Type C_3 $\quad \begin{array}{|c|c|} \hline \bar{3} & \\ \hline 3 & \bar{1} \\ \hline 1 & 2 \\ \hline \end{array} \mapsto \begin{array}{|c|} \hline \bar{3} \\ \hline 3 \\ \hline 1 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \bar{1} \\ \hline 2 \\ \hline \end{array} \mapsto \boxed{\bar{3}} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{\bar{1}} \otimes \boxed{2}$

- alphabet $[\bar{r}] := \{1 < 2 < \ldots < r < \bar{r} < \overline{r-1} < \ldots < \bar{1}\}$
- strictly increasing in columns
- for column $b = b(k) \ldots b(1)$ there is no pair $(z, \bar{z})$ s.t.:

$$z = b(p)\,, \qquad \bar{z} = b(q)\,, \qquad q - p \leq k - z\,.$$

- more complicated rules for rows

Kashiwara–Nakashima tableaux

embed $B(\lambda) \hookrightarrow B(\lambda_1^t) \otimes \cdots \otimes B(\lambda_{\lambda_1}^t) \hookrightarrow B(\square)^{\otimes|\lambda|}$

Type C_r: $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{-r} \xrightarrow{r-1} \cdots \xrightarrow{1} \boxed{-1}$

Example

Type C_3

$$\begin{array}{|c|c|}\hline \bar{3} & \\ \hline 3 & \bar{1} \\ \hline 1 & 2 \\ \hline \end{array} \mapsto \begin{array}{|c|}\hline \bar{3} \\ \hline 3 \\ \hline 1 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \bar{1} \\ \hline 2 \\ \hline \end{array} \mapsto \boxed{\bar{3}} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{\bar{1}} \otimes \boxed{2}$$

- alphabet $[\bar{r}] := \{1 < 2 < \ldots < r < \bar{r} < \overline{r-1} < \ldots < \bar{1}\}$
- strictly increasing in columns
- for column $b = b(k) \ldots b(1)$ there is no pair $(z, \bar{z})$ s.t.:

$$z = b(p)\,, \qquad \bar{z} = b(q)\,, \qquad q - p \leq k - z\,.$$

- more complicated rules for rows

Kashiwara–Nakashima tableaux

embed $B(\lambda) \hookrightarrow B(\lambda_1^t) \otimes \cdots \otimes B(\lambda_{\lambda_1}^t) \hookrightarrow B(\square)^{\otimes |\lambda|}$

Type C_r: $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{-r} \xrightarrow{r-1} \cdots \xrightarrow{1} \boxed{-1}$

Example

Type C_3

$$\begin{array}{|c|c|} \hline \bar{3} & \\ \hline 3 & \bar{1} \\ \hline 1 & 2 \\ \hline \end{array} \mapsto \begin{array}{|c|} \hline \bar{3} \\ \hline 3 \\ \hline 1 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \bar{1} \\ \hline 2 \\ \hline \end{array} \mapsto \boxed{\bar{3}} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{\bar{1}} \otimes \boxed{2}$$

- alphabet $[\bar{r}] := \{1 < 2 < \ldots < r < \bar{r} < \overline{r-1} < \ldots < \bar{1}\}$
- strictly increasing in columns
- for column $b = b(k) \ldots b(1)$ there is no pair $(z, \bar{z})$ s.t.:

$$z = b(p)\,, \qquad \bar{z} = b(q)\,, \qquad q - p \leq k - z\,.$$

- more complicated rules for rows

Kashiwara–Nakashima tableaux

embed $B(\lambda) \hookrightarrow B(\lambda_1^t) \otimes \cdots \otimes B(\lambda_{\lambda_1}^t) \hookrightarrow B(\boxed{})^{\otimes|\lambda|}$

Type C_r: $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{-r} \xrightarrow{r-1} \cdots \xrightarrow{1} \boxed{-1}$

Example

Type C_3 $\begin{array}{|c|c|}\hline \bar{3} & \\ \hline 3 & \bar{1} \\ \hline 1 & 2 \\ \hline \end{array} \mapsto \begin{array}{|c|}\hline \bar{3} \\ \hline 3 \\ \hline 1 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \bar{1} \\ \hline 2 \\ \hline \end{array} \mapsto \boxed{\bar{3}} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{\bar{1}} \otimes \boxed{2}$

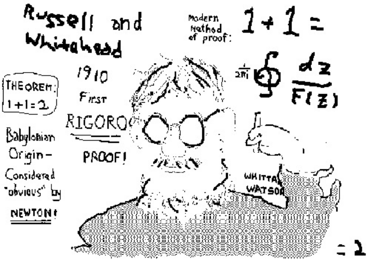
- alphabet $[\bar{r}] := \{1 < 2 < \ldots < r < \bar{r} < \overline{r-1} < \ldots < \bar{1}\}$
- strictly increasing in columns
- for column $b = b(k) \ldots b(1)$ there is no pair $(z, \bar{z})$ s.t.:

$$z = b(p)\,, \qquad \bar{z} = b(q)\,, \qquad q - p \leq k - z\,.$$

- more complicated rules for rows

Sage Days 7 at IPAM

with Nicolas Thiéry
started porting crystal code to SAGE



Dan Bump
uses crystals in number theory

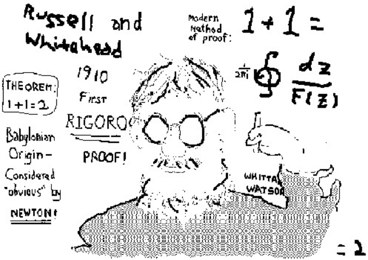
Thematic tutorials

Crystals: http://www.math.ucdavis.edu/~anne/sage/lie/crystals.html

Affine crystals: http://www.math.ucdavis.edu/~anne/sage/lie/affine_crystals.html

Sage Days 7 at IPAM

with Nicolas Thiéry
started porting crystal code to SAGE



Dan Bump
uses crystals in number theory

Thematic tutorials

Crystals: http://www.math.ucdavis.edu/~anne/sage/lie/crystals.html

Affine crystals: http://www.math.ucdavis.edu/~anne/sage/lie/affine_crystals.html

Outline

$U'_q(\widehat{\mathfrak{sl}}_2)$

$P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \delta$, $P' = P/\mathbb{Z}\delta$

$$U'_q(\widehat{\mathfrak{sl}}_2) \longrightarrow U_q(\mathfrak{sl}_2)$$

$$\begin{array}{rcl} e_0, f_1 & \mapsto & f \\ e_1, f_0 & \mapsto & e \\ t_0 & \mapsto & t^{-1} \\ t_1 & \mapsto & t \end{array}$$

2-dim representation $V = Ku_0 \oplus Ku_1$

$$q^h u_0 = q^{\langle h, \Lambda_1 - \Lambda_0 \rangle} u_0$$
$$q^h u_1 = q^{\langle h, \Lambda_0 - \Lambda_1 \rangle} u_1$$
$$e_0 u_k = f_1 u_k = u_{k+1}$$
$$e_1 u_k = f_0 u_k = u_{k-1}$$

crystal graph

$$u_0 \underset{0}{\overset{1}{\rightleftarrows}} u_1$$

$U_q'(\widehat{\mathfrak{sl}}_2)$

$P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \delta$, $P' = P/\mathbb{Z}\delta$

$$U_q'(\widehat{\mathfrak{sl}}_2) \longrightarrow U_q(\mathfrak{sl}_2)$$

$$\begin{array}{rcl} e_0, f_1 & \mapsto & f \\ e_1, f_0 & \mapsto & e \\ t_0 & \mapsto & t^{-1} \\ t_1 & \mapsto & t \end{array}$$

2-dim representation $V = Ku_0 \oplus Ku_1$

$$q^h u_0 = q^{\langle h, \Lambda_1 - \Lambda_0 \rangle} u_0$$

$$q^h u_1 = q^{\langle h, \Lambda_0 - \Lambda_1 \rangle} u_1$$

$$e_0 u_k = f_1 u_k = u_{k+1}$$

$$e_1 u_k = f_0 u_k = u_{k-1}$$

crystal graph

$$u_0 \underset{0}{\overset{1}{\rightleftarrows}} u_1$$

$U'_q(\widehat{\mathfrak{sl}}_2)$

$P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \delta$, $P' = P/\mathbb{Z}\delta$

$$U'_q(\widehat{\mathfrak{sl}}_2) \longrightarrow U_q(\mathfrak{sl}_2)$$

$$\begin{array}{rcl} e_0, f_1 & \mapsto & f \\ e_1, f_0 & \mapsto & e \\ t_0 & \mapsto & t^{-1} \\ t_1 & \mapsto & t \end{array}$$

2-dim representation $V = Ku_0 \oplus Ku_1$

$$q^h u_0 = q^{\langle h, \Lambda_1 - \Lambda_0 \rangle} u_0$$
$$q^h u_1 = q^{\langle h, \Lambda_0 - \Lambda_1 \rangle} u_1$$
$$e_0 u_k = f_1 u_k = u_{k+1}$$
$$e_1 u_k = f_0 u_k = u_{k-1}$$

crystal graph

$$u_0 \underset{0}{\overset{1}{\rightleftarrows}} u_1$$

Affine crystals

Why affine crystals?

- energy function $E : B_N \otimes \cdots \otimes B_1 \to \mathbb{Z}$

$$E(e_i(b)) = E(b) \qquad \text{for } 1 \leq i \leq n$$
$$E(e_0(b)) = E(b) - 1$$

  if e_0 never acts on rightmost step in $b = b_N \otimes \cdots \otimes b_1$.
- one-dimensional sums

$$X(\lambda, B) = \sum_{b \in \mathcal{P}(\lambda, B)} q^{E(b)}$$

- characters of conformal field theories as limits of $X(\lambda, B)$

Affine crystals

Why affine crystals?

- energy function $E : B_N \otimes \cdots \otimes B_1 \to \mathbb{Z}$

$$E(e_i(b)) = E(b) \qquad \text{for } 1 \leq i \leq n$$
$$E(e_0(b)) = E(b) - 1$$

  if e_0 never acts on rightmost step in $b = b_N \otimes \cdots \otimes b_1$.

- one-dimensional sums

$$X(\lambda, B) = \sum_{b \in \mathcal{P}(\lambda, B)} q^{E(b)}$$

- characters of conformal field theories as limits of $X(\lambda, B)$

Affine crystals

Why affine crystals?

- energy function $E : B_N \otimes \cdots \otimes B_1 \to \mathbb{Z}$

$$E(e_i(b)) = E(b) \qquad \text{for } 1 \leq i \leq n$$
$$E(e_0(b)) = E(b) - 1$$

  if e_0 never acts on rightmost step in $b = b_N \otimes \cdots \otimes b_1$.
- one-dimensional sums

$$X(\lambda, B) = \sum_{b \in \mathcal{P}(\lambda, B)} q^{E(b)}$$

- characters of conformal field theories as limits of $X(\lambda, B)$

Affine crystals

Why affine crystals?

- energy function $E : B_N \otimes \cdots \otimes B_1 \to \mathbb{Z}$

$$E(e_i(b)) = E(b) \qquad \text{for } 1 \leq i \leq n$$
$$E(e_0(b)) = E(b) - 1$$

  if e_0 never acts on rightmost step in $b = b_N \otimes \cdots \otimes b_1$.
- one-dimensional sums

$$X(\lambda, B) = \sum_{b \in \mathcal{P}(\lambda, B)} q^{E(b)}$$

- characters of conformal field theories as limits of $X(\lambda, B)$

Energy function

Definition

Local energy function $H : B \otimes B \to \mathbb{Z}$

$$H(e_i(b \otimes b')) = H(b \otimes b') \qquad \text{if } i \neq 0$$

$$H(e_0(b \otimes b')) = H(b \otimes b') + \begin{cases} 1 & \text{if } e_0 \text{ acts right} \\ -1 & \text{if } e_0 \text{ acts left} \end{cases}$$

Definition

Global energy function $E : B^{\otimes N} \to \mathbb{Z}$

$$E(b_N \otimes \cdots \otimes b_1) = \sum_{N > i \geq 1} i \cdot H(b_{i+1} \otimes b_i)$$

Energy function

Definition

Local energy function $H : B \otimes B \to \mathbb{Z}$

$$H(e_i(b \otimes b')) = H(b \otimes b') \qquad \text{if } i \neq 0$$

$$H(e_0(b \otimes b')) = H(b \otimes b') + \begin{cases} 1 & \text{if } e_0 \text{ acts right} \\ -1 & \text{if } e_0 \text{ acts left} \end{cases}$$

Definition

Global energy function $E : B^{\otimes N} \to \mathbb{Z}$

$$E(b_N \otimes \cdots \otimes b_1) = \sum_{N > i \geq 1} i \cdot H(b_{i+1} \otimes b_i)$$

