Ribbon Schur functions with full support and Schur positivity

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Outline

1. Maximal support and Schur positivity

2. Classification of ribbon Schur functions with interval support
The Schur functions are considered to be the most important basis for the ring of symmetric functions.

Given partitions $\mu \subseteq \lambda$, $A := \lambda/\mu$

$$s_A = \sum_{\nu} c_A^\nu s_\nu,$$

where $c_A^\nu$ is the number of SSYT of shape $A$ and content $\nu$, satisfying the Littlewood-Richardson rule.
Schur functions and support

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- Given partitions $\mu \subseteq \lambda$, $A := \lambda/\mu$

\[
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\]

where $c_A^{\nu}$ is the number of SSYT of shape $A$ and content $\nu$, satisfying the Littlewood-Richardson rule.

- $A = 3321/211 = \young(1,1,1,2,2,3,3)$

\[
s_A = s_{2111} + 2s_{221} + s_{311} + s_{32}
\]
Schur functions and support

- $A = \frac{3321}{211} = \square$

- $s_A = s_{2111} + 2s_{221} + s_{311} + s_{32}$

- $r(A)$ is the partition consisting of the row lengths of $A$, and $c(A)$ is defined similarly. The support of $A$, considered as a subposet of the dominance lattice, has a top element $r(A)'$ and a bottom element $c(A)$,

$$s_A = \sum_{c(A) \preceq \nu' \preceq r(A)'} c_A^{\nu'} s_\nu.$$
Schur functions and support

- $A = 3321/211 = \begin{array}{ccc}
  & & \\
  & 1 & \\
 1 & & 1
\end{array}$

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- $\text{supp}(A) = \{ \nu' : c_A^{\nu'} > 0 \}$
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- \( r(A) \) is the partition consisting of the row lengths of \( A \), and \( c(A) \) is defined similarly. The support of \( A \), considered as a subposet of the *dominance lattice*, has a top element \( r(A)' \) and a bottom element \( c(A) \),

\[ s_A = \sum_{c(A) \preceq \nu' \preceq r(A)'} c_A^{\nu'} s_{\nu}. \]

- \( \text{supp}(A) = \{ \nu' : c_A^{\nu'} > 0 \} \subseteq [c(A), r(A)'] = [221; 41] \]
Given skew shapes $A$ and $B$, when is $s_A - s_B$ Schur positive?
Schur positivity

Given skew shapes $A$ and $B$, when is $s_A - s_B$ Schur positive?

$s_A - s_B$ is Schur positive only if $\text{supp}(B) \subseteq \text{supp}(A)$. 
Schur positivity

- Given skew shapes \( A \) and \( B \), when is \( s_A - s_B \) Schur positive?

\[ s_A - s_B \text{ is Schur positive only if } \text{supp}(B) \subseteq \text{supp}(A). \]

- \( A = 3321/211 = \begin{array}{ccc} & & \bigcirc \\ & \bigcirc & \\ \bigcirc & & \end{array} \), \( B = 3311/21 = \begin{array}{ccc} & & \bigcirc \\ \bigcirc & & \end{array} \)

\[ s_A = s_{32} + s_{211} + 2s_{221} + s_{311}, \quad s_B = s_{32} + s_{211} + 1s_{221} + s_{311} \]

\[ \text{supp}A = \text{supp}B, \]

\[ s_A - s_B = s_{221} \text{ is Schur positive but } s_B - s_A = -1s_{221} \text{ is not.} \]
Skew shape equivalences

- Skew shapes yielding the same Schur function
  \( A \) an \( B \) are said to be \textit{Schur equivalent} if \( s_A = s_B \)
  \[ [A] = \{ B : s_A = s_B \} \]

\[
A = \begin{array}{c}
\includegraphics[width=0.5\textwidth]{skew_shape1}
\end{array}
\quad \text{and} \quad
B = \begin{array}{c}
\includegraphics[width=0.5\textwidth]{skew_shape2}
\end{array}
\]

are not Schur equivalent but

\[
A \quad \text{and its antipodal rotation} \quad A^\pi = \begin{array}{c}
\includegraphics[width=0.5\textwidth]{skew_shape3}
\end{array}
\]