Example of energy function

Example

Type A_{n-1}: $B = \{\boxed{1}, \ldots, \boxed{n}\}$

$$H(b \otimes b') = \begin{cases} 1 & \text{if } b > b' \\ 0 & \text{otherwise} \end{cases}$$

descents or inversions

$$E(b_N \otimes \cdots \otimes b_1) = \sum_{N > i \geq 1} i \cdot H(b_{i+1} \otimes b_i)$$

major index

Example of energy function

Example

Type A_{n-1}: $B = \{\boxed{1}, \ldots, \boxed{n}\}$

$$H(b \otimes b') = \begin{cases} 1 & \text{if } b > b' \\ 0 & \text{otherwise} \end{cases}$$

descents or inversions

$$E(b_N \otimes \cdots \otimes b_1) = \sum_{N > i \geq 1} i \cdot H(b_{i+1} \otimes b_i)$$

major index

Example of energy function

Example

Type A_{n-1}: $B = \{\boxed{1}, \ldots, \boxed{n}\}$

$$H(b \otimes b') = \begin{cases} 1 & \text{if } b > b' \\ 0 & \text{otherwise} \end{cases}$$ descents or inversions

$$E(b_N \otimes \cdots \otimes b_1) = \sum_{N > i \geq 1} i \cdot H(b_{i+1} \otimes b_i)$$ major index

$$b = \boxed{3} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{2}$$

Example of energy function

Example

Type A_{n-1}: $B = \{\boxed{1}, \ldots, \boxed{n}\}$

$$H(b \otimes b') = \begin{cases} 1 & \text{if } b > b' \\ 0 & \text{otherwise} \end{cases}$$

descents or inversions

$$E(b_N \otimes \cdots \otimes b_1) = \sum_{N > i \geq 1} i \cdot H(b_{i+1} \otimes b_i)$$

major index

$$b = \boxed{3} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{2}$$

$$E(b) = 4 + \cdots$$

Example of energy function

Example

Type A_{n-1}: $B = \{\boxed{1}, \ldots, \boxed{n}\}$

$$H(b \otimes b') = \begin{cases} 1 & \text{if } b > b' \\ 0 & \text{otherwise} \end{cases}$$

descents or inversions

$$E(b_N \otimes \cdots \otimes b_1) = \sum_{N > i \geq 1} i \cdot H(b_{i+1} \otimes b_i)$$

major index

$$b = \boxed{3} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{2}$$

$$E(b) = 4 + 0 + \cdots$$

Example of energy function

Example

Type A_{n-1}: $B = \{\boxed{1}, \ldots, \boxed{n}\}$

$$H(b \otimes b') = \begin{cases} 1 & \text{if } b > b' \\ 0 & \text{otherwise} \end{cases}$$

descents or inversions

$$E(b_N \otimes \cdots \otimes b_1) = \sum_{N > i \geq 1} i \cdot H(b_{i+1} \otimes b_i)$$

major index

$$b = \boxed{3} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{2}$$

$$E(b) = 4 + 0 + 2 + \cdots$$

Example of energy function

Example

Type A_{n-1}: $B = \{\boxed{1}, \ldots, \boxed{n}\}$

$$H(b \otimes b') = \begin{cases} 1 & \text{if } b > b' \\ 0 & \text{otherwise} \end{cases}$$

descents or inversions

$$E(b_N \otimes \cdots \otimes b_1) = \sum_{N > i \geq 1} i \cdot H(b_{i+1} \otimes b_i)$$

major index

$$b = \boxed{3} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{2}$$

$$E(b) = 4 + 0 + 2 + 0$$

Energy function ... inhomogeneous case

We need the combinatorial R-matrix

$$R : B_1 \otimes B_2 \to B_2 \otimes B_1$$

- affine crystal isomorphism
- maps generators $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$ (one-dimensional weight space)

Energy function ... inhomogeneous case

We need the combinatorial R-matrix

$$R : B_1 \otimes B_2 \longrightarrow B_2 \otimes B_1$$

- affine crystal isomorphism
- maps generators $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$ (one-dimensional weight space)

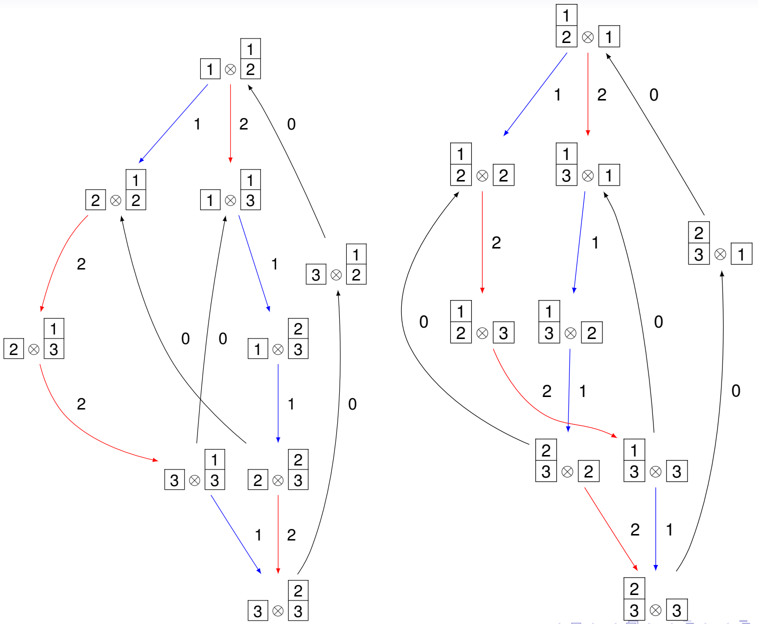
Energy function ... inhomogeneous case

We need the combinatorial R-matrix

$$R : B_1 \otimes B_2 \longrightarrow B_2 \otimes B_1$$

- affine crystal isomorphism
- maps generators $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$ (one-dimensional weight space)

Combinatorial R-matrix



Energy function

Definition

Local energy function $H : B_2 \otimes B_1 \to \mathbb{Z}$

$$H(e_i(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} -1 & \text{if } i = 0 \text{ and LL,} \\ 1 & \text{if } i = 0 \text{ and RR,} \\ 0 & \text{otherwise.} \end{cases}$$

Definition

Global energy function $D^R, D^L : B_N \otimes \cdots \otimes B_1 \to \mathbb{Z}$

$$H^R_{j,i} := H_i R_{i+1} R_{i+2} \cdots R_{j-1} \quad \text{and} \quad H^L_{j,i} := H_{j-1} R_{j-2} R_{j-3} \cdots R_i,$$

$$D^R := \sum_{N \geq j > i \geq 1} H_{j,i} \quad \text{and} \quad D^L := \sum_{N \geq j > i \geq 1} H^L_{j,i}.$$

Set $D := D^L$.

Energy function

Definition

Local energy function $H : B_2 \otimes B_1 \to \mathbb{Z}$

$$H(e_i(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} -1 & \text{if } i = 0 \text{ and LL,} \\ 1 & \text{if } i = 0 \text{ and RR,} \\ 0 & \text{otherwise.} \end{cases}$$

Definition

Global energy function $D^R, D^L : B_N \otimes \cdots \otimes B_1 \to \mathbb{Z}$

$$H^R_{j,i} := H_i R_{i+1} R_{i+2} \cdots R_{j-1} \quad \text{and} \quad H^L_{j,i} := H_{j-1} R_{j-2} R_{j-3} \cdots R_i,$$

$$D^R := \sum_{N \geq j > i \geq 1} H_{j,i} \quad \text{and} \quad D^L := \sum_{N \geq j > i \geq 1} H^L_{j,i}.$$

Set $D := D^L$.

Lusztig involution

involution on classical crystals $S : B(\lambda) \to B(\lambda)$

- maps highest weight vector to lowest weight vector
- $S(e_i) = f_{i^*}$ and $S(f_i) = e_{i^*}$ where $\alpha_{i^*} := -\omega_0(\alpha_i)$.

Example

Type A_n: $i^* = n + 1 - i$ Type C_n: $i^* = i$

Lusztig involution

involution on classical crystals $S : B(\lambda) \to B(\lambda)$

- maps highest weight vector to lowest weight vector
- $S(e_i) = f_{i^*}$ and $S(f_i) = e_{i^*}$ where $\alpha_{i^*} := -\omega_0(\alpha_i)$.

Example

Type A_n: $i^* = n + 1 - i$ Type C_n: $i^* = i$

Lusztig involution

involution on classical crystals $S : B(\lambda) \to B(\lambda)$

- maps highest weight vector to lowest weight vector
- $S(e_i) = f_{i^*}$ and $S(f_i) = e_{i^*}$ where $\alpha_{i^*} := -\omega_0(\alpha_i)$.

Example

Type A_n: $i^* = n + 1 - i$ Type C_n: $i^* = i$

Lusztig involution

involution on classical crystals $S : B(\lambda) \to B(\lambda)$

- maps highest weight vector to lowest weight vector
- $S(e_i) = f_{i^*}$ and $S(f_i) = e_{i^*}$ where $\alpha_{i^*} := -\omega_0(\alpha_i)$.

Example

Type A_n: $i^* = n + 1 - i$ Type C_n: $i^* = i$

Lusztig involution

involution on classical crystals $S : B(\lambda) \to B(\lambda)$

- maps highest weight vector to lowest weight vector
- $S(e_i) = f_{i^*}$ and $S(f_i) = e_{i^*}$ where $\alpha_{i^*} := -\omega_0(\alpha_i)$.

Example

Type A_n: $i^* = n + 1 - i$ Type C_n: $i^* = i$

same as Schützenberger involution in type A

Lusztig involution

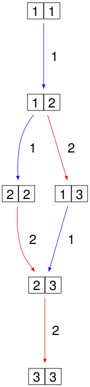
involution on classical crystals $S : B(\lambda) \to B(\lambda)$

- maps highest weight vector to lowest weight vector
- $S(e_i) = f_{i^*}$ and $S(f_i) = e_{i^*}$ where $\alpha_{i^*} := -\omega_0(\alpha_i)$.

Example

Type A_n: $i^* = n + 1 - i$ Type C_n: $i^* = i$

$B(\square\square)$ of type A_2



Affine Lusztig involution

Extend the Lusztig involution to affine crystal by:

$$\tilde{S}(e_0) = f_0 \quad \text{and} \quad \tilde{S}(f_0) = e_0.$$

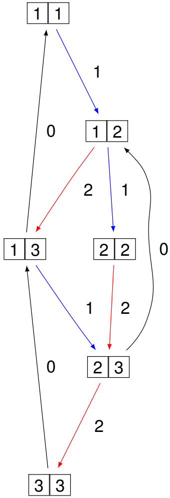
$B^{1,2}$ of type $A_2^{(1)}$

Affine Lusztig involution

Extend the Lusztig involution to affine crystal by:

$$\tilde{S}(e_0) = f_0 \quad \text{and} \quad \tilde{S}(f_0) = e_0.$$

$B^{1,2}$ of type $A_2^{(1)}$



Left and right energy

Henriques–Kamnitzer commutor

$$\begin{aligned} B(\lambda) \otimes B(\mu) &\to B(\mu) \otimes B(\lambda) \\ b \otimes b' &\mapsto S(S(b') \otimes S(b)) \end{aligned}$$

Combinatorial R-matrix

$$\begin{aligned} B_2 \otimes B_1 &\to B_1 \otimes B_2 \\ b \otimes b' &\mapsto \tilde{S}(\tilde{S}(b') \otimes \tilde{S}(b)) \end{aligned}$$

Theorem (Lenart, S., Tingley 2011)

Define

$$\begin{aligned} \tau : \quad B_N \otimes \cdots \otimes B_1 &\to B_1 \otimes \cdots \otimes B_N \\ b_N \otimes \cdots \otimes b_1 &\mapsto S(b_1) \otimes \cdots \otimes S(b_N). \end{aligned}$$

Then

$$D^R(b) = D^L(\tau(b)).$$

Left and right energy

Henriques–Kamnitzer commutor

$$B(\lambda) \otimes B(\mu) \to B(\mu) \otimes B(\lambda)$$
$$b \otimes b' \mapsto S(S(b') \otimes S(b))$$

Combinatorial R-matrix

$$B_2 \otimes B_1 \to B_1 \otimes B_2$$
$$b \otimes b' \mapsto \tilde{S}(\tilde{S}(b') \otimes \tilde{S}(b))$$

Theorem (Lenart, S., Tingley 2011)

Define

$$\tau : \quad B_N \otimes \cdots \otimes B_1 \to B_1 \otimes \cdots \otimes B_N$$
$$b_N \otimes \cdots \otimes b_1 \mapsto S(b_1) \otimes \cdots \otimes S(b_N).$$

Then

$$D^R(b) = D^L(\tau(b)).$$

Left and right energy

Henriques–Kamnitzer commutor

$$\begin{aligned} B(\lambda) \otimes B(\mu) &\to B(\mu) \otimes B(\lambda) \\ b \otimes b' &\mapsto S(S(b') \otimes S(b)) \end{aligned}$$

Combinatorial R-matrix

$$\begin{aligned} B_2 \otimes B_1 &\to B_1 \otimes B_2 \\ b \otimes b' &\mapsto \tilde{S}(\tilde{S}(b') \otimes \tilde{S}(b)) \end{aligned}$$

Theorem (Lenart, S., Tingley 2011)

Define

$$\begin{aligned} \tau: \quad B_N \otimes \cdots \otimes B_1 &\to B_1 \otimes \cdots \otimes B_N \\ b_N \otimes \cdots \otimes b_1 &\mapsto S(b_1) \otimes \cdots \otimes S(b_N). \end{aligned}$$

Then

$$D^R(b) = D^L(\tau(b)).$$

Outline

- **Lecture 1**: Crystal bases, energy function
- **Lecture 2**: Applications of affine crystals (Kirillov–Reshetikhin crystals, charge, Demazure crystals, nonsymmetric Macdonald polynomials)
- **Lecture 3**: k-Schur functions and affine Schubert calculus

Outline

Motivation

$\mathfrak{g}$ Lie algebra/Kac–Moody Lie algebra

- Crystal bases are combinatorial bases for $U_q(\mathfrak{g})$ as $q \to 0$
- Affine finite crystals:
  - appear in 1d sums of exactly solvable lattice models
  - path realization of integrable highest weight $U_q(\mathfrak{g})$-modules
  - fermionic formulas, generalized Kostka polynomials, symmetric functions
  - fusion/quantum cohomology structure constants
- Irreducible finite-dimensional affine $U_q(\mathfrak{g})$-modules classified by Chari-Pressley via Drinfeld polynomials
- HKOTY conjectured that the Kirillov-Reshetikhin modules $W^{r,s}$ have crystal bases