are.
Skew shape equivalences

- Skew shapes yielding the same Schur function
  \( A \) an \( B \) are said to be Schur equivalent if \( s_A = s_B \)
  \([A] = \{B : s_A = s_B\}\)
  \( A = \) \[
  \begin{array}{cccc}
  & & \checkmark & \\
  & \checkmark & & \\
  \checkmark & & & \\
  \end{array}
  \]
  and \( B = \) \[
  \begin{array}{cccc}
  & \checkmark & & \\
  & & \checkmark & \\
  & \checkmark & & \\
  \end{array}
  \]
  are not Schur equivalent but
  \( A \) and its antipodal rotation \( A^\pi = \) \[
  \begin{array}{cccc}
  & & \checkmark & \\
  & \checkmark & & \\
  \checkmark & & & \\
  \end{array}
  \]
  are.

- Skew shapes yielding the same support
  \( A \) an \( B \) are said to be support equivalent if \( \text{supp}A = \text{supp}B \)
  \([A] = \{B : \text{supp}B = \text{supp}A\}\)
  \( A, B \) and \( A^\pi \) are support equivalent.
Partial orders on skew shape classes

- $P_N$ is the poset of all Schur equivalence classes $[A]$ such that $A$ has $N$ boxes.

  $[A] \geq_s [B]$ if $s_A - s_B$ is Schur positive

- $\text{Supp}_N$ is the poset of all support equivalence classes $\lfloor A \rfloor$ such that $A$ has $N$ boxes.

  $\lfloor A \rfloor \geq_{\text{supp}} \lfloor B \rfloor$ if the support of $B$ is contained in that of $A$

  $A = \begin{array}{ccc} \\
    & & \\
    & \& \\
    \& & \\
\end{array}$ and $B = \begin{array}{ccc} \\
    & & \\
    & \& \\
    \& & \\
\end{array}$

  $[B] <_s [A]$ in $P_5$

  $[B] = [A]$ in $\text{Supp}_5$
Maximal supports among connected skew shapes


**Theorem**
An element $\lfloor R \rfloor$ in $\text{Supp}_N$ is a maximal connected element iff $R$ is a ribbon in which the lengths of any two empty rows differ by at most one and the lengths of any two nonempty columns differ by at most one. The $\text{supp} R$ is the full interval.
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\[ R = (2, 3, 3) = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad [R] = [R^\pi] \]

\[ R' = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad [R]' = [R'] \]
Ribbon shapes with full support

- **PROBLEM**: What are the ribbon shapes $R = (r_1, \ldots, r_s)$ whose support consists of the whole interval in the dominance lattice?
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What are the ribbon shapes $R = (r_1, \ldots, r_s)$ with $\text{supp} R = [(r_{k_1}, \ldots, r_{k_s}); (\sum_{j \geq 1} r_j - s + 1, s - 1)]$?

$s - 1$ the number of rows with length two in $R$

$R = (2, 3, 3) = \begin{array}{ccc}
  & & \\
  & X & \\
  X & & \\
\end{array} \quad \text{supp} R = [3^2 2; 6 2]
Ribbon Schur functions with full support
Maximal support and Schur positivity

\[ R = (32522271) \]
\[ (75322221) \leq \xi = (888) \leq (24 - 7, 7). \]
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\[ R = (32522271) \]
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There is vertical space to put the last string of length 8.

\[ \sum_{i=1}^{2} \xi_i - \sum_{i=1}^{2} r_i = (8 + 8) - (7 + 5) = 4 > p = 3, \]
\[ \xi_3 = 8 < \sum_{i \geq 3} r_i - p = 3 + 2 + 2 + 2 + 2 + 1 - 3, \quad (888) \in \text{supp}(R) \]
\[ \xi_3 = 8 = (\sum_{i \geq 3} r_i - p) - 1 \]
There are enough boxes to put the last string of length 7 but not enough vertical space: a row of length two remains.

\[ \begin{align*}
\xi_1 - r_1 + \xi_2 - r_2 & = 7 - 6 + 7 - 6 \\
& \leq 3 - 1 + 3 = 3 \\
\xi_3 & = 7 \\
& \geq 2 + 3 + 2 + 2 - 2 \\
\xi & = (777) \in \text{supp } R
\end{align*} \]

- \( R = (662322) \quad (6^2 \ 3 \ 2^3) \preceq (777) \preceq (21, 21 - 5) \)
$$R = (662322) \ (6^2 \ 3 \ 2^3) \preceq (7 \ 7 \ 7) \preceq (21, 21 - 5)$$
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There are enough boxes to put the last string of length 7 but not enough vertical space: a row of length two remains.