Motivation

$\mathfrak{g}$ Lie algebra/Kac–Moody Lie algebra

- Crystal bases are combinatorial bases for $U_q(\mathfrak{g})$ as $q \to 0$
- Affine finite crystals:
  - appear in 1d sums of exactly solvable lattice models
  - path realization of integrable highest weight $U_q(\mathfrak{g})$-modules
  - fermionic formulas, generalized Kostka polynomials, symmetric functions
  - fusion/quantum cohomology structure constants
- Irreducible finite-dimensional affine $U_q(\mathfrak{g})$-modules classified by Chari-Pressley via Drinfeld polynomials
- HKOTY conjectured that the Kirillov-Reshetikhin modules $W^{r,s}$ have crystal bases

Motivation

$\mathfrak{g}$ Lie algebra/Kac–Moody Lie algebra

- Crystal bases are combinatorial bases for $U_q(\mathfrak{g})$ as $q \to 0$
- Affine finite crystals:
    - appear in 1d sums of exactly solvable lattice models
    - path realization of integrable highest weight $U_q(\mathfrak{g})$-modules
    - fermionic formulas, generalized Kostka polynomials, symmetric functions
    - fusion/quantum cohomology structure constants
- Irreducible finite-dimensional affine $U_q(\mathfrak{g})$-modules classified by Chari-Pressley via Drinfeld polynomials
- HKOTY conjectured that the Kirillov-Reshetikhin modules $W^{r,s}$ have crystal bases

Motivation

$\mathfrak{g}$ Lie algebra/Kac–Moody Lie algebra

- Crystal bases are combinatorial bases for $U_q(\mathfrak{g})$ as $q \to 0$
- Affine finite crystals:
  - appear in 1d sums of exactly solvable lattice models
  - path realization of integrable highest weight $U_q(\mathfrak{g})$-modules
  - fermionic formulas, generalized Kostka polynomials, symmetric functions
  - fusion/quantum cohomology structure constants
- Irreducible finite-dimensional affine $U_q(\mathfrak{g})$-modules classified by Chari-Pressley via Drinfeld polynomials
- HKOTY conjectured that the Kirillov-Reshetikhin modules $W^{r,s}$ have crystal bases

Progress on Kirillov-Reshetikhin crystals ...

- **Existence of KR crystals**
  - Existence of KR crystals for nonexceptional types
    → joint with Masato Okado (arXiv:0706.2224)
- **Combinatorial models for KR crystals**
  - Type A → Shimozono
  - Types $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$
    → AS (arXiv:0704.2046)
  - Types $C_n^{(1)}$, $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$
    → joint with Ghislain Fourier and Masato Okado (arXiv:0810.5067)
  - Type $E_6^{(1)}, \ldots$
    → joint with Brant Jones (arXiv:0909.2442)
- **Perfectness**
  - Perfectness of all nonexceptional KR crystals
    → joint with Ghislain Fourier and Masato Okado (arXiv:0811.1604)

Progress on Kirillov-Reshetikhin crystals ...

- **Existence of KR crystals**
  - Existence of KR crystals for nonexceptional types $\rightarrow$ joint with Masato Okado (arXiv:0706.2224)
- **Combinatorial models for KR crystals**
  - Type A $\rightarrow$ Shimozono
  - Types $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$ $\rightarrow$ AS (arXiv:0704.2046)
  - Types $C_n^{(1)}$, $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$ $\rightarrow$ joint with Ghislain Fourier and Masato Okado (arXiv:0810.5067)
  - Type $E_6^{(1)}, \ldots$ $\rightarrow$ joint with Brant Jones (arXiv:0909.2442)
- **Perfectness**
  - Perfectness of all nonexceptional KR crystals $\rightarrow$ joint with Ghislain Fourier and Masato Okado (arXiv:0811.1604)

Progress on Kirillov-Reshetikhin crystals ...

- **Existence of KR crystals**
  - Existence of KR crystals for nonexceptional types
    $\rightarrow$ joint with Masato Okado (arXiv:0706.2224)
- **Combinatorial models for KR crystals**
  - Type $A \rightarrow$ Shimozono
  - Types $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$
    $\rightarrow$ AS (arXiv:0704.2046)
  - Types $C_n^{(1)}$, $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$
    $\rightarrow$ joint with Ghislain Fourier and Masato Okado (arXiv:0810.5067)
  - Type $E_6^{(1)}, \ldots$
    $\rightarrow$ joint with Brant Jones (arXiv:0909.2442)
- **Perfectness**
  - Perfectness of all nonexceptional KR crystals
    $\rightarrow$ joint with Ghislain Fourier and Masato Okado (arXiv:0811.1604)

...and applications ...

- **Demazure crystals and nonsymmetric Macdonald polynomials**
  - Sanderson, Ion
  - Demazure crystals in terms of KR crystals
    → joint with Ghislain Fourier and Mark Shimozono
    (arXiv:math.QA/0605451)
  - Interpretation of energy function as affine grading
    → joint with Peter Tingley
- **Charge and energy**
  - Nakayashiki and Yamada in type A
  - Definition of charge for type C from Ram-Yip formula, relation to crystal energy
    → joint with Cristian Lenart

...and applications ...

- **Demazure crystals and nonsymmetric Macdonald polynomials**
  - Sanderson, Ion
  - Demazure crystals in terms of KR crystals
    → joint with Ghislain Fourier and Mark Shimozono
    (arXiv:math.QA/0605451)
  - Interpretation of energy function as affine grading
    → joint with Peter Tingley
- **Charge and energy**
  - Nakayashiki and Yamada in type A
  - Definition of charge for type C from Ram-Yip formula, relation to crystal energy
    → joint with Cristian Lenart

Existence of Kirillov-Reshetikhin crystals

Theorem (OS 07)

The Kirillov-Reshetikhin crystals $B^{r,s}$ exist for nonexceptional types.

Proof uses results on characters by Nakajima and Hernandez.
Combinatorial models for these crystals can be constructed using the classical decompositions

$$B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda)$$

and the automorphism σ (i special node $\sigma(i) = 0$)

$$f_0 = \sigma^{-1} \circ f_i \circ \sigma$$
$$e_0 = \sigma^{-1} \circ e_i \circ \sigma$$

or using the virtual crystal construction

Existence of Kirillov-Reshetikhin crystals

Theorem (OS 07)

The Kirillov-Reshetikhin crystals $B^{r,s}$ exist for nonexceptional types.

Combinatorial models for these crystals can be constructed using the classical decompositions

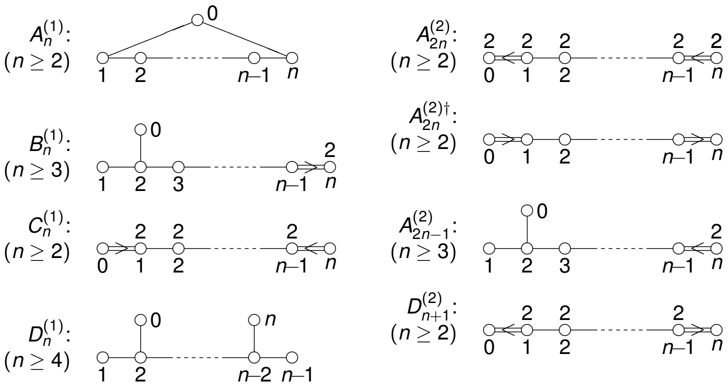
$$B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda)$$

and the automorphism σ (i special node $\sigma(i) = 0$)

$$f_0 = \sigma^{-1} \circ f_i \circ \sigma$$
$$e_0 = \sigma^{-1} \circ e_i \circ \sigma$$

or using the virtual crystal construction

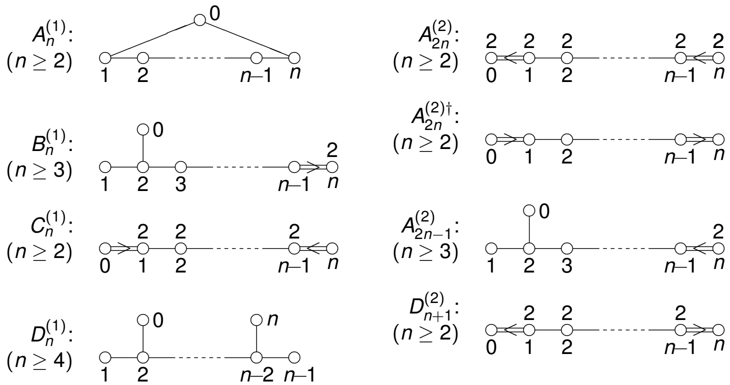
Dynkin diagrams for nonexceptional types



$A_n^{(1)}$ $\emptyset$ $C_n^{(1)}$

$B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}$ $D_{n+1}^{(2)}, A_{2n}^{(2)}$

Dynkin diagrams for nonexceptional types



$A_n^{(1)}$	$\emptyset$	$C_n^{(1)}$	
$B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}$		$D_{n+1}^{(2)}, A_{2n}^{(2)}$	

Type $A_{n-1}^{(1)}$

KMN[2] proved existence of crystals $B^{r,s}$ for Kirillov-Reshetikhin modules $W^{r,s}$

$$B^{r,s} \cong B(s^r) \quad \text{as } \{1, 2, \ldots, n-1\}\text{-crystal}$$

automorphism σ

Promotion operator pr uniquely defined by Shimozono

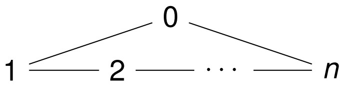
$$\begin{array}{ccc} B^{r,s} & \xrightarrow{\mathrm{pr}} & B^{r,s} \\ \downarrow f_a & & \downarrow f_{a+1} \\ B^{r,s} & \xrightarrow[\mathrm{pr}]{} & B^{r,s} \end{array}$$

$$\langle h_{a+1}, \mathrm{wt}(\mathrm{pr}(b)) \rangle = \langle h_a, \mathrm{wt}(b) \rangle$$

Type $A_{n-1}^{(1)}$

KMN[2] proved existence of crystals $B^{r,s}$ for Kirillov-Reshetikhin modules $W^{r,s}$

$$B^{r,s} \cong B(s^r) \quad \text{as } \{1,2,\ldots,n-1\}\text{-crystal}$$



automorphism σ

Promotion operator pr uniquely defined by Shimozono

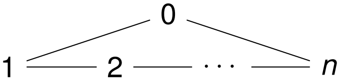
$$\begin{array}{ccc} B^{r,s} & \xrightarrow{\mathrm{pr}} & B^{r,s} \\ \downarrow f_a & & \downarrow f_{a+1} \\ B^{r,s} & \xrightarrow[\mathrm{pr}]{} & B^{r,s} \end{array}$$

$$\langle h_{a+1}\, , \, \mathrm{wt}(\mathrm{pr}(b))\rangle = \langle h_a\, , \, \mathrm{wt}(b)\rangle$$

Type $A_{n-1}^{(1)}$

KMN[2] proved existence of crystals $B^{r,s}$ for Kirillov-Reshetikhin modules $W^{r,s}$

$$B^{r,s} \cong B(s^r) \quad \text{as } \{1,2,\ldots,n-1\}\text{-crystal}$$



automorphism σ

Promotion operator pr uniquely defined by Shimozono

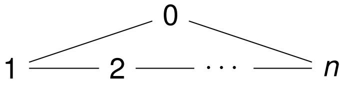
$$\begin{array}{ccc} B^{r,s} & \xrightarrow{\mathrm{pr}} & B^{r,s} \\ f_a \downarrow & & \downarrow f_{a+1} \\ B^{r,s} & \xrightarrow[\mathrm{pr}]{} & B^{r,s} \end{array}$$

$$\langle h_{a+1}\,,\, \mathrm{wt}(\mathrm{pr}(b))\rangle = \langle h_a\,,\, \mathrm{wt}(b)\rangle$$

Type $A_{n-1}^{(1)}$

KMN[2] proved existence of crystals $B^{r,s}$ for Kirillov-Reshetikhin modules $W^{r,s}$

$$B^{r,s} \cong B(s^r) \quad \text{as } \{1, 2, \ldots, n-1\}\text{-crystal}$$



automorphism σ

Promotion operator pr uniquely defined by Shimozono

$$\begin{array}{ccc} B^{r,s} & \xrightarrow{\mathrm{pr}} & B^{r,s} \\ f_a \downarrow & & \downarrow f_{a+1} \\ B^{r,s} & \xrightarrow[\mathrm{pr}]{} & B^{r,s} \end{array}$$

$$\langle h_{a+1}\,,\, \mathrm{wt}(\mathrm{pr}(b))\rangle = \langle h_a\,,\, \mathrm{wt}(b)\rangle$$

Then $\quad e_0 = \mathrm{pr}^{-1} \circ e_1 \circ \mathrm{pr} \quad f_0 = \mathrm{pr}^{-1} \circ f_1 \circ \mathrm{pr}$

Promotion for type A_{n-1}

Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

Promotion for type A_{n-1}

Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

Example

3	4	4
2	3	3
1	2	2

Promotion for type A_{n-1}

Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

Example

3	4	•
2	3	3
1	2	2

Promotion for type A_{n-1}

Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

Example

3	•	4
2	3	3
1	2	2

Promotion for type A_{n-1}

Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

Example

3	3	4
2	•	3
1	2	2

Promotion for type A_{n-1}

Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

Example

3	3	4
2	2	3
1	$\bullet$	2

Promotion for type A_{n-1}

Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

Example

3	3	4
2	2	3
•	1	2

Promotion for type A_{n-1}

Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

Example

3	3	•
2	2	3
•	1	2

Promotion for type A_{n-1}

Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

Example

3	3	3
2	2	•
•	1	2

Promotion for type A_{n-1}

Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

Example

3	3	3
2	2	2
•	1	•

Promotion for type A_{n-1}

Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

Example

3	3	3
2	2	2
•	•	1

Promotion for type A_{n-1}

Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

Example

4	4	4
3	3	3
•	•	2

Promotion for type A_{n-1}

Classical crystal: $B(s^r)$ set of Young tableaux of shape (s^r) over alphabet $\{1, 2, \ldots, n\}$

Promotion:

- Remove righmost n, play jeu de taquin and repeat.
- Increase all entries by one and place 1's in the empty spaces.