\[ \xi_1 - r_1 + \xi_2 - r_2 = 7 - 6 + 7 - 6 \leq 3 - 1 \quad p = 3 \quad \xi_3 = 7 \geq 2 + 3 + 2 + 2 - 2 \]

\[ \xi = (777) \notin \text{supp} R \]
There is not enough vertical space to put a third string of length 7, but there is enough vertical space to put two more strings: one of length 6 and another of length 1.

\[ \xi_1 - r_1 + \xi_2 - r_2 = 7 - 6 + 7 = 2 \]

\[ p = 3 \]

\[ \xi_3 = 6 = 2 + 3 + 2 + 2 \]

\[ (777) \in \text{supp } R \]

\[ (7761) \leq (777) \leq (21, 21 - 5) \]
\[ R = (662322) \ (6^2 \ 3 \ 2^3) \preceq (7 \ 7 \ 6 \ 1) \preceq (7 \ 7 \ 7) \preceq (21, 21 - 5) \]
Ribbon Schur functions with full support
Maximal support and Schur positivity

\[ R = (662322) \preceq (6^2 \, 3 \, 2^3) \preceq (7 \, 7 \, 6 \, 1) \preceq (7 \, 7 \, 7) \preceq (21, 21 - 5) \]
There is not enough vertical space to put a third string of length 7 but there is enough vertical space to put two more strings: one of length 6 and another of length 1.

\[ \xi_1 - r_1 + \xi_2 - r_2 = 7 - 6 + 7 - 6 = 2 \]

\[ p = 3 \]

\[ \xi_3 = 6 = 2 + 3 + 2 + 2 \]

\[ (777) / \in \text{supp } R \]

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\[ R = (662322) (6^2 3 2^3) \leq (7761) \leq (777) \leq (21, 21 - 5) \]
There is not enough vertical space to put a third string of length 7 but there is enough vertical space to put two more strings: one of length 6 and another of length 1.

\[\xi_1 - r_1 + \xi_2 - r_2 = 7 - 6 + 7 - 6 = 2 \quad p = 3 \quad \xi_3 = 6 = 2 + 3 + 2 + 2 - 3\]

\((777) \notin \text{supp} R \quad (7761) \in \text{supp} R\]
$R = (662322) \ (6^2 \ 3 \ 2^3) \preceq (7 \ 7 \ 6 \ 1) \preceq (7 \ 7 \ 7) \preceq (8 \ 7 \ 6) \preceq (21, 21 - 5)$
$R = (662322) (6^2 3^2 2^3) \preceq (7761) \preceq (777) \preceq (876) \preceq (21,21-5)$
$R = (662322) \quad (6^2 \, 3 \, 2^3) \preceq (7 \, 7 \, 6 \, 1) \preceq (7 \, 7 \, 7) \preceq (8 \, 7 \, 6) \preceq (21, 21 - 5)$
Ribbon Schur functions with full support
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\[ R = (662322) (6^2 3 2^3) \preceq (7 7 6 1) \preceq (7 7 7) \preceq (8 7 6) \preceq (21, 21 - 5) \]
\[ R = (662322) \quad (6^2 \ 3 \ 2^3) \preceq (7 \ 7 \ 6 \ 1) \preceq (7 \ 7 \ 7) \preceq (8 \ 7 \ 6) \preceq (21, 21 - 5) \]
\( R = (662322) \)  \((6^2 3 2^3) \preceq (7 7 6 1) \preceq (7 7 7) \preceq (8 7 6) \preceq (21, 21 - 5) \)

\[ \xi_1 - r_1 + \xi_2 - r_2 = 8 - 6 + 7 - 6 = 3 = p = 3 \quad \xi_3 = 6 = 2 + 3 + 2 + 2 - 3 \]

\((777) \notin \text{supp} R, \quad (7761) \in \text{supp} R, \quad (876) \in \text{supp} R \)
Lemma

Let $\xi = (\xi_1, \ldots, \xi_t)$ be a partition in the Schur interval $[(r_{k_1}, \ldots, r_{k_s}); (\sum_{j \geq 1} r_j - s + 1, s - 1)]$ but not in the support of $R$. Then there exists an $1 \leq i \leq t - 1$ such that if $p \geq 1$ is the number of rows with length two among the columns indexed by $S = \{k_{i+1}, \ldots, k_s\}$, one has

$$\xi_{i+1} \geq \sum_{q \in S} r_q - p + 1 \quad \left(\Rightarrow \sum_{j=1}^{i} (\xi_j - r_{k_j}) \leq p - 1\right).$$

(1)

This implies that the number $p$ of rows of length two, among the adjacent columns indexed by $S$ in $R$, can not be shortened by what remains $\sum_{j=1}^{i} (\xi_j - r_{k_j})$. 