Example

4	4	4
3	3	3
1	1	2

Types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$

$$B^{r,s} \cong V^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda) \qquad \text{as } \{1,2,\ldots,n\}\text{-crystal}$$

where Λ is obtained from $s\Lambda_r$ by removing $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$

Dynkin diagram automorphism σ interchanging 0 and 1

$$f_0 = \sigma \circ f_1 \circ \sigma$$
$$e_0 = \sigma \circ e_1 \circ \sigma$$

Theorem (OS 07)

$V^{r,s} \cong B^{r,s}$ as a $\{0,1,\ldots,n\}$-crystal

Types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$

$$B^{r,s} \cong V^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda) \qquad \text{as } \{1,2,\ldots,n\}\text{-crystal}$$

where Λ is obtained from $s\Lambda_r$ by removing 

Dynkin diagram automorphism σ interchanging 0 and 1

$$f_0 = \sigma \circ f_1 \circ \sigma$$
$$e_0 = \sigma \circ e_1 \circ \sigma$$

Theorem (OS 07)

$V^{r,s} \cong B^{r,s}$ as a $\{0,1,\ldots,n\}$-crystal

Types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$

$$B^{r,s} \cong V^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda) \qquad \text{as } \{1,2,\ldots,n\}\text{-crystal}$$

where Λ is obtained from $s\Lambda_r$ by removing 

Dynkin diagram automorphism σ interchanging 0 and 1

$$f_0 = \sigma \circ f_1 \circ \sigma$$
$$e_0 = \sigma \circ e_1 \circ \sigma$$

Theorem (OS 07)

$V^{r,s} \cong B^{r,s}$ as a $\{0,1,\ldots,n\}$-crystal

Type $C_n^{(1)}$

$$B^{r,s} \cong V^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda) \qquad \text{as } \{1,2,\ldots,n\}\text{-crystal}$$

where Λ is obtained from $s\Lambda_r$ by removing $\square\square$

Virtual crystal: ambient crystal $\hat{V}^{r,s} = B^{r,s}$ of type $A_{2n+1}^{(2)}$

Definition

$V^{r,s}$ is the subset of $b \in \hat{V}^{r,s}$ such that $\sigma(b) = b$ such that

$$e_i = \begin{cases} \hat{e}_0 \hat{e}_1 & \text{for } i = 0 \\ \hat{e}_{i+1} & \text{for } 1 \leq i \leq n \end{cases}$$

Theorem (FOS 08)

$V^{r,s} \cong B^{r,s}$ as a $\{0,1,\ldots,n\}$-crystal

Type $C_n^{(1)}$

$$B^{r,s} \cong V^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda) \qquad \text{as } \{1, 2, \ldots, n\}\text{-crystal}$$

where Λ is obtained from $s\Lambda_r$ by removing □□

Virtual crystal: ambient crystal $\hat{V}^{r,s} = B^{r,s}$ of type $A_{2n+1}^{(2)}$

Definition

$V^{r,s}$ is the subset of $b \in \hat{V}^{r,s}$ such that $\sigma(b) = b$ such that

$$e_i = \begin{cases} \hat{e}_0 \hat{e}_1 & \text{for } i = 0 \\ \hat{e}_{i+1} & \text{for } 1 \leq i \leq n \end{cases}$$

Theorem (FOS 08)

$V^{r,s} \cong B^{r,s}$ as a $\{0, 1, \ldots, n\}$-crystal

Type $C_n^{(1)}$

$$B^{r,s} \cong V^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda) \qquad \text{as } \{1,2,\ldots,n\}\text{-crystal}$$

where Λ is obtained from $s\Lambda_r$ by removing $\square\square$

Virtual crystal: ambient crystal $\hat{V}^{r,s} = B^{r,s}$ of type $A_{2n+1}^{(2)}$

Definition

$V^{r,s}$ is the subset of $b \in \hat{V}^{r,s}$ such that $\sigma(b) = b$ such that

$$e_i = \begin{cases} \hat{e}_0 \hat{e}_1 & \text{for } i = 0 \\ \hat{e}_{i+1} & \text{for } 1 \le i \le n \end{cases}$$

Theorem (FOS 08)

$V^{r,s} \cong B^{r,s}$ as a $\{0,1,\ldots,n\}$-crystal

Type $C_n^{(1)}$

$$B^{r,s} \cong V^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda) \qquad \text{as } \{1,2,\ldots,n\}\text{-crystal}$$

where Λ is obtained from $s\Lambda_r$ by removing $\square\square$

Virtual crystal: ambient crystal $\hat{V}^{r,s} = B^{r,s}$ of type $A_{2n+1}^{(2)}$

Definition

$V^{r,s}$ is the subset of $b \in \hat{V}^{r,s}$ such that $\sigma(b) = b$ such that

$$e_i = \begin{cases} \hat{e}_0 \hat{e}_1 & \text{for } i = 0 \\ \hat{e}_{i+1} & \text{for } 1 \leq i \leq n \end{cases}$$

Theorem (FOS 08)

$V^{r,s} \cong B^{r,s}$ as a $\{0,1,\ldots,n\}$-crystal

Outline

Perfectness of KR crystals

Conjecture (HKOTT)

The KR crystal $B^{r,s}$ is perfect if and only if $\frac{s}{c_r}$ is an integer.
If $B^{r,s}$ is perfect, its level is $\frac{s}{c_r}$.

	$(c_1, \ldots, c_n)$
$B_n^{(1)}$	$(1, \ldots, 1, 2)$
$C_n^{(1)}$	$(2, \ldots, 2, 1)$
other nonexceptional	$(1, \ldots, 1)$

Theorem (FOS 08)

If $\mathfrak{g}$ is of nonexceptional type, the Conjecture is true.

Perfectness of KR crystals

Conjecture (HKOTT)

The KR crystal $B^{r,s}$ is perfect if and only if $\frac{s}{c_r}$ is an integer.
If $B^{r,s}$ is perfect, its level is $\frac{s}{c_r}$.

	$(c_1, \ldots, c_n)$
$B_n^{(1)}$	$(1, \ldots, 1, 2)$
$C_n^{(1)}$	$(2, \ldots, 2, 1)$
other nonexceptional	$(1, \ldots, 1)$

Theorem (FOS 08)

If $\mathfrak{g}$ is of nonexceptional type, the Conjecture is true.

Definition of perfectness

$P = \oplus_{i \in I} \mathbb{Z}\Lambda_i$ weight lattice of $\mathfrak{g}$, P^+ set of dominant weights.
$P_\ell^+ = \{\Lambda \in P^+ \mid \mathrm{lev}(\Lambda) = \ell\}$ level ℓ dominant weights

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i$$

Definition

The crystal $\mathcal{B}$ is perfect of level ℓ if:

1. $\mathcal{B} \cong$ crystal graph of a finite-dimensional $U_q'(\mathfrak{g})$-module.
2. $\mathcal{B} \otimes \mathcal{B}$ is connected.

$\exists\, \lambda \in P_0$ such that $\mathrm{wt}(\mathcal{B}) \subset \lambda + \sum_{i \in I_0} \mathbb{Z}_{\le 0}\alpha_i$ and $\exists$ unique element in $\mathcal{B}$ of classical weight λ.

$\forall\, b \in \mathcal{B}$, $\mathrm{lev}(\varepsilon(b)) \ge \ell$.

$\forall\, \Lambda \in P_\ell^+$, $\exists$ unique elements $b_\Lambda, b^\Lambda \in \mathcal{B}$, such that

$$\varepsilon(b_\Lambda) = \Lambda = \varphi(b^\Lambda)$$

Definition of perfectness

$P = \oplus_{i \in I} \mathbb{Z}\Lambda_i$ weight lattice of $\mathfrak{g}$, P^+ set of dominant weights.
$P_\ell^+ = \{\Lambda \in P^+ \mid \mathrm{lev}(\Lambda) = \ell\}$ level ℓ dominant weights

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i$$

Definition

The crystal $\mathcal{B}$ is perfect of level ℓ if:

1. $\mathcal{B} \cong$ crystal graph of a finite-dimensional $U_q'(\mathfrak{g})$-module.
2. $\mathcal{B} \otimes \mathcal{B}$ is connected.
3. $\exists\ \lambda \in P_0$ such that $\mathrm{wt}(\mathcal{B}) \subset \lambda + \sum_{i \in I \setminus \{0\}} \mathbb{Z}_{\leq 0} \alpha_i$ and $\exists$ unique element in $\mathcal{B}$ of classical weight λ.
4. $\forall\ b \in \mathcal{B},\ \ \mathrm{lev}(\varepsilon(b)) \geq \ell$.
5. $\forall\ \Lambda \in P_\ell^+$, $\exists$ unique elements $b_\Lambda, b^\Lambda \in \mathcal{B}$, such that

$$\varepsilon(b_\Lambda) = \Lambda = \varphi(b^\Lambda)$$

Definition of perfectness

$P = \oplus_{i \in I} \mathbb{Z}\Lambda_i$ weight lattice of $\mathfrak{g}$, P^+ set of dominant weights.
$P_\ell^+ = \{\Lambda \in P^+ \mid \mathrm{lev}(\Lambda) = \ell\}$ level ℓ dominant weights

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i$$

Definition

The crystal $\mathcal{B}$ is perfect of level ℓ if:

1. $\mathcal{B} \cong$ crystal graph of a finite-dimensional $U_q'(\mathfrak{g})$-module.
2. $\mathcal{B} \otimes \mathcal{B}$ is connected.
3. $\exists\, \lambda \in P_0$ such that $\mathrm{wt}(\mathcal{B}) \subset \lambda + \sum_{i \in I \setminus \{0\}} \mathbb{Z}_{\leq 0}\alpha_i$ and $\exists$ unique element in $\mathcal{B}$ of classical weight λ.
4. $\forall\, b \in \mathcal{B}, \ \mathrm{lev}(\varepsilon(b)) \geq \ell$.
5. $\forall\, \Lambda \in P_\ell^+$, $\exists$ unique elements $b_\Lambda, b^\Lambda \in \mathcal{B}$, such that

$$\varepsilon(b_\Lambda) = \Lambda = \varphi(b^\Lambda)$$

Definition of perfectness

$P = \oplus_{i \in I} \mathbb{Z}\Lambda_i$ weight lattice of $\mathfrak{g}$, P^+ set of dominant weights.
$P^+_\ell = \{\Lambda \in P^+ \mid \mathrm{lev}(\Lambda) = \ell\}$ level ℓ dominant weights

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i$$

Definition

The crystal $\mathcal{B}$ is perfect of level ℓ if:

1. $\mathcal{B} \cong$ crystal graph of a finite-dimensional $U'_q(\mathfrak{g})$-module.
2. $\mathcal{B} \otimes \mathcal{B}$ is connected.
3. $\exists\ \lambda \in P_0$ such that $\mathrm{wt}(\mathcal{B}) \subset \lambda + \sum_{i \in I \setminus \{0\}} \mathbb{Z}_{\leq 0}\alpha_i$ and $\exists$ unique element in $\mathcal{B}$ of classical weight λ.
4. $\forall\ b \in \mathcal{B},\ \ \mathrm{lev}(\varepsilon(b)) \geq \ell$.
5. $\forall\ \Lambda \in P^+_\ell$, $\exists$ unique elements $b_\Lambda, b^\Lambda \in \mathcal{B}$, such that

$$\varepsilon(b_\Lambda) = \Lambda = \varphi(b^\Lambda)$$

Definition of perfectness

$P = \oplus_{i \in I} \mathbb{Z}\Lambda_i$ weight lattice of $\mathfrak{g}$, P^+ set of dominant weights.
$P_\ell^+ = \{\Lambda \in P^+ \mid \mathrm{lev}(\Lambda) = \ell\}$ level ℓ dominant weights

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i$$

Definition

The crystal $\mathcal{B}$ is perfect of level ℓ if:

1. $\mathcal{B} \cong$ crystal graph of a finite-dimensional $U_q'(\mathfrak{g})$-module.
2. $\mathcal{B} \otimes \mathcal{B}$ is connected.
3. $\exists\ \lambda \in P_0$ such that $\mathrm{wt}(\mathcal{B}) \subset \lambda + \sum_{i \in I \setminus \{0\}} \mathbb{Z}_{\leq 0}\alpha_i$ and $\exists$ unique element in $\mathcal{B}$ of classical weight λ.
4. $\forall\ b \in \mathcal{B},\ \ \mathrm{lev}(\varepsilon(b)) \geq \ell$.
5. $\forall\ \Lambda \in P_\ell^+$, $\exists$ unique elements $b_\Lambda, b^\Lambda \in \mathcal{B}$, such that

$$\varepsilon(b_\Lambda) = \Lambda = \varphi(b^\Lambda)$$

Definition of perfectness

$P = \oplus_{i \in I} \mathbb{Z}\Lambda_i$ weight lattice of $\mathfrak{g}$, P^+ set of dominant weights.
$P^+_\ell = \{\Lambda \in P^+ \mid \mathrm{lev}(\Lambda) = \ell\}$ level ℓ dominant weights

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i$$

Definition

The crystal $\mathcal{B}$ is perfect of level ℓ if:

1. $\mathcal{B} \cong$ crystal graph of a finite-dimensional $U_q'(\mathfrak{g})$-module.
2. $\mathcal{B} \otimes \mathcal{B}$ is connected.
3. $\exists\ \lambda \in P_0$ such that $\mathrm{wt}(\mathcal{B}) \subset \lambda + \sum_{i \in I \setminus \{0\}} \mathbb{Z}_{\leq 0}\alpha_i$ and $\exists$ unique element in $\mathcal{B}$ of classical weight λ.
4. $\forall\ b \in \mathcal{B},\ \ \mathrm{lev}(\varepsilon(b)) \geq \ell$.
5. $\forall\ \Lambda \in P^+_\ell$, $\exists$ unique elements $b_\Lambda, b^\Lambda \in \mathcal{B}$, such that

$$\varepsilon(b_\Lambda) = \Lambda = \varphi(b^\Lambda)$$

Example: $B^{1,2}$ of type $C_2^{(1)}$

$$B^{1,2} \cong B(2\Lambda_1) \oplus B(0).$$

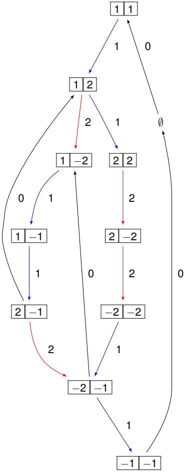
Bijection $\varepsilon : B^{1,2}_{\min} \to P_1^+$ given by:

b	$\varepsilon(b) = \varphi(b)$
$\emptyset$	Λ_0
$\boxed{1}\boxed{\bar{1}}$	Λ_1
$\boxed{2}\boxed{\bar{2}}$	Λ_2

Example: $B^{1,2}$ of type $C_2^{(1)}$

$$B^{1,2} \cong B(2\Lambda_1) \oplus B(0).$$

Bijection $\varepsilon : B^{1,2}_{\min} \to P_1^+$ given by:



b	$\varepsilon(b) = \varphi(b)$
$\emptyset$	Λ_0
1 $\bar{1}$	Λ_1
2 $\bar{2}$	Λ_2

Example: $B^{1,1}$ of type $C_3^{(1)}$

$$B^{1,1} \cong B(\Lambda_1)$$

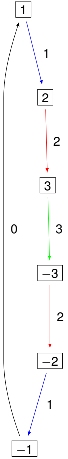
$B^{1,1}$ is not perfect. ε is not a bijection:

b	$\varepsilon(b)$
$\boxed{1}$	Λ_0
$\boxed{2}$, $\boxed{\bar{1}}$	Λ_1
$\boxed{3}$, $\boxed{\bar{2}}$	Λ_2
$\boxed{\bar{3}}$	Λ_3

Example: $B^{1,1}$ of type $C_3^{(1)}$

$$B^{1,1} \cong B(\Lambda_1)$$

$B^{1,1}$ is not perfect. ε is not a bijection:



b		$\varepsilon(b)$
$\boxed{1}$		Λ_0
$\boxed{2}$,	$\boxed{\bar{1}}$	Λ_1
$\boxed{3}$,	$\boxed{\bar{2}}$	Λ_2
$\boxed{\bar{3}}$		Λ_3

Kyoto path model

$B(\Lambda)$ highest weight infinite-dimensional crystal of type $\mathfrak{g}$
$u_\Lambda \in B(\Lambda)$ highest weight vector

Theorem (KMN²)

$\Lambda \in P_s^+$
$B^{r_1,\ell c_{r_1}}, B^{r_2,\ell c_{r_2}}, \ldots$ *perfect of level-ℓ*

$$\Phi : B(\Lambda) \cong \cdots \otimes B^{r_2,\ell c_{r_2}} \otimes B^{r_1,\ell c_{r_1}} \otimes B(\widetilde{\Lambda})$$

$\mathcal{B}$ perfect
$\mathcal{B}_{\min} = \{b \in \mathcal{B} \mid \mathrm{lev}(\varepsilon(b)) = \ell\}$
$\varepsilon, \varphi : \mathcal{B}_{\min} \to P_\ell^+$ are bijections
Induced automorphism $\tau = \varphi \circ \varepsilon^{-1}$ on P_ℓ^+
Ground state $\Phi(u_\Lambda) = \cdots \otimes b_{\tau^2(\Lambda)} \otimes b_{\tau(\Lambda)} \otimes b_\Lambda$

Kyoto path model

$B(\Lambda)$ highest weight infinite-dimensional crystal of type $\mathfrak{g}$
$u_\Lambda \in B(\Lambda)$ highest weight vector

Theorem (KMN²)

$\Lambda \in P_s^+$
$B^{r_1,\ell c_{r_1}}, B^{r_2,\ell c_{r_2}}, \ldots$ *perfect of level-*ℓ

$$\Phi : B(\Lambda) \cong \cdots \otimes B^{r_2,\ell c_{r_2}} \otimes B^{r_1,\ell c_{r_1}} \otimes B(\widetilde{\Lambda})$$

$\mathcal{B}$ perfect
$\mathcal{B}_{\text{min}} = \{b \in \mathcal{B} \mid \mathrm{lev}(\varepsilon(b)) = \ell\}$
$\varepsilon, \varphi : \mathcal{B}_{\text{min}} \to P_\ell^+$ are bijections
Induced automorphism $\tau = \varphi \circ \varepsilon^{-1}$ on P_ℓ^+
Ground state $\Phi(u_\Lambda) = \cdots \otimes b_{\tau^2(\Lambda)} \otimes b_{\tau(\Lambda)} \otimes b_\Lambda$

Kyoto path model

$B(\Lambda)$ highest weight infinite-dimensional crystal of type $\mathfrak{g}$
$u_\Lambda \in B(\Lambda)$ highest weight vector

Theorem (KMN^2)

$\Lambda \in P_s^+$
$B^{r_1,\ell c_{r_1}}, B^{r_2,\ell c_{r_2}}, \ldots$ *perfect of level-ℓ*

$$\Phi : B(\Lambda) \cong \cdots \otimes B^{r_2,\ell c_{r_2}} \otimes B^{r_1,\ell c_{r_1}} \otimes B(\widetilde{\Lambda})$$

$\mathcal{B}$ perfect
$\mathcal{B}_{\min} = \{b \in \mathcal{B} \mid \mathrm{lev}(\varepsilon(b)) = \ell\}$
$\varepsilon, \varphi : \mathcal{B}_{\min} \to P_\ell^+$ are bijections
Induced automorphism $\tau = \varphi \circ \varepsilon^{-1}$ on P_ℓ^+
Ground state $\Phi(u_\Lambda) = \cdots \otimes b_{\tau^2(\Lambda)} \otimes b_{\tau(\Lambda)} \otimes b_\Lambda$

Outline

Demazure crystals

Demazure module:

$$V_w(\lambda) := U_q(\mathfrak{g})^{>0} \cdot u_{w(\lambda)}$$

Demazure crystal: $w = s_{i_N} \cdots s_{i_1}$ fixed reduced expression

$$B_w(\lambda) = f_w(u_\lambda)$$

where $f_w(b) := \{ f_{i_N}^{m_N} \cdots f_{i_1}^{m_1}(b) \mid m_k \in \mathbb{Z}_{\geq 0} \}$

Demazure crystals

Demazure module:

$$V_w(\lambda) := U_q(\mathfrak{g})^{>0} \cdot u_{w(\lambda)}$$

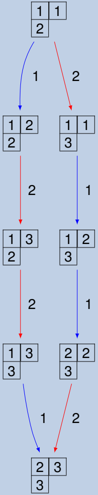
Demazure crystal: $w = s_{i_N} \cdots s_{i_1}$ fixed reduced expression

$$B_w(\lambda) = f_w(u_\lambda)$$

where $f_w(b) := \{ f_{i_N}^{m_N} \cdots f_{i_1}^{m_1}(b) \mid m_k \in \mathbb{Z}_{\geq 0} \}$

Example

Type A_2



$$B_{s_2 s_1}\left(\ytableaushort{\ \ ,\ }\right) = \left\{ \ytableaushort{11,2}, \ytableaushort{12,2}, \ytableaushort{13,2}, \ytableaushort{13,3}, \ytableaushort{11,3} \right\}$$

Affine Demazure crystals

Demazure crystal: $B_w(\lambda) = B_v(\tau(\lambda))$
where $w = v\tau \in \widetilde{W}$ affine extended Weyl group

Theorem (Fourier,S.,Shimozono 2006; S., Tingley 2011)

$B = B^{r_N, \ell c_{r_N}} \otimes \cdots \otimes B^{r_1, \ell c_{r_1}}$
$\lambda = -(c_{r_1}\omega_{r_1^*} + \cdots + c_{r_N}\omega_{r_N^*})$
$t_\lambda = v\tau \in \widetilde{W}$ *translation by* λ
Then there is a unique isomorphism of affine crystals

$$j : B(\ell\Lambda_{\tau(0)}) \to B \otimes B(\ell\Lambda_0),$$

which satisfies

$$j(u_{\ell\Lambda_{\tau(0)}}) = u_B \otimes u_{\ell\Lambda_0}$$

$$j\left(B_v(\ell\Lambda_{\tau(0)})\right) = B \otimes u_{\ell\Lambda_0}.$$

Affine Demazure crystals

Demazure crystal: $B_w(\lambda) = B_v(\tau(\lambda))$
where $w = v\tau \in \widetilde{W}$ affine extended Weyl group

Theorem (Fourier,S.,Shimozono 2006; S., Tingley 2011)

$B = B^{r_N,\ell c_{r_N}} \otimes \cdots \otimes B^{r_1,\ell c_{r_1}}$
$\lambda = -(c_{r_1}\omega_{r_1^*} + \cdots + c_{r_N}\omega_{r_N^*})$
$t_\lambda = v\tau \in \widetilde{W}$ *translation by* λ
Then there is a unique isomorphism of affine crystals

$$j : B(\ell\Lambda_{\tau(0)}) \to B \otimes B(\ell\Lambda_0),$$

which satisfies

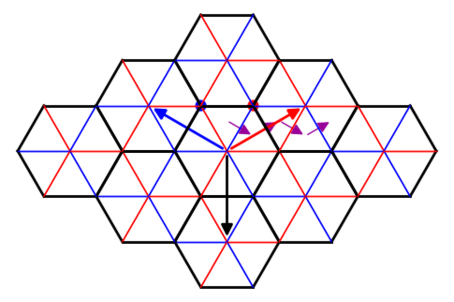
$$j(u_{\ell\Lambda_{\tau(0)}}) = u_B \otimes u_{\ell\Lambda_0}$$

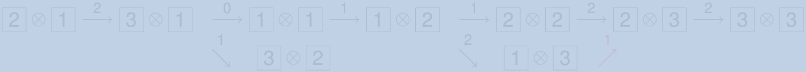
$$j\left(B_v(\ell\Lambda_{\tau(0)})\right) = B \otimes u_{\ell\Lambda_0}.$$

Example

Demazure crystal: $B_{-2\omega_2}(\Lambda_0)$ of type $A_2^{(1)}$

$$t_{-2\omega_2} = s_2 s_1 s_0 s_2 \tau \qquad \text{with } \tau : 0 \to 2 \to 1 \to 0$$

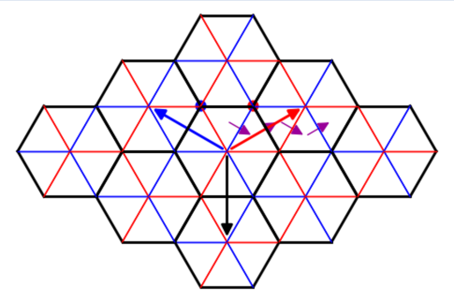




Example

Demazure crystal: $B_{-2\omega_2}(\Lambda_0)$ of type $A_2^{(1)}$

$$t_{-2\omega_2} = s_2 s_1 s_0 s_2 \tau \qquad \text{with } \tau : 0 \to 2 \to 1 \to 0$$



$$\boxed{2} \otimes \boxed{1} \xrightarrow{2} \boxed{3} \otimes \boxed{1} \xrightarrow{0} \boxed{1} \otimes \boxed{1} \xrightarrow{1} \boxed{1} \otimes \boxed{2} \xrightarrow{1} \boxed{2} \otimes \boxed{2} \xrightarrow{2} \boxed{2} \otimes \boxed{3} \xrightarrow{2} \boxed{3} \otimes \boxed{3}$$

$$\boxed{3} \otimes \boxed{1} \xrightarrow{1} \boxed{3} \otimes \boxed{2} \qquad \boxed{1} \otimes \boxed{2} \xrightarrow{2} \boxed{1} \otimes \boxed{3} \xrightarrow{1} \boxed{2} \otimes \boxed{3}$$

Demazure arrows

Definition

$B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$, fix $\ell \geq \lceil s_k/c_k \rceil$ for all $1 \leq k \leq N$

f_i on $b \in B$ is an ℓ-**Demazure arrow** if $\varphi_i(b) > 0$ and

1. $i \in I \setminus \{0\}$ or
2. $i = 0$ and $\varepsilon_0(b) \geq \ell$.

Energy

Theorem (S., Tingley 2011)

$B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$, ℓ as above, $b \in B$

1. *$\varepsilon_0(b) \geq \ell$ implies $D^R(f_0(b)) = D^R(b) + 1$;*
2. *$\varphi_0(b) \geq \ell$ implies $D^L(e_0(b)) = D^L(b) + 1$.*

Remark

Works even in nonperfect setting!

Demazure arrows

Definition

$B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$, fix $\ell \geq \lceil s_k/c_k \rceil$ for all $1 \leq k \leq N$

f_i on $b \in B$ is an ℓ-**Demazure arrow** if $\varphi_i(b) > 0$ and

1. $i \in I \setminus \{0\}$ or
2. $i = 0$ and $\varepsilon_0(b) \geq \ell$.

Energy

Theorem (S., Tingley 2011)

$B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$, ℓ as above, $b \in B$

1. *$\varepsilon_0(b) \geq \ell$ implies $D^R(f_0(b)) = D^R(b) + 1$;*
2. *$\varphi_0(b) \geq \ell$ implies $D^L(e_0(b)) = D^L(b) + 1$.*

Remark

Works even in nonperfect setting!