Ribbon Schur functions with full support
Maximal support and Schur positivity

Ribbon shape LR fillings
**Theorem**

Let \( R = (r_1, \ldots, r_s), \ s \geq 2, \) be a ribbon. Then

\[
\text{supp} R \subseteq \left((r_{k_1}, \ldots, r_{k_s}); (\sum_{j \geq 1} r_j - s + 1, s - 1)\right)
\]

if and only if for some \( 1 \leq i \leq s - 2 \) with \( p > 0 \) rows of length two among the columns indexed by \( \{k_{i+1}, \ldots, k_s\} \), there exist \( g_1, \ldots, g_i \geq 0 \) with \( \sum_{j=1}^i g_j = p - 1 \), such that

\[
\begin{align*}
    r_{k_1} + g_1 &\geq \sum_{j=i+1}^s r_{k_j} - p + 1 \\
    & \vdots \\
    r_{k_{i-1}} + g_{i-1} &\geq \sum_{j=i+1}^s r_{k_j} - p + 1 \\
    r_{k_i} + g_i &\geq \sum_{j=i+1}^s r_{k_j} - p + 1
\end{align*}
\]

Moreover \( (r_{k_1} + g_1, \ldots, r_{k_i} + g_i, \sum_{j=i+1}^s r_{k_j} - p + 1) \notin \text{supp} R \).
\[ R = (662322) \ (6^2 \ 3 \ 2^3) \leq (7 \ 7 \ 7) \leq (21, 21 - 5) \]
\[ R = (662322) \ (6^2 \ 3 \ 2^3) \preceq (7 \ 7 \ 7) \preceq (21, 21 - 5) \]
\[ R = (662322) \quad (6^2 \, 3 \, 2^3) \preceq (7 \, 7 \, 7) \preceq (21, 21 - 5) \]

\[ \xi = (777) \notin \text{supp} R \]

\[ \xi_1 - r_1 + \xi_2 - r_2 = 7 - 6 + 7 - 6 \leq 3 - 1 \quad \xi_3 = 7 \geq 2 + 3 + 2 + 2 - 2 \]

\[ r_1 + 1 = 6 + 1 \geq 2 + 3 + 2 + 2 - 2 \]

\[ r_2 + 1 = 6 + 1 \geq 2 + 3 + 2 + 2 - 2 \]
\[ R = (662322) \quad (6^2 3 2^3) \preceq (7 7 7) \preceq (21, 21 - 5) \]

\[ \xi = (777) \notin \text{supp} R \]
\[ \xi_1 - r_1 + \xi_2 - r_2 = 7 - 6 + 7 - 6 \leq 3 - 1 \quad \xi_3 = 7 \geq 2 + 3 + 2 + 2 - 2 \]
\[ r_1 + 1 = 6 + 1 \geq 2 + 3 + 2 + 2 - 2 \]
\[ r_2 + 1 = 6 + 1 \geq 2 + 3 + 2 + 2 - 2 \]

\[ (876) \in \text{supp} R \]
Examples

- Ribbons whose column and row lengths differ in one unity have full support $[(t^m, (t - 1)^n); (mt + n(t - 1) - m - n + 1, m + n - 1)]$.

- The support of a ribbon $R = (r_1, r_2, r_3)$ has full interval except when $r = (r_1, r_2, r_3)$ or $r = (r_2, r_3, r_1)$ with $r_1 \geq r_2 + r_3$.

- $R = (6222276), (7662222)$.

\[
6 + 2 \geq 2 + 2 + 2 + 2 - 2 \\
7, 6 \geq 2 + 2 + 2 + 2 - 2 .
\]

Then $\xi = (6 + 2, 7, 6, 2 + 2 + 2 + 2 - 2) \notin supp(R)$. 