Demazure arrows

Definition

$B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$, fix $\ell \geq \lceil s_k/c_k \rceil$ for all $1 \leq k \leq N$

f_i on $b \in B$ is an ℓ-**Demazure arrow** if $\varphi_i(b) > 0$ and

1. $i \in I \setminus \{0\}$ or
2. $i = 0$ and $\varepsilon_0(b) \geq \ell$.

Energy

Theorem (S., Tingley 2011)

$B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$, ℓ as above, $b \in B$

1. *$\varepsilon_0(b) \geq \ell$ implies $D^R(f_0(b)) = D^R(b) + 1$;*
2. *$\varphi_0(b) \geq \ell$ implies $D^L(e_0(b)) = D^L(b) + 1$.*

Remark

Works even in nonperfect setting!

Energy in Demazure crystal

Theorem (S., Tingley 2011)

$B = B^{r_N,\ell c_{r_N}} \otimes \cdots \otimes B^{r_1,\ell c_{r_1}}$, $b \in B$

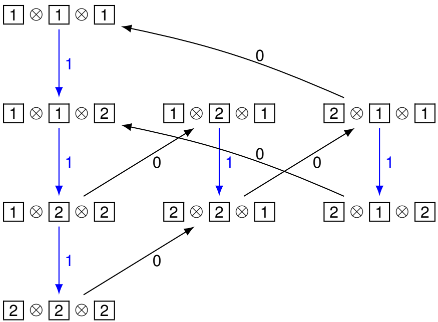
$D^R(b) - D^R(u_B) =$ *number of Demazure* e_0 *arrows from* b *to* u_B

Energy in Demazure crystal

Theorem (S., Tingley 2011)

$B = B^{r_N,\ell c_{r_N}} \otimes \cdots \otimes B^{r_1,\ell c_{r_1}}$, $b \in B$

$D^R(b) - D^R(u_B) =$ *number of Demazure* e_0 *arrows from* b *to* u_B



Energy in nonperfect setting

Theorem (Kang, Kashiwara, Misra, Miwa, Nakashima, Nakayashiki 1992)

$B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$, *fix* $\ell \geq \lceil s_k/c_k \rceil$ *for all* $1 \leq k \leq N$ $\Lambda \in P_\ell^+$. *Then*

$$B \otimes B(\Lambda) \cong \bigoplus_{\Lambda'} B(\Lambda'),$$

where sum is over finite collection of (not necessarily distinct) $\Lambda' \in P_\ell^+$.

Theorem (S., Tingley 2011)

$B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$, ℓ *as above,* $b \in B$

$D^R(b) - D^R(u_B^b) =$ *number of Demazure* e_0 *arrows from* b *to* u_B^b

Energy in nonperfect setting

Theorem (Kang, Kashiwara, Misra, Miwa, Nakashima, Nakayashiki 1992)

$B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$, fix $\ell \geq \lceil s_k/c_k \rceil$ for all $1 \leq k \leq N$ $\Lambda \in P_\ell^+$. Then

$$B \otimes B(\Lambda) \cong \bigoplus_{\Lambda'} B(\Lambda'),$$

where sum is over finite collection of (not necessarily distinct) $\Lambda' \in P_\ell^+$.

Theorem (S., Tingley 2011)

$B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$, ℓ as above, $b \in B$

$D^R(b) - D^R(u_B^b) =$ number of Demazure e_0 arrows from b to u_B^b

Nonsymmetric Macdonald polynomials

Sanderson, Ion: Relation between nonsymmetric Macdonald polynomials and Demazure characters

$$E_\lambda(q,0) = q^c \, \mathrm{ch}(V_{t_\lambda}(\Lambda_0))|_{e^\delta = q,\ e^{\Lambda_0}=1}$$

Example

$$E_{(0,0,2)}(q,0) = x_1^2 + (q+1)x_1x_2 + x_2^2 + (q+1)x_1x_3 + (q+1)x_2x_3 + x_3^2$$

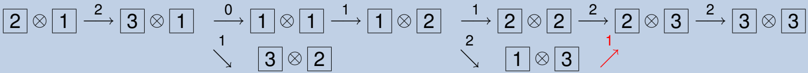
Nonsymmetric Macdonald polynomials

Sanderson, Ion: Relation between nonsymmetric Macdonald polynomials and Demazure characters

$$E_\lambda(q,0) = q^c \,\mathrm{ch}(V_{t_\lambda}(\Lambda_0))|_{e^\delta=q,\ e^{\Lambda_0}=1}$$

Example

$$E_{(0,0,2)}(q,0) = x_1^2+(q+1)x_1x_2+x_2^2+(q+1)x_1x_3+(q+1)x_2x_3+x_3^2$$



Outline

Charge

1979 Lascoux, Schützenberger charge for type A
1995 Lascoux, Leclerc, Thibon from geometry of crystal
1997 Nakayashiki, Yamada relation charge and energy type A
2000 Kuniba, Misra, Okado, Takagi, Uchiyama relation of cyclage to f_0 in Demazure crystals of type A
2001 Shimozono/ S., Warnaar generalization of charge to Littlewood-Richardson tableaux in type A
2005 Lecouvey conjectural charge for types B, C, D

Definitions of charge

- from reading word, cyclage graph
- catabolism

Approach: charge from Ram–Yip formula for Macdonald polynomials from alcove paths via quantum Bruhat order

- gives charge directly on Kashiwara–Nakashima tableaux for single columns

Charge

1979 Lascoux, Schützenberger charge for type A

1995 Lascoux, Leclerc, Thibon from geometry of crystal

1997 Nakayashiki, Yamada relation charge and energy type A

2000 Kuniba, Misra, Okado, Takagi, Uchiyama relation of cyclage to f_0 in Demazure crystals of type A

2001 Shimozono/ S., Warnaar generalization of charge to Littlewood-Richardson tableaux in type A

2005 Lecouvey conjectural charge for types B, C, D

Definitions of charge

- from reading word, cyclage graph
- catabolism

Approach: charge from Ram–Yip formula for Macdonald polynomials from alcove paths via quantum Bruhat order

- gives charge directly on Kashiwara–Nakashima tableaux for single columns

Charge

1979 Lascoux, Schützenberger charge for type A
1995 Lascoux, Leclerc, Thibon from geometry of crystal
1997 Nakayashiki, Yamada relation charge and energy type A
2000 Kuniba, Misra, Okado, Takagi, Uchiyama relation of cyclage to f_0 in Demazure crystals of type A
2001 Shimozono/ S., Warnaar generalization of charge to Littlewood-Richardson tableaux in type A
2005 Lecouvey conjectural charge for types B, C, D

Definitions of charge

- from reading word, cyclage graph
- catabolism

Approach: charge from Ram–Yip formula for Macdonald polynomials from alcove paths via quantum Bruhat order

- gives charge directly on Kashiwara–Nakashima tableaux for single columns

Charge

1979 Lascoux, Schützenberger charge for type A
1995 Lascoux, Leclerc, Thibon from geometry of crystal
1997 Nakayashiki, Yamada relation charge and energy type A
2000 Kuniba, Misra, Okado, Takagi, Uchiyama relation of cyclage to f_0 in Demazure crystals of type A
2001 Shimozono/ S., Warnaar generalization of charge to Littlewood-Richardson tableaux in type A
2005 Lecouvey conjectural charge for types B, C, D

Definitions of charge

- from reading word, cyclage graph
- catabolism

Approach: charge from Ram–Yip formula for Macdonald polynomials from alcove paths via quantum Bruhat order

- gives charge directly on Kashiwara–Nakashima tableaux for single columns

Charge

1979 Lascoux, Schützenberger charge for type A
1995 Lascoux, Leclerc, Thibon from geometry of crystal
1997 Nakayashiki, Yamada relation charge and energy type A
2000 Kuniba, Misra, Okado, Takagi, Uchiyama relation of cyclage to f_0 in Demazure crystals of type A
2001 Shimozono/ S., Warnaar generalization of charge to Littlewood-Richardson tableaux in type A
2005 Lecouvey conjectural charge for types B, C, D

Definitions of charge

- from reading word, cyclage graph
- catabolism

Approach: charge from Ram–Yip formula for Macdonald polynomials from alcove paths via quantum Bruhat order

- gives charge directly on Kashiwara–Nakashima tableaux for single columns

Charge

1979 Lascoux, Schützenberger charge for type A
1995 Lascoux, Leclerc, Thibon from geometry of crystal
1997 Nakayashiki, Yamada relation charge and energy type A
2000 Kuniba, Misra, Okado, Takagi, Uchiyama relation of cyclage to f_0 in Demazure crystals of type A
2001 Shimozono/ S., Warnaar generalization of charge to Littlewood-Richardson tableaux in type A
2005 Lecouvey conjectural charge for types B, C, D

Definitions of charge

- from reading word, cyclage graph
- catabolism

Approach: charge from Ram–Yip formula for Macdonald polynomials from alcove paths via quantum Bruhat order

- gives charge directly on Kashiwara–Nakashima tableaux for single columns

Charge

1979 Lascoux, Schützenberger charge for type A
1995 Lascoux, Leclerc, Thibon from geometry of crystal
1997 Nakayashiki, Yamada relation charge and energy type A
2000 Kuniba, Misra, Okado, Takagi, Uchiyama relation of cyclage to f_0 in Demazure crystals of type A
2001 Shimozono/ S., Warnaar generalization of charge to Littlewood-Richardson tableaux in type A
2005 Lecouvey conjectural charge for types B, C, D

Definitions of charge

- from reading word, cyclage graph
- catabolism

Approach: charge from Ram–Yip formula for Macdonald polynomials from alcove paths via quantum Bruhat order

- gives charge directly on Kashiwara–Nakashima tableaux for single columns

Charge

1979 Lascoux, Schützenberger charge for type A
1995 Lascoux, Leclerc, Thibon from geometry of crystal
1997 Nakayashiki, Yamada relation charge and energy type A
2000 Kuniba, Misra, Okado, Takagi, Uchiyama relation of cyclage to f_0 in Demazure crystals of type A
2001 Shimozono/ S., Warnaar generalization of charge to Littlewood-Richardson tableaux in type A
2005 Lecouvey conjectural charge for types B, C, D

Definitions of charge

- from reading word, cyclage graph
- catabolism

Approach: charge from Ram–Yip formula for Macdonald polynomials from alcove paths via quantum Bruhat order

- gives charge directly on Kashiwara–Nakashima tableaux for single columns

Charge

1979 Lascoux, Schützenberger charge for type A
1995 Lascoux, Leclerc, Thibon from geometry of crystal
1997 Nakayashiki, Yamada relation charge and energy type A
2000 Kuniba, Misra, Okado, Takagi, Uchiyama relation of cyclage to f_0 in Demazure crystals of type A
2001 Shimozono/ S., Warnaar generalization of charge to Littlewood-Richardson tableaux in type A
2005 Lecouvey conjectural charge for types B, C, D

Definitions of charge

- from reading word, cyclage graph
- catabolism

Approach: charge from Ram–Yip formula for Macdonald polynomials from alcove paths via quantum Bruhat order

- gives charge directly on Kashiwara–Nakashima tableaux for single columns

Charge type A

Charge à la Lascoux and Schützenberger:
w word of partition content μ

Example

$\mu = (3,3,3,1)$

1132214323

$\text{charge}(1132214323) = 1 + 2 + 3 = 6$

Charge type A

Charge à la Lascoux and Schützenberger:
w word of partition content μ

Example

$\mu = (3, 3, 3, 1)$

1132214323

$\mathrm{charge}(1132214323) = 1 + 2 + 3 = 6$

Charge type A

Charge à la Lascoux and Schützenberger:
w word of partition content μ

Example

$\mu = (3, 3, 3, 1)$

1132214323 charge contribution 1

$\mathrm{charge}(1132214323) = 1 + 2 + 3 = 6$

Charge type A

Charge à la Lascoux and Schützenberger:
w word of partition content μ

Example

$\mu = (3, 3, 3, 1)$

1132214323 charge contribution 1

11 2 323

$\text{charge}(1132214323) = 1 + 2 + 3 = 6$

Charge type A

Charge à la Lascoux and Schützenberger:
w word of partition content μ

Example

$\mu = (3, 3, 3, 1)$

1132214323 charge contribution 1

11 2 323

$\text{charge}(1132214323) = 1 + 2 + 3 = 6$

Charge type A

Charge à la Lascoux and Schützenberger:
w word of partition content μ

Example

$\mu = (3, 3, 3, 1)$

1132214323	charge contribution 1
11 2 323	charge contribution 2

$\text{charge}(1132214323) = 1 + 2 + 3 = 6$

Charge type A

Charge à la Lascoux and Schützenberger:
w word of partition content μ

Example

$\mu = (3, 3, 3, 1)$

1132214323	charge contribution 1
11 2 323	charge contribution 2
1 2 3	

$\mathrm{charge}(1132214323) = 1 + 2 + 3 = 6$

Charge type A

Charge à la Lascoux and Schützenberger:
w word of partition content μ

Example

$\mu = (3,3,3,1)$

1132214323	charge contribution 1
11 2 323	charge contribution 2
1 2 3	

$\text{charge}(1132214323) = 1 + 2 + 3 = 6$

Charge type A

Charge à la Lascoux and Schützenberger:
w word of partition content μ

Example

$\mu = (3, 3, 3, 1)$

1132214323	charge contribution 1
11 2 323	charge contribution 2
1 2 3	charge contribution 3

$\text{charge}(1132214323) = 1 + 2 + 3 = 6$

Charge type A

Charge à la Lascoux and Schützenberger:
w word of partition content μ

Example

$\mu = (3,3,3,1)$

1132214323	charge contribution 1
11 2 323	charge contribution 2
1 2 3	charge contribution 3

$\mathrm{charge}(1132214323) = 1 + 2 + 3 = 6$

Charge on KN tableaux - type A

$$B_\mu := \bigotimes_{i=1}^{\mu_1} B^{\mu_i^t,1}$$

circular order $\prec_i$: $\quad i \prec_i i+1 \prec_i \cdots \prec_i n \prec_i 1 \prec_i \cdots \prec_i i-1$
construct reordered c from $b \in B_\mu$

Example

$$b = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & 2 \\ \hline 5 & 3 & 2 & \\ \cline{1-3} 6 & 4 & 4 & \\ \cline{1-3} \end{array} \qquad \text{and} \qquad c = \begin{array}{|c|c|c|c|}\hline 3 & 3 & 4 & 2 \\ \hline 5 & 2 & 2 & \\ \cline{1-3} 6 & 4 & 1 & \\ \cline{1-3} \end{array}$$

$$\mathrm{cw}(b) = \begin{pmatrix} 6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\ 1 & 1 & 3 & 2 & 2 & 1 & 4 & 3 & 2 & 3 \end{pmatrix}$$

Charge on KN tableaux - type A

$$B_\mu := \bigotimes_{i=1}^{\mu_1} B^{\mu_i^t,1}$$

circular order $\prec_i$: $i \prec_i i+1 \prec_i \cdots \prec_i n \prec_i 1 \prec_i \cdots \prec_i i-1$

construct reordered c from $b \in B_\mu$

Example

$$b = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & 2 \\ \hline 5 & 3 & 2 & \\ \cline{1-3} 6 & 4 & 4 & \\ \cline{1-3}\end{array} \quad \text{and} \quad c = \begin{array}{|c|c|c|c|}\hline 3 & 3 & 4 & 2 \\ \hline 5 & 2 & 2 & \\ \cline{1-3} 6 & 4 & 1 & \\ \cline{1-3}\end{array}$$

$$\mathrm{cw}(b) = \begin{pmatrix} 6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\ 1 & 1 & 3 & 2 & 2 & 1 & 4 & 3 & 2 & 3 \end{pmatrix}$$

Charge on KN tableaux - type A

$$B_\mu := \bigotimes_{i=1}^{\mu_1} B^{\mu_i^t,1}$$

circular order $\prec_i$: $i \prec_i i+1 \prec_i \cdots \prec_i n \prec_i 1 \prec_i \cdots \prec_i i-1$

construct reordered c from $b \in B_\mu$

Example

$$b = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & 2 \\ \hline 5 & 3 & 2 \\ \cline{1-3} 6 & 4 & 4 \\ \cline{1-3}\end{array} \quad \text{and} \quad c = \begin{array}{|c|c|c|c|}\hline 3 & 3 & 4 & 2 \\ \hline 5 & 2 & 2 \\ \cline{1-3} 6 & 4 & 1 \\ \cline{1-3}\end{array}$$

$$\mathrm{cw}(b) = \begin{pmatrix} 6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\ 1 & 1 & 3 & 2 & 2 & 1 & 4 & 3 & 2 & 3 \end{pmatrix}$$

Charge on KN tableaux - type A

Example

$$b = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & 2 \\ \hline 5 & 3 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3} 6 & 4 & 4 & \multicolumn{1}{c}{} \\ \cline{1-3}\end{array} \quad \text{and} \quad c = \begin{array}{|c|c|c|c|}\hline 3 & 3 & 4 & 2 \\ \hline 5 & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3} 6 & 4 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3}\end{array}$$

$$\mathrm{cw}(b) = \begin{pmatrix} 6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\ 1 & 1 & 3 & 2 & 2 & 1 & 4 & 3 & 2 & 3 \end{pmatrix}$$

$$\sum_{\gamma \in \mathrm{Des}(c)} \mathrm{arm}(\gamma) = \mathrm{charge}(\mathrm{cw}_2(b))$$

Charge on KN tableaux - type A

Example

$$b = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & 2 \\ \hline 5 & 3 & 2 \\ \cline{1-3} 6 & 4 & 4 \\ \cline{1-3}\end{array} \quad \text{and} \quad c = \begin{array}{|c|c|c|c|}\hline 3 & 3 & 4 & 2 \\ \hline 5 & 2 & 2 \\ \cline{1-3} 6 & 4 & 1 \\ \cline{1-3}\end{array}$$

$$\mathrm{cw}(b) = \begin{pmatrix} 6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\ 1 & 1 & 3 & 2 & 2 & 1 & 4 & 3 & 2 & 3 \end{pmatrix}$$

$$\sum_{\gamma \in \mathrm{Des}(c)} \mathrm{arm}(\gamma) = \mathrm{charge}(\mathrm{cw}_2(b))$$

Charge on KN tableaux - type A

Example

$$b = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & 2 \\ \hline 5 & 3 & 2 \\ \cline{1-3} 6 & 4 & 4 \\ \cline{1-3}\end{array} \quad \text{and} \quad c = \begin{array}{|c|c|c|c|}\hline \color{red}{3} & \color{red}{3} & \color{red}{4} & \color{red}{2} \\ \hline \color{blue}{5} & \color{blue}{2} & \color{blue}{2} \\ \cline{1-3} 6 & 4 & 1 \\ \cline{1-3}\end{array}$$

$$\mathrm{cw}(b) = \begin{pmatrix} 6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\ 1 & \color{blue}{1} & \color{red}{3} & 2 & \color{red}{2} & \color{red}{1} & \color{red}{4} & \color{blue}{3} & \color{blue}{2} & 3 \end{pmatrix}$$

$$\sum_{\gamma \in \mathrm{Des}(c)} \mathrm{arm}(\gamma) = \mathrm{charge}(\mathrm{cw}_2(b))$$

Charge on KN tableaux - type A

Example

$$b = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & 2 \\ \hline 5 & 3 & 2 & \\ \cline{1-3} 6 & 4 & 4 & \\ \cline{1-3}\end{array} \quad \text{and} \quad c = \begin{array}{|c|c|c|c|}\hline 3 & 3 & \mathbf{4} & 2 \\ \hline \mathbf{5} & 2 & 2 & \\ \cline{1-3} \mathbf{6} & \mathbf{4} & 1 & \\ \cline{1-3}\end{array}$$

$$\mathrm{cw}(b) = \begin{pmatrix} 6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\ 1_2 & 1_2 & 3_1 & 2_1 & 2 & 1 & 4 & 3 & 2 & 3 \end{pmatrix}$$

$$\sum_{\gamma \in \mathrm{Des}(c)} \mathrm{arm}(\gamma) = \mathrm{charge}(\mathrm{cw}_2(b))$$

Charge on KN tableaux - type A

Example

$$b = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & 2 \\ \hline 5 & 3 & 2 & \\ \cline{1-3} 6 & 4 & 4 & \\ \cline{1-3}\end{array} \quad \text{and} \quad c = \begin{array}{|c|c|c|c|}\hline 3 & 3 & \mathbf{4} & 2 \\ \hline \mathbf{5} & 2 & 2 & \\ \cline{1-3} \mathbf{6} & \mathbf{4} & 1 & \\ \cline{1-3}\end{array}$$

$$\mathrm{cw}(b) = \begin{pmatrix} 6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\ 1_2 & 1_2 & 3_1 & 2_1 & 2 & 1 & 4 & 3 & 2 & 3 \end{pmatrix}$$

$$\sum_{\gamma \in \mathrm{Des}(c)} \mathrm{arm}(\gamma) = \mathrm{charge}(\mathrm{cw}_2(b))$$

Charge type C

Example

$$
\begin{array}{|c|}\hline \overline{5} \\ \hline \overline{3} \\ \hline \overline{2} \\ \hline \overline{1} \\ \hline \end{array}
\otimes
\begin{array}{|c|}\hline 3 \\ \hline \overline{4} \\ \hline \overline{3} \\ \hline \end{array}
\otimes
\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \overline{3} \\ \hline \end{array}\,.
$$

Doubling the columns and cyclically reordered:

$$
\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1' & 2 & 2' & 3 & 3' \\ \hline \end{array}
$$

$$
\begin{array}{|c|c|c|c|c|c|}\hline \overline{5} & \overline{5} & 2 & 3 & 1 & 1 \\ \hline \overline{3} & \overline{3} & \overline{4} & \overline{4} & 2 & 3 \\ \hline \overline{2} & \overline{2} & \overline{3} & \overline{2} & \overline{3} & \overline{2} \\ \hline \overline{1} & \overline{1} \\ \cline{1-2} \end{array}
\qquad
c = \begin{array}{|c|c|c|c|c|c|}\hline \overline{5} & \overline{5} & \overline{4} & \overline{4} & \overline{3} & \overline{2} \\ \hline \overline{3} & \overline{3} & \overline{3} & \overline{\mathbf{2}} & 1 & 1 \\ \hline \overline{2} & \overline{\mathbf{2}} & 2 & \mathbf{3} & 2 & 3 \\ \hline \overline{1} & \overline{1} \\ \cline{1-2} \end{array}
$$

Charge type C

Example

$$\begin{array}{|c|}\hline \overline{5} \\ \hline \overline{3} \\ \hline \overline{2} \\ \hline \overline{1} \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 3 \\ \hline \overline{4} \\ \hline \overline{3} \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \overline{3} \\ \hline \end{array} .$$

Doubling the columns and cyclically reordered:

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1' & 2 & 2' & 3 & 3' \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|}\hline \overline{5} & \overline{5} & 2 & 3 & 1 & 1 \\ \hline \overline{3} & \overline{3} & \overline{4} & \overline{4} & 2 & 3 \\ \hline \overline{2} & \overline{2} & \overline{3} & \overline{2} & \overline{3} & \overline{2} \\ \hline \overline{1} & \overline{1} \\ \cline{1-2} \end{array}$$

$$c = \begin{array}{|c|c|c|c|c|c|}\hline \overline{5} & \overline{5} & \overline{4} & \overline{4} & \overline{3} & \overline{2} \\ \hline \overline{3} & \overline{3} & \overline{3} & \overline{\mathbf{2}} & 1 & 1 \\ \hline \overline{2} & \overline{\mathbf{2}} & 2 & \mathbf{3} & 2 & 3 \\ \hline \overline{1} & \overline{1} \\ \cline{1-2} \end{array}$$

Charge type C

Example

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1' & 2 & 2' & 3 & 3' \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|}\hline \bar{5} & \bar{5} & 2 & 3 & 1 & 1 \\ \hline \bar{3} & \bar{3} & \bar{4} & \bar{4} & 2 & 3 \\ \hline \bar{2} & \bar{2} & \bar{3} & \bar{2} & \bar{3} & \bar{2} \\ \hline \bar{1} & \bar{1} \\ \cline{1-2} \end{array}$$

$$c = \begin{array}{|c|c|c|c|c|c|}\hline \bar{5} & \bar{5} & \bar{4} & \bar{4} & \bar{3} & \bar{2} \\ \hline \bar{3} & \bar{3} & \bar{3} & \bar{\mathbf{2}} & 1 & 1 \\ \hline \bar{2} & \bar{\mathbf{2}} & 2 & \mathbf{3} & 2 & 3 \\ \hline \bar{1} & \bar{1} \\ \cline{1-2} \end{array}$$

$$\mathrm{charge}(b) = (2+2+4)/2 = 4$$

$$\mathrm{charge}(b) = \frac{1}{2} \sum_{\gamma \in \mathrm{Des}(c)} \mathrm{arm}(\gamma)$$

Charge type C

Example

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1' & 2 & 2' & 3 & 3' \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|}\hline \bar{5} & \bar{5} & 2 & 3 & 1 & 1 \\ \hline \bar{3} & \bar{3} & \bar{4} & \bar{4} & 2 & 3 \\ \hline \bar{2} & \bar{2} & \bar{3} & \bar{2} & \bar{3} & \bar{2} \\ \hline \bar{1} & \bar{1} \\ \cline{1-2} \end{array}$$

$$c = \begin{array}{|c|c|c|c|c|c|}\hline \bar{5} & \bar{5} & \bar{4} & \bar{4} & \bar{3} & \bar{2} \\ \hline \bar{3} & \bar{3} & \bar{3} & \bar{\mathbf{2}} & 1 & 1 \\ \hline \bar{2} & \bar{\mathbf{2}} & 2 & \mathbf{3} & 2 & 3 \\ \hline \bar{1} & \bar{1} \\ \cline{1-2} \end{array}$$

$$\mathrm{charge}(b) = (2+2+4)/2 = 4$$

$$\mathrm{charge}(b) = \frac{1}{2} \sum_{\gamma \in \mathrm{Des}(c)} \mathrm{arm}(\gamma)$$

Relation between charge and energy

Theorem (Lenart, S. 2011)

$B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ of type $A_n^{(1)}$ or type $C_n^{(1)}$
Then for $b \in B$

$$D(b) = -\mathrm{charge}(b)$$

Idea of proof:

- show $D(e_i b) = D(b)$ and $\mathrm{charge}(e_i b) = \mathrm{charge}(b)$ for $i = 1, 2 \ldots, n$
- show $D(e_0 b) = D(b) + 1$ and $\mathrm{charge}(e_0 b) = \mathrm{charge}(b) + 1$ if $\varphi_0(b) \geq 1$ (Demazure arrow)
- for type $C_n^{(1)}$ for every b there exists path to type $A_n^{(1)}$ using Demazure arrows

Relation between charge and energy

Theorem (Lenart, S. 2011)

$B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ of type $A_n^{(1)}$ or type $C_n^{(1)}$
Then for $b \in B$

$$D(b) = -\mathrm{charge}(b)$$

Idea of proof:

- show $D(e_i b) = D(b)$ and $\mathrm{charge}(e_i b) = \mathrm{charge}(b)$ for $i = 1, 2 \ldots, n$
- show $D(e_0 b) = D(b) + 1$ and $\mathrm{charge}(e_0 b) = \mathrm{charge}(b) + 1$ if $\varphi_0(b) \geq 1$ (Demazure arrow)
- for type $C_n^{(1)}$ for every b there exists path to type $A_n^{(1)}$ using Demazure arrows

Relation between charge and energy

Theorem (Lenart, S. 2011)

$B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ of type $A_n^{(1)}$ or type $C_n^{(1)}$
Then for $b \in B$

$$D(b) = -\mathrm{charge}(b)$$

Idea of proof:

- show $D(e_i b) = D(b)$ and $\mathrm{charge}(e_i b) = \mathrm{charge}(b)$ for $i = 1, 2 \ldots, n$
- show $D(e_0 b) = D(b) + 1$ and $\mathrm{charge}(e_0 b) = \mathrm{charge}(b) + 1$ if $\varphi_0(b) \geq 1$ (Demazure arrow)
- for type $C_n^{(1)}$ for every b there exists path to type $A_n^{(1)}$ using Demazure arrows

Relation between charge and energy

Theorem (Lenart, S. 2011)

$B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ of type $A_n^{(1)}$ or type $C_n^{(1)}$
Then for $b \in B$

$$D(b) = -\mathrm{charge}(b)$$

Idea of proof:

- show $D(e_i b) = D(b)$ and $\mathrm{charge}(e_i b) = \mathrm{charge}(b)$ for $i = 1, 2 \ldots, n$
- show $D(e_0 b) = D(b) + 1$ and $\mathrm{charge}(e_0 b) = \mathrm{charge}(b) + 1$ if $\varphi_0(b) \geq 1$ (Demazure arrow)
- for type $C_n^{(1)}$ for every b there exists path to type $A_n^{(1)}$ using Demazure arrows

Outline

Crystals

Affine crystals

KR crystals

Perfectness

Demazure crystals

Charge

Affine Schubert calculus

Progress on ... affine Schubert calculus

- **Symmetric functions and geometry:**
  - k-Schur functions, affine Stanley symmetric functions
    → joint with Thomas Lam and Mark Shimozono for type C
    (arXiv:0710.2720)
  - K-theory of the affine Grassmannian, stable affine Grothendieck polynomials
    → joint with Thomas Lam and Mark Shimozono
    (arXiv:0901.1506)
  - Murnaghan–Nakayama rule for k-Schur functions
    → joint with Jason Bandlow and Mike Zabrocki

Progress on ... affine Schubert calculus

- **Symmetric functions and geometry:**
  - k-Schur functions, affine Stanley symmetric functions
    $\rightarrow$ joint with Thomas Lam and Mark Shimozono for type C (arXiv:0710.2720)
  - K-theory of the affine Grassmannian, stable affine Grothendieck polynomials
    $\rightarrow$ joint with Thomas Lam and Mark Shimozono (arXiv:0901.1506)
  - Murnaghan–Nakayama rule for k-Schur functions
    $\rightarrow$ joint with Jason Bandlow and Mike Zabrocki

Progress on ... affine Schubert calculus

- **Symmetric functions and geometry:**
  - k-Schur functions, affine Stanley symmetric functions
    $\rightarrow$ joint with Thomas Lam and Mark Shimozono for type C (arXiv:0710.2720)
  - K-theory of the affine Grassmannian, stable affine Grothendieck polynomials
    $\rightarrow$ joint with Thomas Lam and Mark Shimozono (arXiv:0901.1506)
  - Murnaghan–Nakayama rule for k-Schur functions
    $\rightarrow$ joint with Jason Bandlow and Mike Zabrocki

Schubert calculus

- **Enumerative Geometry:** counting subspaces satisfying certain intersection conditions (Hilbert's 15th problem) Schubert, Pieri, Giambelli,... 1874
- **Cohomology:** computations in cohomology ring of the Grassmannian $H^*(G/P)$ with $G = SL_n(\mathbb{C})$ and $P \subset G$ maximal parabolic 1950's
- **Symmetric Functions:** cohomology ring of Grassmannian (with its natural Schubert basis) same as the algebra of symmetric functions (with Schur basis) 1950's
- **Combinatorics:** multiplication of Schubert basis governed by Littlewood-Richardson rule 1970's

Schubert calculus

- **Enumerative Geometry:** counting subspaces satisfying certain intersection conditions (Hilbert's 15th problem) Schubert, Pieri, Giambelli,... 1874
- **Cohomology:** computations in cohomology ring of the Grassmannian $H^*(G/P)$ with $G = SL_n(\mathbb{C})$ and $P \subset G$ maximal parabolic 1950's
- **Symmetric Functions:** cohomology ring of Grassmannian (with its natural Schubert basis) same as the algebra of symmetric functions (with Schur basis) 1950's
- **Combinatorics:** multiplication of Schubert basis governed by Littlewood-Richardson rule 1970's

Schubert calculus

- **Enumerative Geometry:** counting subspaces satisfying certain intersection conditions (Hilbert's 15th problem) Schubert, Pieri, Giambelli,... 1874
- **Cohomology:** computations in cohomology ring of the Grassmannian $H^*(G/P)$ with $G = SL_n(\mathbb{C})$ and $P \subset G$ maximal parabolic 1950's
- **Symmetric Functions:** cohomology ring of Grassmannian (with its natural Schubert basis) same as the algebra of symmetric functions (with Schur basis) 1950's
- **Combinatorics:** multiplication of Schubert basis governed by Littlewood-Richardson rule 1970's

Schubert calculus

- **Enumerative Geometry:** counting subspaces satisfying certain intersection conditions (Hilbert's 15th problem) Schubert, Pieri, Giambelli,... 1874
- **Cohomology:** computations in cohomology ring of the Grassmannian $H^*(G/P)$ with $G = SL_n(\mathbb{C})$ and $P \subset G$ maximal parabolic 1950's
- **Symmetric Functions:** cohomology ring of Grassmannian (with its natural Schubert basis) same as the algebra of symmetric functions (with Schur basis) 1950's
- **Combinatorics:** multiplication of Schubert basis governed by Littlewood-Richardson rule 1970's

Affine Schubert calculus

Definition

G affine Kac–Moody group
$P \subset G$ maximal parabolic subgroup
G/P affine Grassmannian Gr

Example: $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$
affine Grassmannian $Gr = SL_{k+1}(\mathcal{K})/SL_{k+1}(\mathcal{O})$

Theorem (Lam)

Schubert bases of $H_*(Gr)$ and $H^*(Gr)$ are given by k-Schur functions and affine Stanley symmetric functions of Lascoux, Lapointe, Morse

Structure constants include genus zero Gromov-Witten invariants or fusion coefficients

Affine Schubert calculus

Definition

G affine Kac–Moody group
$P \subset G$ maximal parabolic subgroup
G/P affine Grassmannian Gr

Example: $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$
affine Grassmannian $Gr = SL_{k+1}(\mathcal{K})/SL_{k+1}(\mathcal{O})$

Theorem (Lam)

Schubert bases of $H_*(Gr)$ and $H^*(Gr)$ are given by k-Schur functions and affine Stanley symmetric functions of Lascoux, Lapointe, Morse

Structure constants include genus zero Gromov-Witten invariants or fusion coefficients

Affine Schubert calculus

Definition

G affine Kac–Moody group
$P \subset G$ maximal parabolic subgroup
G/P affine Grassmannian Gr

Example: $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$
affine Grassmannian $Gr = SL_{k+1}(\mathcal{K})/SL_{k+1}(\mathcal{O})$

Theorem (Lam)

Schubert bases of $H_*(Gr)$ and $H^*(Gr)$ are given by k-Schur functions and affine Stanley symmetric functions of Lascoux, Lapointe, Morse

Structure constants include genus zero Gromov-Witten invariants or fusion coefficients

Affine Schubert calculus

Definition

G affine Kac–Moody group
$P \subset G$ maximal parabolic subgroup
G/P affine Grassmannian Gr

Example: $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$
affine Grassmannian $Gr = SL_{k+1}(\mathcal{K})/SL_{k+1}(\mathcal{O})$

Theorem (Lam)

Schubert bases of $H_*(Gr)$ and $H^*(Gr)$ are given by k-Schur functions and affine Stanley symmetric functions of Lascoux, Lapointe, Morse

Structure constants include genus zero Gromov-Witten invariants or fusion coefficients

nilHecke algebra

Definition (nilHecke algebra)

The nilHecke algebra

- generators $A_1, \ldots, A_{n-1}$
- relations

$$\begin{aligned} A_iA_j &= A_jA_i && \text{for } |i-j| \geq 2 \\ A_iA_{i+1}A_i &= A_{i+1}A_iA_{i+1} \\ A_i^2 &= 0 \end{aligned}$$

Stanley symmetric functions for other types

- For each Weyl group W one can construct a new nilHecke algebra by taking the associated graded $\mathbb{C}[W]$.
- Finding Stanley symmetric functions for each W is equivalent to finding a particular commutative subalgebra of the nilHecke algebra.

Theorem (Lam; LSS 07)

Schubert bases of $H_*(Gr)$ and $H^*(Gr)$ are given by k-Schur functions and affine Stanley symmetric functions for type $A_n^{(1)}$ and $C_n^{(1)}$.

Theorem (LSS 09)

Schubert bases of $K_*(Gr)$ and $K^*(Gr)$ are given by K-k-Schur functions and affine stable Grothendieck polynomials.

Stanley symmetric functions for other types

- For each Weyl group W one can construct a new nilHecke algebra by taking the associated graded $\mathbb{C}[W]$.
- Finding Stanley symmetric functions for each W is equivalent to finding a particular commutative subalgebra of the nilHecke algebra.

Theorem (Lam; LSS 07)

Schubert bases of $H_*(Gr)$ and $H^*(Gr)$ are given by k-Schur functions and affine Stanley symmetric functions for type $A_n^{(1)}$ and $C_n^{(1)}$.

Theorem (LSS 09)

Schubert bases of $K_*(Gr)$ and $K^*(Gr)$ are given by K-k-Schur functions and affine stable Grothendieck polynomials.

Stanley symmetric functions for other types

- For each Weyl group W one can construct a new nilHecke algebra by taking the associated graded $\mathbb{C}[W]$.
- Finding Stanley symmetric functions for each W is equivalent to finding a particular commutative subalgebra of the nilHecke algebra.

Theorem (Lam; LSS 07)

Schubert bases of $H_*(Gr)$ and $H^*(Gr)$ are given by k-Schur functions and affine Stanley symmetric functions for type $A_n^{(1)}$ and $C_n^{(1)}$.

Theorem (LSS 09)

Schubert bases of $K_*(Gr)$ and $K^*(Gr)$ are given by K-k-Schur functions and affine stable Grothendieck polynomials.

Stanley symmetric functions for other types

- For each Weyl group W one can construct a new nilHecke algebra by taking the associated graded $\mathbb{C}[W]$.
- Finding Stanley symmetric functions for each W is equivalent to finding a particular commutative subalgebra of the nilHecke algebra.

Theorem (Lam; LSS 07)

Schubert bases of $H_*(Gr)$ and $H^*(Gr)$ are given by k-Schur functions and affine Stanley symmetric functions for type $A_n^{(1)}$ and $C_n^{(1)}$.

Theorem (LSS 09)

Schubert bases of $K_*(Gr)$ and $K^*(Gr)$ are given by K-k-Schur functions and affine stable Grothendieck polynomials.

Relation to KR crystals

k-Schur functions	$s_\lambda^{(k)}$
Structure coefficients	$s_\lambda^{(k)} s_\mu^{(k)} = \sum_\nu c_{\lambda\mu}^{k,\nu} s_\nu^{(k)}$

Observation: (inspired by Postnikov and Stroppel/Korff)

- s_λ evaluated at crystal operators acting on $B^{1,k}$ of type $A_{n-1}^{(1)}$ yields fusion coefficients
- s_λ evaluated at crystal operators acting on $B^{n,1}$ of type $A_{n+k-1}^{(1)}$ yields quantum cohomology structure coefficients

Relation to KR crystals

k-Schur functions	$s_\lambda^{(k)}$
Structure coefficients	$s_\lambda^{(k)} s_\mu^{(k)} = \sum_\nu c_{\lambda\mu}^{k,\nu} s_\nu^{(k)}$

Observation: (inspired by Postnikov and Stroppel/Korff)

- s_λ evaluated at crystal operators acting on $B^{1,k}$ of type $A_{n-1}^{(1)}$ yields fusion coefficients
- s_λ evaluated at crystal operators acting on $B^{n,1}$ of type $A_{n+k-1}^{(1)}$ yields quantum cohomology structure coefficients

Relation to KR crystals

k-Schur functions	$s_\lambda^{(k)}$
Structure coefficients	$s_\lambda^{(k)} s_\mu^{(k)} = \sum_\nu c_{\lambda\mu}^{k,\nu} s_\nu^{(k)}$

Observation: (inspired by Postnikov and Stroppel/Korff)

- s_λ evaluated at crystal operators acting on $B^{1,k}$ of type $A_{n-1}^{(1)}$ yields fusion coefficients
- s_λ evaluated at crystal operators acting on $B^{n,1}$ of type $A_{n+k-1}^{(1)}$ yields quantum cohomology structure coefficients

Relation to KR crystals

k-Schur functions	$s_\lambda^{(k)}$
Structure coefficients	$s_\lambda^{(k)} s_\mu^{(k)} = \sum_\nu c_{\lambda\mu}^{k,\nu} s_\nu^{(k)}$

Observation: (inspired by Postnikov and Stroppel/Korff)

- s_λ evaluated at crystal operators acting on $B^{1,k}$ of type $A_{n-1}^{(1)}$ yields fusion coefficients
- s_λ evaluated at crystal operators acting on $B^{n,1}$ of type $A_{n+k-1}^{(1)}$ yields quantum cohomology structure coefficients